III(D) Finally we are ready to consider models that include vector bosons associated with local gauge invariance. For simplicity let's first consider U(1) gauge invariance that is free non-interacting photons A^μ.

The observables of the theory depend only on $\vec{E}$ & $\vec{B}$ i.e. $F^{\mu\nu}$. Hence we require that when expressed in terms of the potential A^μ they be gauge invariant

$$U^{-1}(\Lambda) A^\mu(x) U(\Lambda) = A^\mu(x) + \partial^\mu \Lambda(x)$$

and indeed $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ is gauge invariant

$$U^{-1} F^{\mu\nu} U = F^{\mu\nu}.$$

The dynamics should also preserve the gauge invariance since if we start with a gauge invariant quantity we do not want it to evolve into something that is non-invariant — therefore not physical. Hence the invariant Lagrangian is given by

$\mathcal{L}_{inv} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ (again we stop at dimension 4 for renormalizability reasons)

The Euler-Lagrange equations simply yield

$$\partial_\mu F^{\mu\nu} = 0 = (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu$$

and the ETCR involve the momenta

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -F^{0\mu}$$

As we recall we have 2 related problems due to the gauge invariance

1) $\Pi^0 = 0$ so A^0 has no canonical momentum — how do we quantize?

2) $\partial_\mu F^{\mu\nu} = 0$ does not determine A^μ uniquely since $(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu)$ is a non-invertible differential operator.

That is the gauge invariance requires us to deal with equivalence classes of fields, each class defined as potentials related by a gauge transformation;
if $\exists$ a $\Lambda(x)$ $\ni$ $A'_\mu = A_\mu + \partial_\mu \Lambda$ then A_μ and A'_μ are in the same class. Physical quantities are given in terms of equivalence classes only, whichever representative from an equiv. class we use to evaluate the observable it results in the same observation. In particular recall in QED we specified a member of the equivalence class by "choosing a gauge". We did this in 2 ways, first in the Coulomb gauge given by $\vec{\nabla} \cdot \vec{A} = 0$ and the Lorentz gauge $\partial_\mu A^\mu = 0$. In the Coulomb gauge A_0 and A_3 were eliminated as constrained fields and the Hamiltonian as well as all observables were written in terms of the physical transverse photon degrees of freedom A_1, A_2 and their canonically conjugate momentum. The price we paid for dealing with physical quantities only was that we gave up manifest Lorentz invariance; The matrix elements of observables had Lorentz covariance to be proven. In fact we carried

out such a proof for the S-matrix elements in QED. We showed that the Coulomb interaction we had in the Hamiltonian cancelled the non-Lorentz covariant parts of the photon propagator to result in Lorentz invariant matrix elements.

So we decided it was preferable to avoid such a complicated procedure and instead we choose a gauge which was Lorentz invariant, for example the Lorentz gauge $\partial_\mu A^\mu = 0$. Indeed all observables were manifestly covariant but the price we paid was that A^μ created 2 additional unphysical degrees of freedom, A_0 the scalar photon and A_3 the longitudinal photon and the ~~Fock~~ space $\mathcal{V}$ of states was no longer a Hilbert space. Gupta and Bleuler determined how to characterize the physical subspace $\mathcal{H}_{phys}$ of states in a Lorentz invariant manner:

If $\partial_\mu A^{\mu +} |\phi\rangle = 0$ then $|\phi\rangle \in \mathcal{H}_{phys}$.

We then showed explicitly that the physical state matrix elements of observables was gauge invariant — it gave the same expression as in the Coulomb gauge.

Hence the physical state S-matrix elements were the same.

From a more general point of view we must show that, no matter what gauge choice we might have made, when physical states scatter only physical states can be produced, i.e. we start with no scalar photons we cannot end up with scalar photons. That is we are able to express the physical S-operator in terms of the in-fields of the physical particles alone. More directly stated the physical state S-matrix elements themselves must be unitary in $\mathcal{H}_{phys}$. Cryptically we have physical states $\{|n\,phys\rangle\}$ and unphysical state $\{|m\,unphys\rangle\}$. The S-operator is (pseudo-) unitary in the indefinite metric Fock space $\mathcal{V}$ i.e. $SS^\dagger = 1$, when we sum over all states.

For S to be unitary in $\mathcal{H}_{phys}$ we must show that

$$\sum_n \langle \psi\, phys | S | n\, phys \rangle \langle n\, phys | S^\dagger | \phi\, phys \rangle = \langle \psi\, phys | \phi\, phys \rangle$$

If this is the case then S when acting on $\mathcal{H}_{phys}$ can be written in terms of the physical in-fields only & that is exactly what S would be if found in the Coulomb gauge. Hence gauge invariance of S boils down to showing that S is unitary on $\mathcal{H}_{phys}$, we then have a unique physical state matrix element with the required properties.

To show this recall that $SS^{\dagger} = 1$ in $\mathcal{V}$ So

$$\sum_n' S|n\,phys\rangle\langle n\,phys|S^{\dagger}$$

$$+\sum_m' S|m\,unphys\rangle\langle m\,unphys|S^{\dagger} = 1$$

Taking physical state matrix elements, we find we must have

$$\sum_m' \langle \psi\,phys|S|m\,unphys\rangle\langle m\,unphys|S^{\dagger}|\phi\,phys\rangle = 0$$

if S is to be unitary on $\mathcal{H}_{phys}$.

That is If a unitary (preserves probability) S_{phys} (S restricted to $\mathcal{H}_{phys}$) can be written as

$$S_{phys} = \sum_n' |\text{phys } n \text{ in}\rangle\langle \text{phys } n \text{ out}|$$

then we must have that the $\sum'_{\text{unphysical}}$ states vanish in the unitarity equation above.

Now recall that we can use the LSZ reduction formulae to write this in terms of time ordered functions. Specifically the ~~the~~ physical state projector can be written as

$$\mathcal{P} = :e^{\int dx\,dy\, \phi^{phys}_{in}(x) K(x,y) \frac{\delta}{i\delta J(y)}}:$$

where ϕ^{phys}_{in} denotes all the physical components of the in-fields i.e. $A^{in}_{1,2}$, ψ_{in}, $\bar{\psi}_{in}$ etc. $K(x,y)$ are the free in-field inverse 2-point functions and J the sources. Then the generalized unitarity conditions are

$$\overrightarrow{\mathcal{P}}\, Z[J]\, e^{\Omega}\, \bar{Z}[J]\, \overleftarrow{\mathcal{P}} = 1$$

where
$$\Omega = \int dx \cdots dw\, \frac{\overleftarrow{\delta}}{i\delta J(x)} K(x,y) \Delta^{+}(y,z) K(z,w) \frac{\overrightarrow{\delta}}{i\delta \bar{J}(w)}$$

~~other~~ with Δ^+ the 2-point Wightman function and we are summing over all physical and unphysical sources.

If S is to be unitary on $\mathcal{H}_{phys}$ then we must have

$$\vec{\mathcal{Q}}\, Z[J]\, e^{\mathcal{D}_{unphys}} \bar{Z}[J]\, \overleftarrow{\mathcal{Q}} = 0$$

where $\mathcal{D}_{unphys} = \int dx \dots dw \frac{\overleftarrow{\delta}}{i\delta J(x)} K(x,y) \Delta^+_{unphys}(y,z) K(z,w) \frac{\overrightarrow{\delta}}{i\delta J(w)}$

with Δ^+_{unphys} the unphysical commutators only — hence $\mathcal{D}_{unphys}$ involves derivatives wrt the unphysical fields' sources only.

Gupta-Bleuler showed that $\partial_\lambda A^\lambda$ is the unphysical field; and its differential operator is just ∂_x^2 since it's massless, so we must show that

$$\partial^2_{y_1} \cdots \partial^2_{y_m} \left. \langle 0 | T\, \phi^{phys}(x_1) \cdots \phi^{phys}(x_n)\, \partial_{\lambda_1} A^{\lambda_1}(y_1) \cdots \partial_{\lambda_m} A^{\lambda_m}(y_m) | 0 \rangle \right|_{\text{mass shell}} = 0$$

if S is to be unitary on $\mathcal{H}_{phys}$.

But this is precisely what the gauge invariance of the Green functions implies. Recall in the general Stückelberg gauge the Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\lambda A^\lambda)^2 - J^\mu A_\mu$$

with field equations

$$\partial_\mu F^{\mu\nu}(x) + \frac{1}{\alpha}\partial^\nu \partial_\lambda A^\lambda(x) = J^\nu .$$

Hence $\frac{1}{\alpha}\partial^2 \partial_\lambda A^\lambda = 0$ where $J^\nu = e\bar{\psi}\gamma^\nu\psi$ is conserved and $\partial_\lambda A^\lambda$ is a free field.

As usual when we calculate the consequences of this for the time ordered functions we have contact terms on the right hand side —

$$\partial_\nu \langle 0|T[\partial_\mu F^{\mu\nu}(x) + \frac{1}{\alpha}\partial^\nu\partial_\lambda A^\lambda(x) - J^\nu(x)]\,X|0\rangle$$
$$= \sum_{i=1}^{l} (+i)\partial_x^{\mu_i}\delta^4(x-x^i)\langle 0|T A^{\mu_1}(x_1)\cdots \cancel{A^{\mu_i}(x_i)} \cdots \bar{\psi}(z_n)|0\rangle$$

Applying the Fermion field equations to calculate $\partial_\nu J^\nu$ we obtain the gauge Ward identity (more on this later)

$$\frac{1}{\alpha}\partial_x^2 \langle 0|T\partial_\lambda A^\lambda(x)\,X|0\rangle$$
$$= \sum_{i=1}^{l} (+i)\partial_x^{\mu_i}\delta^4(x-x^i)\langle 0|T A^{\mu}(x_1)\cdots \cancel{A^{\mu_i}(x_i)}\cdots \bar{\psi}(z_n)|0\rangle$$
$$+ \sum_{j=1}^{m} -ie\,\delta^4(x-y_j)\langle 0|T X|0\rangle$$
$$+ \sum_{k=1}^{n} ie\,\delta^4(x-z_k)\langle 0|T X|0\rangle$$

But on mass shell the RHS is zero so

$$\frac{1}{2}\partial_x^2 \langle 0|T\,\partial_\lambda A^\lambda(x)\,X(0)\rangle\Big|_{m.s.} = 0 .$$

Thus S is gauge invariant and unitary on $\mathcal{H}_{phys}$, it is the same as the Coulomb gauge S operator.

Of course we have made use of the Stückelberg gauge Lagrangian in order to quantize QED and we relied on perturbation theory to define the Green functions.

We would like to express the generating functional in terms of a path integral instead. Let us first proceed intuitively then more rigorously in defining it.

If we were to proceed naively the Green function generating functional would be represented by

$$Z[J] \equiv \int [dA^\mu]\, e^{i\int dx(\mathcal{L} + J_\mu A^\mu)}$$

with $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ so that

$$\int dx\, -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} = \int d^4x\, \tfrac{1}{2} A_\mu (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu$$

Thus

$$Z[J] = \int [dA^\mu]\, e^{\frac{i}{2}\int dx \{A_\mu (\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu) A_\nu + J_\mu A^\mu\}}$$

$$= \frac{1}{\det(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu)}\, e^{-\frac{i}{2}\int dx\,dy\, J_\mu(x) [\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu]^{-1}(x,y) J_\nu}$$

Unfortunately $(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu)$ is a projection operator hence $\det(\ ) = 0$ and $[\ ]^{-1}$ does not exist. (Recall in momentum space we have the equation

$$K^{\mu\nu} \equiv a\,(P_T^{\mu\nu}(p)) + b\,(P_L^{\mu\nu}(p))$$

with $P_T^{\mu\nu} \equiv \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)$, $P_L^{\mu\nu} = \frac{p^\mu p^\nu}{p^2}$

So $P_{T\atop L}^{\mu\nu} P_{T\atop L\,\nu}{}^{\lambda} = P_{T\atop L}^{\mu\lambda}$; $P_T^{\mu\nu} P_{L\nu}{}^{\lambda} = 0$

and

$$P_T^{\mu\nu} + P_L^{\mu\nu} = g^{\mu\nu} .$$

have an inverse

$$K^{\mu\nu^{-1}} = \frac{1}{a} P_T^{\mu\nu} + \frac{1}{b} P_L^{\mu\nu}$$

Hence if $(\partial_x^2 g^{\mu\nu} - \partial_x^\mu \partial_x^\nu) \Delta_{F\nu}{}^{\lambda}(x-y) = +i g^{\mu\lambda} \delta^4(x-y)$

In momentum space we have

$$-p^2\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) \tilde{\Delta}_{F\nu}{}^{\lambda}(p) = i g^{\mu\lambda}$$

but $P_T^{\mu\nu}$ has no inverse!! So $\tilde{\Delta}_F$ is indeterminate.)

The point being that the action is invariant under gauge transformations (it is said to be constant on the orbits of the gauge group). That is for fixed A^μ and $g(x) = e^{\Lambda(x)}$ ranging over all G

$\mathcal{L}(A_\mu^g)$ is constant where $A_\mu^g \equiv U^{-1}(g) A_\mu U(g) + U^{-1}(g) \partial_\mu U(g)$

Hence the path integral diverges since for those variations of A^μ along the orbits, the action does not provide any Gaussian damping. The divergence is just proportional to the volume of the orbits $\int[dg(x)]$ and this should be factored out before defining $Z[J]$. That is we should define $Z[J]$ by not summing over all gauge field configurations but only over equivalence classes. That is, over one element from each orbit, the gauge invariance of $\mathcal{L}$ guarantees the same answer independent of the el. chosen. Thus we must specify a hypersurface in field space which intersects each orbit only once and we then define the integral over the hypersurface. Hence

if $f(A^\mu)=0$ is the equation of the hypersurface then $f(A_\mu^g)=0$ must have a unique solution for g for each A_μ if the hypersurface intersects each orbit only once.

Consider the integral

$$\int [dg]\, \delta[f(A_\mu^g)] \equiv \Delta_f^{-1}[A_\mu]$$

where recall $U(g) = e^{\Lambda(x)}$ and $g(x)$ are parameterized by the phase $\Lambda(x)$ & hence we have that $[dg] = [d\Lambda]$

Also $[dgg'] = [dg]$ since $U(g)U(g') = e^{\Lambda+\Lambda'} = U(gg')$

and $[d\Lambda] = [d(\Lambda+\Lambda')]$

hence $[dg]$ is the group invariant or Hurwitz measure.

$\Delta_f^{-1}[A_\mu]$ is gauge invariant since

$$\Delta_f^{-1}[A_\mu^g] = \int [dg']\, \delta[f(A_\mu^{gg'})]$$

$$= \int [dgg']\, \delta[f(A_\mu^{gg'})]$$

$$= \int [dg'']\, \delta[f(A_\mu^{g''})] = \Delta_f^{-1}[A_\mu].$$

So consider

$$Z[0] = \int [dA_\mu] \left(\Delta_f[A_\mu] \int [dg]\, \delta[f(A_\mu^g)] \right) e^{i\int dx \mathcal{L}_{inv}}$$

$$= \int [dg] \int [dA_\mu] \Delta_f[A_\mu] \delta[f(A_\mu^g)] e^{i\int dx \mathcal{L}_{inv}}$$

Now change integration variables $A_\mu = A_\mu^{(g^{-1})}$, but $[dA_\mu] = [dA_\mu^{g^{-1}}]$ and $\mathcal{L}_{inv}(A_\mu) = \mathcal{L}_{inv}(A_\mu^{g^{-1}})$

So with $\Delta_f[A_\mu] = \Delta_f[A_\mu^{g^{-1}}]$ we have

$$Z[0] = \int [dg] \left(\int [dA_\mu] \Delta_f[A_\mu] \delta[f(A_\mu)] e^{i\int dx \mathcal{L}_{inv}} \right)$$

Hence the orbit volume factors out $\int [dg]$ and the remaining integral is independent of g hence we <u>define</u> the correct $Z[J]$ by

$$\boxed{Z[J] \equiv \frac{\int [dA_\mu]\, \Delta_f[A_\mu]\, \delta[f(A_\mu)]\, e^{i\int dx (\mathcal{L}_{inv}(A_\mu) + J_\mu A^\mu)}}{\int [dA_\mu]\, \Delta_f[A_\mu]\, \delta[f(A_\mu)]\, e^{i\int dx \mathcal{L}_{inv}(A_\mu)}}}$$

Now we can evaluate $\Delta_f[A_\mu]$ by expanding $f(A_\mu^g)$ about A_μ for infinitesimal Λ

$$f(A_\mu^g) = f(A_\mu + \partial_\mu \Lambda)$$

$$= f(A_\mu) + \underbrace{\int d^4y\, M_f(x,y)\, \Lambda(y)}_{= \text{gauge variation of } f(A_\mu)} + O(\Lambda^2)$$

So

$$\Delta_f^{-1}[A_\mu] = \int [dg]\, \delta[f(A_\mu^g)]$$

$$= \int [d\Lambda]\, \delta\left[f(A_\mu) + \int d^4y\, M_f(x,y)\, \Lambda(y)\right]$$

but in $Z[J]$, Δ_f is mult. by $\delta[f(A_\mu)]$

So we only need integral at $f(A_\mu) = 0$

$$\Delta_f^{-1}[A_\mu] = \int [d\Lambda]\, \delta\left[\int dy\, M_f(x,y)\Lambda(y)\right]$$

Changing variable $\Lambda'(x) = \int dy\, M_f(x,y)\Lambda(y)$

$$\Rightarrow \quad [d\Lambda] = \frac{1}{\det M_f}[d\Lambda']$$

Hence $\boxed{\Delta_f[A_\mu] = \det M_f}$

where M_f is given by the gauge variation of the gauge fixing term $f(A_\mu)$.

Hence we find

$$\boxed{Z[J] = \frac{\int [dA_\mu]\, \delta[f(A_\mu)](\det M_f)\, e^{i\int dx\left(\mathcal{L}_{inv}(A_\mu) + J_\mu A^\mu\right)}}{\int [dA_\mu]\, \delta[f(A_\mu)](\det M_f)\, e^{i\int dx\, \mathcal{L}_{inv}(A_\mu)}}}$$

In short we integrate over all A^g_μ but choose one g for each orbit

$$Z\{0\} = \int [dA^g]\, \delta[g - g_0] \cdot e^{i\int dx\, \mathcal{L}_{inv}}$$

but $\delta[g-g_0] = \delta[e^{\Lambda} - e^{\Lambda_0}] = \delta[(\Lambda - \Lambda_0)]$ and $[dA^g] = [dA]$

$$Z[0] = \int [dA]\, \delta[\Lambda - \Lambda_0]\, e^{i\int dx\, \mathcal{L}_{inv}}$$

Now $f(A^g) = 0$ determines Λ uniquely to be Λ

$$0 = f(A^g) = f(A) + \int dy\, M_f(x,y)\,(\Lambda(y) - \Lambda_0(y))$$

$$\Rightarrow \delta[f(A)] = \delta[M_f(\Lambda - \Lambda_0)] = \frac{1}{\det M_f}\,\delta[\Lambda - \Lambda_0]$$

$$\Rightarrow \delta[\Lambda - \Lambda_0] = (\det M_f)\,\delta[f(A_\mu)]$$

Hence

$$Z\{0\} = \int [dA_\mu]\, \delta[f(A_\mu)]\, (\det M_f)\, e^{i\int dx\, \mathcal{L}_{inv}}.$$

Examples: 1) Lorentz gauge

$$f(A) = \partial_\mu A^\mu$$

$$f(A_\mu + \partial_\mu \Lambda) = \partial_\mu A^\mu + \partial^2 \Lambda$$

$$\Rightarrow \boxed{M_f(x,y) = \partial_x^2 \delta^4(x-y)}$$

Now $\det M_f = \det \partial_x^2$ which is just a (∞) constant and cancels out in $Z[J]$

2) Coulomb gauge

$$f(A) = \vec{\nabla} \cdot \vec{A}$$

$$f(A + \partial\Lambda) = \vec{\nabla} \cdot \vec{A} + \nabla^2 \Lambda$$

$$\Rightarrow M_f(x,y) = \nabla_x^2 \delta^4(x-y)$$

again $\det M_f = \det \nabla_x^2$ a constant which factors out of $Z[J]$.

3) Stückelberg gauge

$$f(A) = \partial_\mu A^\mu - a(x) \qquad a(x) = \text{arbitrary function}$$

$$M_f = \partial_x^2 \, \delta^4(x-y) \quad \text{again}$$

and $\det \partial_x^2$ factors out of $Z[J]$

ignoring renormalization i.e. $J \to Z^{1/2} J$ effects — have more on this later

Now path integral is independent of the gauge choice, hence we can sum over $a(x)$ with a gaussian damping

$$Z[J] = \frac{\int [dA_\mu][da]\, e^{-\frac{i}{2\alpha}\int dx\, a^2(x)}\, \delta[f(A)-a]\,(\det M_f)\, e^{i\int dx\,(\mathcal{L}_{inv}(A) + J_\mu A^\mu)}}{N}$$

$$= \frac{\int [dA_\mu]\,(\det M_f)\, e^{i\int d^4x\,\left(\mathcal{L}_{inv}(A_\mu) - \frac{1}{2\alpha} f(A)^2 + J_\mu A^\mu\right)}}{N}$$

$$\boxed{Z[J] = \frac{\int [dA_\mu]\, e^{i\int d^4x\,\left(\mathcal{L}_{inv}(A_\mu) - \frac{1}{2\alpha}(\partial_\lambda A^\lambda)^2 + J_\mu A^\mu\right)}}{\int [dA_\mu]\, e^{i\int dx\,\left(\mathcal{L}_{inv} - \frac{1}{2\alpha}(\partial_\lambda A^\lambda)^2\right)}}}$$

4) Veltman Gauge (Non-linear gauge):

$$f(A) = \partial_\lambda A^\lambda + \frac{\beta}{2} A_\lambda A^\lambda$$

$$f(A+\partial\Lambda) = \partial^2\Lambda + \beta A_\mu \partial^\mu \Lambda \quad (+f(A))$$

hence

$$M_f(x,y) = \left(\partial_x^2 + \beta A_\mu(x)\partial_x^\mu\right)\delta^4(x-y)$$

and

$$\det M_f = \det\left(\partial^2 + \beta A_\mu \partial^\mu\right)$$

this is no longer a constant but depends upon the photon field and hence cannot be factored out of the integral and cancelled in $Z[J]$!

However we can recall that for anti-commuting complex fields we have

$$\int [dc][d\bar{c}]\, e^{i\int dx\,dy\, \bar{c}(x) K(x,y) c(y) + i\int dx (\bar{\xi} c + \bar{c}\xi)}$$

$$= (\det iK)\, e^{-i\int dx\,dy\, \bar{\xi}(x) K^{-1}(x,y)\, \xi(y)}$$

where $C(x)$ is a one component Grassmann variable and similarly $\bar{C}(x)$

Thus we may re-write the $\det M_f$ by means of this trick

$$(\det M_f) = \int [dc][d\bar{c}] e^{i\int d^4x d^4y\, \bar{C}(x) M_f(x,y) C(y)}$$

This is known as the Fadeev-Popov trick and determinant. $C, \bar{C}$ are ϕ-π ghost fields — they are anti-commuting complex scalar fields.

Hence in general we have that

$$Z[J] = \frac{\int [dA_\mu][dc][d\bar{c}] e^{i\int dx [\mathcal{L} + J_\mu A^\mu]}}{\int [dA_\mu][dc][d\bar{c}]\, e^{i\int dx \mathcal{L}}}$$

with the $\mathcal{L} = \mathcal{L}_{inv} + \mathcal{L}_g + \mathcal{L}_{\phi\pi}$

where $\mathcal{L}_{inv}$ is gauge invariant

$$\mathcal{L}_g = -\frac{1}{2\alpha} f(A_\mu)^2$$

$$\mathcal{L}_{\phi\pi} = \int dy\, \bar{C}(x) M_f(x,y) C(y)$$

for the Veltman gauge we have

$$\mathcal{L}_g = -\frac{1}{2\alpha}\left(\partial_\lambda A^\lambda + \frac{\beta}{2} A_\lambda A^\lambda\right)^2$$

$$\mathcal{L}_{\phi\pi} = -\partial_\mu \bar{c}\, \partial^\mu c + \beta \bar{c} A_\mu \partial^\mu c \, .$$

Note: $-\partial_\mu \bar{c}\, \partial^\mu c \Rightarrow$ ghosts states have negative norm!!

$$\langle 0 | T\, \tilde{C}(p)\, \bar{C}(0) | 0 \rangle = \frac{-i}{p^2 + i\epsilon}$$

-1 residue $\Rightarrow$ $|\vec{k}, \phi\text{-}\pi\rangle$ has norm -1

$$\langle \phi\text{-}\pi, \vec{k} | \vec{p}, \phi\pi \rangle = -\frac{1}{(2\pi)^3 2\omega_k}\, \delta^3(\vec{p} - \vec{k}) \, .$$

So 2 more unphysical degrees of freedom. now $\partial_\lambda A^\lambda$ is not free - the A_0, A_3 modes cancel against $c, \bar{c}$ modes and S is unitary in $\mathcal{H}_{phys}$. - more on this later.

H.W. What are Feynman Rules?

Besides this intuitive derivation of $Z[J]$ given by ϕ-π they also derived this form of the path integral from the quantization of QED in the Coulomb gauge. Hence they rigorously derived the above results. We will do the same in the non-abelian case rather than spend time on QED now.

Finally let's consider the gauge transformation of $Z[J]$ since we still need to show that the unphysical modes cancel from the unitarity sum.

Now make the change of variables in the path integral (go to 402' first)

$$A'_\mu = A_\mu + \partial_\mu \Lambda$$

Then

$$\delta \mathcal{L}_{inv} = \mathcal{L}_{inv}(A') - \mathcal{L}_{inv}(A) = 0$$

$$\delta \mathcal{L}_g = \mathcal{L}_g(A') - \mathcal{L}_g(A) = -\frac{1}{\alpha} f(A)(\partial^2 \Lambda + \beta A_\lambda \partial^\lambda$$

$$\delta \mathcal{L}_{\phi\pi} = \beta \bar{c}\, \partial_\mu c\, \partial^\mu \Lambda$$

In general let's consider a change of variables in $Z[J]$

$$Z[J] = \int [d\varphi]\, e^{i\int dx\, (\mathcal{L}(\varphi) + J\varphi)}$$

$\varphi' = \varphi + \delta\varphi$, $[d\varphi'] = \mathcal{J}[d\varphi]$ where $\mathcal{J}$ is the Jacobian

then $\mathcal{L}(\varphi') = \mathcal{L}(\varphi + \delta\varphi) \equiv \mathcal{L}(\varphi) + \delta\mathcal{L}$

Hence

$$Z[J] = \int [d\varphi']\, e^{i\int dx\, (\mathcal{L}(\varphi') + J\varphi')}$$
$$= \int [d\varphi]\, \mathcal{J}\, e^{i\int dx\, [\mathcal{L}(\varphi) + J\varphi]}\, e^{i\int dx\, (\delta\mathcal{L} + J\delta\varphi)}$$

Now for unitary transformations $\mathcal{J} = 1$

So

$$Z[J] = \int [d\varphi]\, e^{i\int dx\, [\mathcal{L} + J\varphi)} \left(1 + i\int dx\, (\delta\mathcal{L} + J\delta\varphi)\right)$$

$\Rightarrow$ QAP

$$\boxed{i\int dx\, \delta\mathcal{L}\, Z[J] = -i\int dx\, J\, \delta\varphi\left[\tfrac{\delta}{i\delta J}\right] Z[J]}$$

So

$$Z[J] = \frac{\int [dA_\mu][dc][d\bar{c}]\, e^{i\int dx\left[\mathcal{L}+\delta\mathcal{L}+J_\mu A^\mu + J_\mu \partial^\mu \Lambda\right]}}{N}$$

$$= \frac{\int [dA_\mu][dc][d\bar{c}]}{N}\, e^{i\int dx\left[\mathcal{L}+J_\mu A^\mu\right]}\left(1+i\int dx\left(\delta\mathcal{L} + J_\mu \partial^\mu \Lambda\right)\right)$$

$\Rightarrow$ (example of Schwinger's Quantum Action Principle)

$$\boxed{\begin{aligned} 0 = \int [dA_\mu][dc][d\bar{c}]\, & e^{i\int dx\left[\mathcal{L}+J_\mu A^\mu\right]} \times \\ & \times \left[i\int dx\left(\delta\mathcal{L} + J_\mu \partial^\mu \Lambda\right)\right] \end{aligned}}$$

Since Λ is arbitrary we have integrands equal
$\Rightarrow$

$$\boxed{\begin{aligned} \partial_\mu J^\mu Z[J] = & -\frac{1}{\alpha}\partial_x^2 f(A)(x)\, Z[J] \\ & + \frac{\beta}{\alpha}\partial_x^\lambda \left(A_\lambda(x) f(A)(x)\right) Z[J] \\ & - \beta\, \partial_\mu\left(\bar{c}\,\partial^\mu c\right)(x)\, Z[J] \end{aligned}}$$

for $\beta = 0$ the Stückelberg gauge we have

$\partial_\mu J^\mu_{(x)} Z[J] = -\frac{1}{\alpha} \partial_x^2 (\partial_\lambda A^\lambda)_{(x)} Z[J]$

and on shell $\Rightarrow$

$$\partial_{y_1}^2 \cdots \partial_{y_m}^2 \langle 0 | T \partial_{\lambda_1} A^{\lambda_1}(y_1) \cdots \partial_{\lambda_m} A^{\lambda_m}(y_m) \, X \, |0\rangle \Big|_{ms} = 0.$$

For $\beta \neq 0$ we must invert

$[-\partial_x^2 + \beta \partial^\lambda A_\lambda] f$ to show that

the unphysical photon modes cancel the unphysical ϕ-π ghost modes also and to obtain $\langle phys | f | phys \rangle = 0$!. This is most ~~easily~~ accomplished by using the techniques of BRS symmetry discovered by Becchi-Rouet-Stora.

Before confronting this problem let's note that for non-abelian gauge symmetries we will proceed in a similar manner

There are D gauge (Yang-Mills) fields

A_μ^a, $a = 1, \ldots, D$ $\quad D = \dim$ of group $\mathcal{G}$.

The invariant Lagrangian is given by

$$\mathcal{L}_{inv} = -\tfrac{1}{4} F_{\mu\nu}^{\;a} F^{a\mu\nu}$$

with $F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_\mu^b A_\nu^c$

with g = gauge coupling constant

f_{abc} = group structure constants.

The infinitesimal gauge transformations are parametrized by D constants λ^a

$$\delta(\lambda) A_\mu^a = \partial_\mu \lambda^a + f_{abc} \lambda^b A_\mu^c$$

i.e.

$$\mathcal{U}^{-1}(\lambda) iT^a A_\mu^a(x) \mathcal{U}(\lambda) \equiv U(\lambda) A_\mu^a iT^a U^{-1}(\lambda)$$

$$+ U(\lambda) \partial_\mu U^{-1}(\lambda)$$

$$= [\partial_\mu \lambda^a + f_{abc} \lambda^b A_\mu^c] iT^a$$

with T^a some Rep. matrices of G.

Hence we will need D-gauge fixing conditions $f^a(A)=0$ to choose a hypersurface which uniquely determines g, a group element

$$f^a(A^g)=0.$$

The ϕ-π quantization procedure then results in

$$Z\{J^a_\mu\} = \int [dA^a_\mu]\, \delta[f^a(A_\mu)] \det M_f \times$$
$$\times\, e^{i\int dx\,(\mathcal{L}_{inv} + J^a_\mu A^{a\mu})}$$

$$= \int [dA^a_\mu][dc^a][d\bar{c}^a]\, e^{i\int dx\,(\mathcal{L} + J^a_\mu A^{a\mu})}$$

with
$$\mathcal{L} = \mathcal{L}_{inv} + \mathcal{L}_g + \mathcal{L}_{\phi\pi} \quad ; \quad \mathcal{L}_{inv} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$
$$\mathcal{L}_g = -\tfrac{1}{2\alpha} f^a f^a \quad ; \quad \mathcal{L}_{\phi\pi} = \int d^4y\, \bar{c}^a(x) M^{ab}_f(x,y)\, c^b(y)$$

and

$$\int d^4y\, M_f^{ab}(x,y)\, \lambda^b(y) = \delta(\lambda) f^a(A)$$
$$= f^a(A_\mu + \delta A_\mu) - f^a(A_\mu) .$$

However before we explore this and BRS invariance in more detail let's back up and consider the description of symmetries in general in our functional formulation of field theory first.