We are interested in defining integrals over real functions $\varphi(x)$ and functionals $F[\varphi]$:

$$\int [d\varphi]\, F[\varphi] .$$

One way to define such an integral is to imagine space-time divided up into cells V_i labelled by index i with $\varphi(x)$ replaced by the average of $\varphi(x)$ over the cell volume

$$\varphi_i = \frac{1}{\epsilon^4} \int_{V_i^4} d^4x\, \phi(x)$$

with each φ_i as an independent variable
Then $F[\varphi]$ becomes an ordinary function $F(\varphi_i)$ of the set of variables φ_i; The integral is then defined as the limit of ordinary integrals over φ_i as the cell volume goes to zero, when the limit exists:

$$\int [d\varphi]\, F[\varphi] = \lim_{\substack{\epsilon \to 0 \\ n \to \infty}} \int d\varphi_1 \cdots d\varphi_n\, F(\varphi_1 \cdots \varphi_n \, ; \epsilon)$$

Alternatively we could also expand the functions $\varphi(x)$ in terms of a complete orthonormal set of functions

$$\varphi(x) = \sum_{\alpha=1}^{\infty} \varphi_\alpha f_\alpha(x)$$

$$\sum_\alpha f_\alpha(x) f_\alpha(x') = \delta^4(x-x') \quad , \int d^4x\, f_\alpha(x) f_\beta(x) = \delta_{\alpha\beta}$$

Then we can regard the φ_α as indep. variables
then the functional integral is defined as

$$\int [\mathcal{D}\varphi]\, F\{\varphi\} = \lim_{n\to\infty} \int d\varphi_1 \cdots d\varphi_n\, F(\varphi_1, \dots, \varphi_n) .$$

We can only show that this integral exists for gaussian functionals all others are open questions.

Consider

$$\int \left[\frac{d\varphi}{\sqrt{2\pi}}\right] e^{-\frac{1}{2}\int d^4x\, \varphi(x)\varphi(x)}$$

$$= \lim_{n\to\infty} \int_{-\infty}^{+\infty} \frac{d\varphi_1}{\sqrt{2\pi}} \cdots \frac{d\varphi_n}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\int d^4x \sum_{\alpha,\beta=1}^{n} \varphi_\alpha \varphi_\beta f_\alpha(x) f_\beta(x)}$$

$$= \lim_{n\to\infty} \int_{-\infty}^{+\infty} \frac{d\varphi_1}{\sqrt{2\pi}} \cdots \frac{d\varphi_n}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\sum_{\alpha=1}^{n} \varphi_\alpha^2}$$

$$= \lim_{n\to\infty} \prod_{\alpha=1}^{n} \left[\int_{-\infty}^{+\infty} \frac{d\varphi_\alpha}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\varphi_\alpha^2} \right]$$

but $\displaystyle\int \frac{d\varphi}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\varphi^2} = 1$

So $$\boxed{\int \left[\frac{d\varphi}{\sqrt{2\pi}}\right] e^{-\frac{1}{2}\int d^4x\, \varphi(x)\varphi(x)} = 1}$$

Assuming the functional integral depends linearly on the integrand and is translationally invariant

$$\int [d\varphi] F[\varphi] = \int [d\varphi] F[\varphi + J]$$

we find

$$\int \left[d \tfrac{\varphi}{\sqrt{2\pi}}\right] e^{-\frac{1}{2}\int (\varphi+J)^2_{(x)} d^4x} = 1$$

$$\Rightarrow \quad e^{+\frac{1}{2}\int d^4x\, J(x) J(x)} = \int \left[d \tfrac{\varphi}{\sqrt{2\pi}}\right] e^{-\frac{1}{2}\int d^4x \left[\varphi^2 + 2J\varphi\right]}$$

replacing $J \to iJ$ in this identity, we obtain

$$e^{-\frac{1}{2}\int d^4x\, J(x) J(x)} = \int \left[d \tfrac{\varphi}{\sqrt{2\pi}}\right] e^{-\frac{1}{2}\int d^4x \left[\varphi^2 - 2iJ\varphi\right]}$$

So we can evaluate any functional $F[\varphi]$ by

$$\boxed{\begin{aligned} &\int \left[d \tfrac{\varphi}{\sqrt{2\pi}}\right] F[\varphi]\, e^{\int d^4x \left[-\frac{1}{2}\varphi^2 + iJ\varphi\right]} \\ &\quad = F\left[\frac{\delta}{i\delta J}\right] \int \left[d \tfrac{\varphi}{\sqrt{2\pi}}\right] e^{\int dx \left[-\frac{1}{2}\varphi^2 + iJ\varphi\right]} \\ &\quad = F\left[\frac{\delta}{i\delta J}\right] e^{-\frac{1}{2}\int dx\, J(x) J(x)} \end{aligned}}$$

Further we imagine changing variables

$$\varphi'(x) = \int K(x,y)\,\varphi(y)\,d^4y \qquad K(x,y) \text{ indep. of } \varphi$$

$\Rightarrow$

$$\sum_\beta{}' \varphi'_\beta f_\beta(x) = \int dy \quad K(x,y) \sum_\beta{}' \varphi_\beta f_\beta(y)$$

So,

$$\sum_\beta{}' \int dx\, \varphi'_\beta f_\alpha(x) f_\beta(x) = \int dx\, dy \sum_\beta{}' \quad f_\alpha(x) f_\beta(y)\, \varphi_\beta\, K(x,y)$$

$$\varphi'_\alpha = \sum_\beta{}' \left(\int dx\,dy\, f_\alpha(x)\, K(x,y) \quad f_\beta(y)\right)\varphi_\beta$$

$$\boxed{\varphi'_\alpha \equiv \sum_\beta{}' K_{\alpha\beta}\,\varphi_\beta}$$

So for

$$[d\varphi] = d\varphi_1 \cdots d\varphi_n \quad , \quad [d\varphi'] = d\varphi'_1 \cdots d\varphi'_n$$

we find

$$[d\varphi'] = (\det K_{\alpha\beta})\,[d\varphi]$$

with the $n \to \infty$ understood.

writing $K(x,y)$ as $\mathbb{K}_{xy}$ a continuous index matrix we have (as well as $f_\alpha(x) = f_{\alpha x}$ etc)

$$K_{\alpha\beta} = f_{\alpha x}\,\mathbb{K}_{xy}\,f_{\beta y}$$

or in matrix notation $\quad K = f\,k\,f^T$

Since $\quad f_{\alpha x} f_{\beta x} = \delta_{\alpha\beta} \qquad f f^T = 1$

$$\Rightarrow f^T = f^{-1}$$

thus $\quad \det K = \det(f k f^T) = \det(f k f^{-1})$

$$= \det k\,.$$

hence with $\quad \varphi' = k\varphi$ we have

$$[d\,k\varphi] = (\det k)[d\varphi]\,.$$

So we can evaluate integrals of the form

$$\int \left[d\frac{\varphi}{\sqrt{2\pi}}\right] e^{\int d^4x \int d^4y \left[-\frac{1}{2}\varphi(x)K(x,y)\varphi(y)\right] + i\int d^4x\, J(x)\varphi(x)}\, F[\varphi]$$

$$= F\left[\frac{\delta}{i\delta J}\right] \int \left[d\frac{\varphi}{\sqrt{2\pi}}\right] e^{-\frac{1}{2}\int dx\,dy\,dz\left[\varphi(x)K^{1/2}(x,y)K^{1/2}(y,z)\varphi(z)\right] + i\int dx\,dy\,dz\, J(x)K^{-1/2}(x,y)K^{1/2}(y,z)\varphi(z)}$$

$$= F\left[\frac{\delta}{i\delta J}\right] \int \left[d\frac{\varphi}{\sqrt{2\pi}}\right] e^{-\frac{1}{2}\left[\varphi^T k^{1/2} k^{1/2}\varphi\right] + i\,J k^{-1/2} k^{1/2}\varphi}$$

So letting $\quad \varphi' = k^{1/2}\varphi$

we find

$$= F\left[\frac{\delta}{i\delta J}\right]\int\left[\frac{d\varphi'}{\sqrt{2\pi}}\right]\left(\frac{1}{\det k}\right)^{1/2} e^{-\frac{1}{2}\varphi'^T\varphi' + i(Jk^{-1/2})^T\varphi'}$$

$$= F\left[\frac{\delta}{i\delta J}\right]\left(\frac{1}{\det k^{1/2}}\right) e^{-\frac{1}{2}(Jk^{-1/2})^T(k^{-1/2}J)}$$

$$= F\left[\frac{\delta}{i\delta J}\right]\left(\frac{1}{\det k^{1/2}}\right) e^{-\frac{1}{2}\int dx\,dy\, J(x) K^{-1}(x,y) J(y)}$$

$$= \int\left[d\frac{\varphi}{\sqrt{2\pi}}\right] e^{\int dx\,dy\left[-\frac{1}{2}\varphi(x)K(x,y)\varphi(y)\right] + i\int dx\, J(x)\varphi(x)} F[\varphi] .$$

Note $\left(\frac{1}{\det k^{1/2}}\right)$ is just a constant.

Finally we can introduce the functional fourier transform $\tilde{F}[\varphi]$

$$F[J] = \int\left[d\frac{\varphi}{\sqrt{2\pi}}\right] e^{i\int dx\, J(x)\varphi(x)}\, \tilde{F}[\varphi]$$

$$= \lim_{n\to\infty}\int_{-\infty}^{+\infty} \frac{d\varphi_1}{\sqrt{2\pi}} \cdots \frac{d\varphi_n}{\sqrt{2\pi}}\, \tilde{F}(\varphi_1 \cdots \varphi_n)\, e^{iJ_\alpha\varphi_\alpha}$$

with the inverse

$$\tilde{F}[\varphi] = \lim_{n\to\infty} \int_{-\infty}^{+\infty} \frac{dJ_1}{\sqrt{2\pi}} \cdots \frac{dJ_n}{\sqrt{2\pi}} F(J_1 \cdots J_n)\, e^{-iJ_\alpha \varphi_\alpha}$$

$$= \int \left[\frac{dJ}{\sqrt{2\pi}}\right] F[J]\, e^{-i\int dx\, J(x)\varphi(x)}$$

as usual the functional δ-function can be found

$$F[J] = \int \left[d\frac{\varphi}{\sqrt{2\pi}}\right] e^{i\int dx\, J(x)\varphi(x)} \int \left[\frac{dJ'}{\sqrt{2\pi}}\right] F[J']\, e^{-i\int dy\, J'(y)\varphi(y)}$$

$$= \int \left[\frac{dJ'}{\sqrt{2\pi}}\right] F[J'] \int \left[\frac{d\varphi}{\sqrt{2\pi}}\right] e^{i\int dx\,(J(x)-J'(x))\varphi(x)}$$

$$= \int \frac{dJ_1'}{\sqrt{2\pi}} \cdots \frac{dJ_n'}{\sqrt{2\pi}} F(J_1' \cdots J_n')\, (\sqrt{2\pi})^n \delta(J_1 - J_1') \cdots \delta(J_n - J_n')$$

So

$$F[J] = \int \left[\frac{dJ'}{\sqrt{2\pi}}\right] F[J']\, \delta[J - J']$$

with

$$\delta[J - J'] = \lim_{n\to\infty} (2\pi)^{n/2} \delta(J_1 - J_1') \cdots \delta(J_n - J_n')$$

$$= \int \left[\frac{d\varphi}{\sqrt{2\pi}}\right] e^{i\int dx\,(J(x)-J'(x))\varphi(x)}$$

So we can finally return to our problem of solving the assumed dynamics of our model:

$$\left\{-\left[Z\delta^2+(m^2+a)\right]\frac{\delta}{i\delta J(x)}-\left(\frac{\lambda+c}{3!}\right)\frac{\delta^3}{(i\delta J(x))^3}\right\}Z[J]=-J(x)Z[J]$$

Now let

$$Z[J]=\int\left[\frac{d\varphi}{\sqrt{2\pi}}\right]e^{i\int dx\,J(x)\varphi(x)}\tilde{Z}[\varphi]$$

So $$\frac{\delta}{i\delta J(x)}Z[J]=\int\left[\frac{d\varphi}{\sqrt{2\pi}}\right]e^{i\int dy\,J(y)\varphi(y)}\varphi(x)\tilde{Z}[\varphi]$$

etc.

Further

$$\int\left[\frac{d\varphi}{\sqrt{2\pi}}\right] e^{i\int dx\, J(x)\varphi(x)} \frac{\delta}{\delta\varphi(x)} \tilde{Z}[\varphi]$$

$$= \int\left[\frac{d\varphi}{\sqrt{2\pi}}\right] e^{i\int dx\, J(x)\varphi(x)} \frac{\delta}{\delta\varphi(x)} \int\left[\frac{dJ'}{\sqrt{2\pi}}\right] Z[J'] e^{-i\int dx\, J'(x)\varphi(x)}$$

$$= \int\left[\frac{d\varphi}{\sqrt{2\pi}}\right]\left[\frac{dJ'}{\sqrt{2\pi}}\right](-i J'(x))\, Z[J']\, e^{i\int dx\,(J(x)-J'(x))\varphi(x)}$$

$$= \int\left[\frac{dJ'}{\sqrt{2\pi}}\right](-i J'(x)) Z[J']\, \delta[J-J']$$

$$= -i J(x)\, Z[J] \,.$$

Thus our eq. of motion become

$$\int\left[\frac{d\varphi}{\sqrt{2\pi}}\right] e^{+i\int dy\, J(y)\varphi(y)} \Big\{ -[Z\partial_x^2 + (m^2+a)]\varphi(x) - \frac{(\lambda+c)}{3!}\varphi^3(x)\Big\} \tilde{Z}[\varphi]$$

$$= \int\left[\frac{d\varphi}{\sqrt{2\pi}}\right] e^{+i\int dy\, J(y)\varphi(y)} \left(-i\frac{\delta}{\delta\varphi(x)} \tilde{Z}[\varphi]\right)$$

Hence

$$\boxed{\begin{aligned}&-i\frac{\delta}{\delta\varphi(x)}\tilde{Z}[\varphi]\\ &=\left\{-\left[Z\partial_x^2+(m^2+a)\right]\varphi(x)-\frac{(\lambda+c)}{3!}\varphi^3(x)\right\}\tilde{Z}[\varphi]\end{aligned}}$$

Thus we can solve this linear DE for $\tilde{Z}[\varphi]$

$$\tilde{Z}[\varphi]=e^{-i\int d^4x\left\{\frac{1}{2}\varphi(x)\left[Z\partial_x^2+(m^2+a)\right]\varphi(x)+\frac{(\lambda+c)}{4!}\varphi^4(x)\right\}}N^{-1}$$

$$=N^{-1}e^{+i\int d^4x\left\{\frac{1}{2}Z\partial_\mu\varphi\partial^\mu\varphi-\frac{1}{2}(m^2+a)\varphi^2-\frac{(\lambda+c)}{4!}\varphi^4\right\}}$$

throwing away surface terms

$$=N^{-1}e^{+i\int d^4x\,\mathcal{L}[\varphi]}$$

with N a normalization factor

hence we find

$$\boxed{Z[J]=N^{-1}\int\left[\frac{d\varphi}{\sqrt{2\pi}}\right]e^{i\int d^4x\left[\mathcal{L}[\varphi]+J(x)\varphi(x)\right]}}$$

Since we desire $Z[0]=1$ we have

$$N = \int\left[\frac{d\varphi}{\sqrt{2\pi}}\right] e^{i\int dx[\mathcal{L}]}.$$

So

$$\boxed{Z[J] = \frac{\int\left[\frac{d\varphi}{\sqrt{2\pi}}\right] e^{i\int dx[\mathcal{L}[\varphi]+J\varphi]}}{\int\left[\frac{d\varphi}{\sqrt{2\pi}}\right] e^{i\int dx\,\mathcal{L}[\varphi]}}}$$

This is Feynman path integral rep. for the generating functional.

Remarks: 1)

(If we can make sense out of path integral)

This solution is more general than our perturbative GML expansion. The GML exp. is contained in this solution.

$$Z[J] = \frac{1}{N} e^{i\int dx\,\mathcal{L}_I\left[\frac{\delta}{i\delta J}\right]} \int\left[\frac{d\varphi}{\sqrt{2\pi}}\right] e^{i\int dx[\mathcal{L}_{in}[\varphi]+J\varphi]}$$

Now

$$Z_{in}[J] = \int\left[\frac{d\varphi}{\sqrt{2\pi}}\right] e^{i\int dx\,\mathcal{L}_{in}[\varphi]+J\varphi} \frac{1}{N_{in}}$$

with $Z_{in}[0]=1$

So

$$Z_{in}[J] = \frac{1}{N_{in}}\int \left[\frac{d\varphi}{\sqrt{2\pi}}\right] e^{-\frac{1}{2}\int dx\left[\varphi(x)\, i(\partial_x^2+m^2)\varphi(x)\right] + i\int dx\, J\varphi}$$

(Note $K(x,y)$ is originally real, analytically continue to $i(\partial_x^2+m^2)+\epsilon$ ← ϵ is for convergence of Gaussian so this is $i(\partial^2+m^2-i\epsilon)$ the Feynman propagator boundary condition)

Thus with $i(\partial_x^2+m^2)\delta^4(x-y) \equiv K(x,y) = k_{xy}$

we have
$$\int d^4y\, K(x,y)\, K^{-1}(y,z) = \delta^4(x-z)$$
$$= i(\partial_x^2+m^2)K^{-1}(x,z) = \delta^4(x-z)$$

$$\Rightarrow \quad \boxed{(\partial_x^2+m^2)K^{-1}(x,z) = -i\delta^4(x-z)}$$

$$\Rightarrow \quad K^{-1}(x,z) = \Delta_F(x-z)$$

Thus

$$Z_{in}[J) = \frac{1}{N_{in}}\left[\frac{1}{\det K(x,y)^{+1/2}}\right] e^{-\frac{1}{2}\int dx\,dy\, J(x)\Delta_F(x-y)J(y)}$$

& $\frac{1}{N_{in}} = \det K^{1/2} \quad \Rightarrow$

$$\boxed{Z_{in}[J] = e^{-\frac{1}{2}\int dx\,dy\, J(x)\Delta_F(x-y)\, J(y)}}$$

(This is an alternate derivation of Wick's Theorem!)

So

$$\boxed{Z[J] = \frac{N_{in}}{N} e^{i\int dx \mathcal{L}_I\left[\frac{\delta}{i\delta J}\right]} Z_{in}[J]}$$

$\boxed{\text{\& } N_{in} = N \quad \text{if we exclude vac. bubbles}}$

$$\text{So} \quad Z[J] = e^{i\int dx \mathcal{L}_I\left[\frac{\delta}{i\delta J}\right]} \cdot e^{-\frac{1}{2}\int dx dy\, J(x) \Delta_F(x-y) J(y)}$$

just the GML expansion.

$$= \langle 0 | T e^{i\int d^4x (\mathcal{L}_I[\phi_{in}] + J\phi_{in})} |0\rangle$$

$$= \langle 0 | T e^{i\int d^4x J\phi} |0\rangle .$$

Hence we have our same Feynman diagram pert. expansion. Let's recall the properties of $Z[J]$ in terms of this path integral representation

2) By construction

$Z[J]$ obeys the Euler-Lagrange eq's.

$$E[\phi](x) Z[J] = -J(x) Z[J] .$$

with

$$E[\phi](x) \equiv \frac{\delta^{\varepsilon}\mathcal{L}}{\delta^{\varepsilon}\phi(x)} = -\left(Z\partial_x^2 + (m^2+a)\right)\phi(x)$$
$$- \frac{(\lambda+c)}{3!}\phi^3(x)$$

which can be checked explicitly.

3) The n-point function is given by

$$\langle 0|T\,\phi(x_1)\cdots\phi(x_n)|0\rangle = \frac{\delta^n}{i^n\delta J(x_1)\cdots\delta J(x_n)} Z[J]\Big|_{J=0}$$

$$= \frac{\int[d\varphi]\,\varphi(x_1)\cdots\varphi(x_n)\,e^{i\int d^4x\,\mathcal{L}(x)}}{\int[d\varphi]\,e^{i\int dx\,\mathcal{L}(x)}}$$

or with the convention that $[d\varphi]$ contains the normalization factors so that $Z[0]=1$ we have

$$\langle 0|T\,\phi(x_1)\cdots\phi(x_n)|0\rangle = \int[d\varphi]\,\varphi(x_1)\cdots\varphi(x_n)\,e^{i\int d^4x\,\mathcal{L}(x)}.$$

4) Since we are dealing w/ T^* operators note that

$$\partial^{(\mu_1)}_{x_1} \cdots \partial^{(\mu_n)}_{x_n} \langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle$$

$$= \int [d\varphi]\, \partial^{(\mu_1)} \varphi(x_1) \cdots \partial^{(\mu_n)} \varphi(x_n)\, e^{i\int d^4x\, \mathcal{L}(x)}$$

$$= \langle 0 | T\, \partial^{(\mu_1)} \phi(x_1) \cdots \partial^{(\mu_n)} \phi(x_n) | 0 \rangle .$$

5) Finally we have for the arbitrarily inserted Green Function with $B_i = B_i(\phi)$

$$B_1(x_1) \cdots B_\ell(x_\ell)\, Z[J]$$

$$= \langle 0 | T B_1(x_1) \cdots B_\ell(x_\ell)\, e^{i\int dx\, J(x) \phi(x)} | 0 \rangle$$

$$= \frac{\int [d\varphi]\, B_1(\varphi(x_1)) \cdots B_\ell(\varphi(x_\ell))\, e^{i\int dx [\mathcal{L}(x) + J\varphi]}}{\int [d\varphi]\, e^{i\int dx\, \mathcal{L}(x)}}$$

6) The tree approximation ($\hbar \to 0$); in perturbation theory this corresponds to including only graphs with no loops in calculating $Z[J]$ since the power of $\hbar$ counts the # of loops in the graph.

If we include $\hbar$ we have that

$$Z[J] = \int [d\varphi]\, e^{\frac{i}{\hbar}\int dx\,[\mathcal{L} + J\varphi]}$$

$$= \int [d\varphi]\, e^{\frac{i}{\hbar} S^J}$$

we expect $Z = \sum_{n=0}^{\infty} \hbar^n Z^{(n)}$ so that $\hbar^0$ is the dominant effect.

Suppose $\varphi = \varphi_0 + \varphi'$ where φ_0 is determined by the stationary phase condition on the path integral for $\hbar \to 0$ and φ' contains the quantum $\hbar \neq 0$ corrections (often called radiative corrections). Since $\varphi = \varphi_0 + \varphi'$ we have $[d\varphi] = [d\varphi']$

So

$$Z[J] = e^{\frac{i}{\hbar} S^J(\varphi_0)} \int [d\varphi'] \times$$

$$\times e^{\frac{i}{\hbar}\int dx \left[\mathcal{L}\{\varphi_0+\varphi'\} - \mathcal{L}[\varphi_0] + J\varphi'\right]}$$

$$= e^{\frac{i}{\hbar} S^J(\varphi_0)} \int [d\varphi'] e^{\frac{i}{\hbar}\int dx\,dy \left[\frac{1}{2}\varphi'(x)\varphi'(y) \frac{\delta^2 S[\varphi']}{\delta\varphi'(x)\delta\varphi'(y)} + \cdots\right]}$$

with $\left.\dfrac{\delta S^J}{\delta \varphi}\right|_{\varphi=\varphi^0} = 0$.

Hence on $\hbar \to 0$ $\varphi \to \varphi_0$ and

$$Z[J] \longrightarrow Z^{(0)}[J] = e^{\frac{i}{\hbar} S^J[\varphi_0]}$$

with $\dfrac{\delta S^J[\varphi_0]}{\delta \varphi_0(x)} = 0$ that is $\varphi_0 = \varphi_0[J]$.

$$= \frac{\partial^\varepsilon \mathcal{L}(\varphi_0)}{\partial^\varepsilon \varphi_0} + J$$