Having introduced asymptotic fields describing our system's behavior at initial and final times when the interacting particles are spatially widely separated, we would like to introduce a Heisenberg picture interacting field $\phi(x)$ to interpolate between these two extremes. By our first axiom we already know how the field must evolve in time i.e. by the Heisenberg equations of motion. In a sense what we are doing now is postulating something about the initial (or final) conditions the interacting field is to obey.

We had seen in the introduction that the in- and out-fields are related to the interacting field by means of the Yang-Feldman equation, for instance

$$\phi(x) = \phi_{in}(x) - \int_{-\infty}^{+\infty} d^4y \, \Delta_R(x-y) \, j(y)$$

with $\Delta_R(x-y) = -\theta(x^0-y^0)\,\Delta(x-y)$.

These are like integrated equations of motion with the boundary condition that as $x^0 \to -\infty$ we find

$$\lim_{x^0 \to -\infty} \phi(x) = \phi_{in}(x) + \lim_{x^0 \to -\infty} \int_{-\infty}^{x^0} d^4y \, \Delta(x-y) j(y)$$

$$\to \phi_{in}(x) \quad \text{since the}$$

region of integration vanishes. So we find that

$$\lim_{x^0 \to -\infty} \phi(x) = \phi_{in}(x)$$

and similarly $\lim_{x^0 \to +\infty} \phi(x) = \phi_{out}(x)$

are reasonable boundary conditions to postulate for our interacting fields.

Since these are operators we must be somewhat more specific about what this above limiting procedure really means. That is in what sense does $\phi(x)$ as an operator converge to the $\phi_{\substack{in \\ out}}(x)$ operators.

Haag first proposed that this should be understood in the sense of strong operator convergence $\phi(x) \to \phi_{in}(x)$ as $x^0 \to -\infty$ means

$$\lim_{x^0 \to -\infty} \| (\phi(x) - \phi_{in}(x)) |\psi\rangle \| = 0$$

for arbitrary $|\psi\rangle$ in the domain of ϕ & ϕ_{in}.

(Recall $\| |\psi\rangle \| = \langle\psi|\psi\rangle^{1/2}$, the norm of $|\psi\rangle$).

However, LSZ showed that this ~~has~~ two difficulties associated with it.

First, as we already know the states $\phi(x)|\psi\rangle$ and $\phi_{in}(x)|\psi\rangle$ do not have finite norm. For instance

$$\langle 0|\phi_{in}(x)\phi_{in}(y)|0\rangle = i\Delta(x-y) \to \infty \text{ as } x \to y.$$

This difficulty is easily overcome by using wavepackets as we introduced in the discussion of the last axiom.

Hence defining

$$\phi_\alpha(t) \equiv i\int_{x^0=t} d^3x\, f^*_\alpha(x) \overleftrightarrow{\partial_0} \phi(x) \equiv a_\alpha(t)$$

and, as we had,

$$\phi^{in}_\alpha \equiv i\int d^3x\, f^*_\alpha(x) \overleftrightarrow{\partial_0} \phi_{in}(x) = a^{in}_\alpha$$

(Note since $\phi_{in}(x)$ as well as $f_\alpha(x)$ obeys the Klein-Gordon equation a^{in}_α is independent of the time, but $\phi(x)$ is not a free field and so $a_\alpha(t)$ depends on time)

with $f_\alpha(x)$ normalized positive energy solutions to the K-G eq., i.e. wavepackets

we have that $a_\alpha(t)|\Psi\rangle$ and $a^{in}_\alpha|\Psi\rangle$ have finite norm (as is assumed of $|\Psi\rangle$ i.e. $|\Psi\rangle \in \mathcal{H}$)

The second difficulty LSZ pointed out was with the strong convergence criteria itself

$$\lim_{t \to -\infty} \| (a_\alpha(t) - a_\alpha^{in}) |\Psi\rangle \| = 0 .$$

If we choose $|\Psi\rangle = |0\rangle$ then this implies that $\langle 0 | \phi_\alpha(t) \phi_\beta(t) | 0 \rangle \to \langle 0 | \phi_\alpha^{in} \phi_\beta^{in} | 0 \rangle$

but $\langle 0 | \phi(x) \phi(y) | 0 \rangle = i \Delta'(x-y)$, depending only on the difference $x^0 - y^0$

hence this should be insensitive to the limit and we have that

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = \langle 0 | \phi_{in}(x) \phi_{in}(y) | 0 \rangle$$

in general. This is untrue from perturbative calculations, hence the strong operator convergence must be replaced.

LSZ concluded that weak operator

convergence is the correct convergence for the field operator

$\phi_\alpha(t)$ weakly converges to ϕ_α^{in} as $t \to -\infty$ if

$$\lim_{t \to -\infty} |\langle \chi | (a_\alpha(t) - a_\alpha^{in}) | \psi \rangle| = 0$$

for all pairs of vectors $|\chi\rangle, |\psi\rangle$ in the domain of $a_\alpha(t)$ and a_α^{in}. This is the sense in which

$$\lim_{x^0 \to -\infty} \phi(x) = \phi_{in}(x) \quad \text{is}$$

to be understood. Sometimes we write

$$\phi(x) \xrightarrow[x^0 \to -\infty]{LSZ} \phi_{in}(x) \quad \text{to}$$

mean

$$\lim_{t \to -\infty} \langle \chi | a_\alpha(t) | \psi \rangle = \langle \chi | a_\alpha^{in} | \psi \rangle .$$

Similarly the final time boundary condition is

$$\phi(x) \xrightarrow[x^0 \to +\infty]{LSZ} \phi_{out}(x) \text{, that is}$$

$$\lim_{t \to +\infty} \langle \chi | a_\alpha(t) | \psi \rangle = \langle \chi | a_\alpha^{out} | \psi \rangle .$$

Before stating our third axiom then, we should allow for one more generality. Namely, the interacting field $\phi(x)$ contains in it the description of the complicated interaction dynamics & the field configurations describe not just the creation or annihilation of a single asymptotic particle as the ϕ_{in} & ϕ_{out} fields do but contained in it is not only the possibility of single particle creation & annihilation – but also creation of pairs, particle-anti-particle annihilation, creation of pairs of pairs, and so on. In other words $\phi(x)$ acting on the vacuum creates much more than just one additional asymptotic particle as do ϕ_{in}, ϕ_{out}, but creates pairs, pairs of pairs and all sorts of multi-particle states. Hence

in the remote past we expect

$$\phi(x) \xrightarrow[x^0 \to -\infty]{LSZ} Z^{1/2}\, \phi_{in}(x)$$

where the factor $Z^{1/2}$ expresses the fact that ϕ creates 1 particle + pairs from $|0\rangle$.

(This is expected from the results of bare field renormalization theory where the full propagator went like $\frac{iZ}{p^2-m^2+i\epsilon}$ + ;

i.e. $Z^{1/2}$ represents the fact that the one particle state is only part of what $\phi|0\rangle$ contains.

Z is called the wavefunction re-normalization factor. So we finally have

Axiom 3: The Asymptotic Condition

There exists a covariant Heisenberg field $\phi(x)$ (called the interpolating or interacting field) which obeys the asymptotic conditions

$$\lim_{t\to-\infty} \langle\chi|a_\alpha(t)|\psi\rangle = Z^{1/2}\langle\chi|a_\alpha^{in}|\psi\rangle$$

$$\lim_{t\to+\infty} \langle\chi|a_\alpha(t)|\psi\rangle = Z^{1/2}\langle\chi|a_\alpha^{out}|\psi\rangle,$$

where $a_\alpha(t) \equiv i\int_{x^0=t} d^3x\, f_\alpha^*(x)\overleftrightarrow{\partial_0}\phi(x)$

$$a_\alpha^{in} = i\int d^3x\, f_\alpha^*(x)\overleftrightarrow{\partial_0}\phi_{in}(x) \quad \text{and}$$

$f_\alpha(x)$ is any normalizable solution to the K-G equation.

(If we use the renormalized field $\phi_R = Z^{-1/2}\phi$, then the $Z^{1/2}$ factor is eliminated.)

<u>Note</u>: By covariant we <u>assume</u> $\phi(x)$ carries the same representation of the Poincaré group that ϕ_{in} or ϕ_{out} does. In the scalar case here

$$U(\Lambda,a)\,\phi(x)\,U^{-1}(\Lambda,a) = \phi(\Lambda x + a)$$

in particular

$$-i\partial_\mu \phi(x) = [P_\mu, \phi(x)],$$

the Heisenberg equations of motion.

Also we recall from the introduction that the perturbative expansion for the S-matrix could only be understood after <u>we</u> renormalized the Feynman integrals. This was accomplished by forcing the in- and out-fields (and iP fields) to have the physical mass by adding a mass counterterm to the interaction. As well we allowed the coupling constants in the theory to be renormalized also. And finally we saw that we must also express the perturbation expansion in terms of the renormalized field operator (eq. 1.1.58)

$\phi_R = Z^{-1/2}\phi$ if we desire to use the second method of renormalized perturbation theory. Otherwise using bare but regularized quantities we found the external lines carried the renormalized single particle wavefunctions (eq. 1.1.60)

$$u_R = Z^{+1/2} u \, .$$

Using the bare renormalization program in the Yang-Feldman equation derivation (and the careful application of the adiabatic hypothesis) we argued it should more correctly read

$$\phi(x) = Z^{1/2}\phi_{\substack{in\\out}}(x) - \int d^4y\, \Delta_{\substack{R\\A}}(x-y)\, j(y)$$

$$= Z^{1/2}\phi_{\substack{in\\out}}(x) - \int d^4y\, \Delta_{\substack{R\\A}}(x-y)(\partial_y^2+m^2)\phi(y) .$$

Even if we use renormalized P.t. but with a Z that normalized the propagator residue to a finite value $\neq 1$, we would have the explicit Z_R factors i.e. let $\phi_R = Z_R^{-1/2} Z^{-1/2}\phi$, then

$$\phi_R(x) = Z_R^{1/2}\phi_{\substack{in\\out}}(x) - \int d^4y\, \Delta_{\substack{R\\A}}(x-y)(\partial_y^2+m^2)\phi_R(y) .$$

All consistent with the asymptotic condition of axiom 3.

We can now derive the all important reduction formulae relating the S-matrix elements to VEV of T-products of interacting fields.

So we are interested in finding

$$S_{\alpha\beta} \equiv \langle \alpha_1 \ldots \alpha_m \, out | \beta_1 \ldots \beta_n \, in \rangle$$

where

$$|\beta_1 \ldots \beta_n \, in\rangle = \frac{1}{\underbrace{\sqrt{P_{\beta_1 \ldots \beta_n}}}_{\equiv P_n^\beta}} a^{in\dagger}_{\beta_1} \cdots a^{in\dagger}_{\beta_n} |0\rangle$$

$$|\alpha_1 \ldots \alpha_m \, out\rangle = \frac{1}{\underbrace{\sqrt{P_{\alpha_1 \ldots \alpha_m}}}_{\equiv P_m^\alpha}} a^{out\dagger}_{\alpha_1} \cdots a^{out\dagger}_{\alpha_m} |0\rangle$$

So

$$S_{\alpha\beta} = \langle \{\alpha\} out | a^{in\dagger}_{\beta_n} |\beta_1 \ldots \beta_{n-1} \, in\rangle \left(\frac{P_n^\beta}{P_{n-1}^\beta}\right)$$

in order to simplify the algebra wlog we can take all β_i different and all α_i different so that $P_{\beta_1 \ldots \beta_n} = 1 = P_{\alpha_1 \ldots \alpha_m}$.

Hence

$$S_{\alpha\beta} = \langle \{\alpha\} \text{out} | a^{in\,\dagger}_{\beta_n} | \beta_1 \cdots \beta_{n-1} \text{ in} \rangle$$

by the asymptotic condition this is simply

$$= \lim_{t \to -\infty} Z^{-1/2} \langle \{\alpha\} \text{out} | a^{\dagger}_{\beta_n}(t) | \beta_1 \cdots \beta_{n-1} \text{ in} \rangle$$

$$= \lim_{t \to -\infty} -i \int_{x^0 = t} d^3x \, f_{\beta_n}(x) \overleftrightarrow{\partial}_{x^0} Z^{-1/2} \times$$

$$\times \langle \{\alpha\} \text{out} | \phi(x) | \beta_1 \cdots \beta_{n-1} \text{ in} \rangle$$

Now we can write the $\int_{x^0=t} d^3x$ integral as the surface of a d^4x integral

$$\lim_{t \to -\infty} \int_{x^0 = t} d^3x \cdots = -\int_{-\infty}^{+\infty} d^4x \, \partial_{x^0} \cdots$$

$$+ \lim_{t \to +\infty} \int_{x^0 = t} d^3x \cdots$$

So

$$S_{\alpha\beta} = +i\int d^4x\, \partial_{x^0}\Big[Z^{-1/2} f_{\beta_n}(x) \overleftrightarrow{\partial}_{x^0} \langle \{\alpha\}\text{out} | \phi(x) | \beta_1 \cdots \beta_{n-1}\, \text{in}\rangle \Big]$$

$$-i \lim_{t\to +\infty} \int_{x^0 = t} d^3x\, Z^{-1/2} f_{\beta_n}(x) \overleftrightarrow{\partial}_{x^0} \langle \{\alpha\}\text{out} | \phi(x) | \beta_1 \cdots \beta_{n-1}\, \text{in}\rangle$$

but applying the outgoing asymptotic condition to the second term on the RHS

$$\lim_{t\to +\infty} \langle \chi | a^{+}_{\beta_n}(t) | \psi\rangle = Z^{1/2} \langle \chi | a^{+\text{out}}_{\beta_n} | \psi\rangle$$

we have

$$S_{\alpha\beta} = +i\int d^4x\, \partial_{x^0}\Big[Z^{-1/2} f_{\beta_n}(x) \overleftrightarrow{\partial}_{x^0} \langle \{\alpha\}\text{out} | \phi(x) | \beta_1 \cdots \beta_{n-1}\, \text{in}\rangle \Big]$$
$$+ \langle \{\alpha\}\text{out} | a^{\text{out}+}_{\beta_n} | \beta_1 \cdots \beta_{n-1}\, \text{in}\rangle$$

performing the ∂_{x^0} derivatives we have

$$S_{\alpha\beta} = \langle \{\alpha\} \text{out} | a^{\text{out}\dagger}_{\beta_n} | \beta_1 \cdots \beta_{n-1} \text{ in} \rangle$$

$$+ i \int d^4x \, Z^{-1/2} f_{\beta_n}(x) \, \partial^2_{x^0} \langle \{\alpha\} \text{out} | \phi(x) | \beta_1 \cdots \beta_{n-1} \text{ in} \rangle$$

$$- i \int d^4x \, Z^{-1/2} \ddot{f}_{\beta_n}(x) \langle \{\alpha\} \text{out} | \phi(x) | \beta_1 \cdots \beta_{n-1} \text{ in} \rangle$$

But $f_{\beta_n}(x)$ obeys the KG equation

$$(\partial^2 + m^2) f_{\beta_n}(x) = 0 = (\partial_0^2 - \nabla^2 + m^2) f_{\beta_n}$$

thus

$$-\ddot{f}_{\beta_n}(x) = -\nabla^2 f_{\beta_n}(x) + m^2 f_{\beta_n}(x).$$

So we secure

$$S_{\alpha\beta} = \langle \{\alpha\} \text{out} | a^{\text{out}\dagger}_{\beta_n} | \beta_1 \cdots \beta_{n-1} \text{ in} \rangle$$

$$+ i \int d^4x \, Z^{-1/2} f_{\beta_n}(x) \partial^2_{x^0} \langle \{\alpha\} \text{out} | \phi(x) | \beta_1 \cdots \beta_{n-1} \text{ in} \rangle$$

$$+ i \int d^4x \, Z^{-1/2} (-\nabla^2 + m^2) f_{\beta_n}(x) \, \langle \{\alpha\} \text{out} | \phi(x) | \beta_1 \cdots \beta_{n-1} \text{ in} \rangle$$

Now since f_{β_n} is normalizable we can integrate the ∇^2 by parts and throw away the surface terms to obtain

$$\boxed{\begin{aligned} S_{\alpha\beta} &= \langle\{\alpha\}\text{out}|\, a^{\text{out}\dagger}_{\beta_n}\, |\beta_1 \cdots \beta_{n-1}\, \text{in}\rangle \\ &\quad + i\int d^4x\, Z^{-1/2} f_{\beta_n}(x)\left(\partial_x^2 + m^2\right)\langle\{\alpha\}\text{out}|\phi(x)|\beta_1\cdots\beta_{n-1}\text{in}\rangle \end{aligned}}$$

Thus we have "reduced out" a particle from the in-state and replaced it with an interpolating field. The first term on the RHS the in-particle goes straight through and becomes an out-particle. Hence $a^{\text{out}\dagger}_{\beta_n}$ acts to remove a particle from $\langle\{\alpha\}\text{out}|$ or give zero if no α_i equals β_n.

Similarly we could have reduced out an outgoing particle to obtain

$$\boxed{S_{\alpha\beta} = \langle \alpha_1 \cdots \alpha_{m-1}\, \text{out} |\, a^{in}_{\alpha_m} | \{\beta\}\, \text{in}\rangle + i \int d^4x\, f^{*}_{\alpha_m}(x)\, Z^{-1/2} (\partial_x^2 + m^2) \langle \alpha_1 \cdots \alpha_{m-1}\, \text{out} | \phi(x) | \{\beta\}\, \text{in}\rangle}$$

These are the <u>LSZ Reduction Formulae</u>.

We next continue reducing out the particles from the in- & out- states

for example we consider

$$\langle \{\alpha\}\, \text{out} | \phi(x) | \beta_1 \cdots \beta_{n-1}\, \text{in}\rangle$$

$$= \langle \{\alpha\}\, \text{out} | \phi(x)\; a^{in\,+}_{\beta_{n-1}} | \beta_1 \cdots \beta_{n-2}\, \text{in}\rangle$$

$$= \lim_{t \to -\infty} Z^{-1/2} \langle \{\alpha\}\, \text{out} | \phi(x)\, a^{+}_{\beta_{n-1}}(t) | \beta_1 \cdots \beta_{n-2}\, \text{in}\rangle$$

$$= \lim_{t \to -\infty} -i Z^{-1/2} \int_{y^0 = t} d^3y\, f_{\beta_{n-1}}(y) \overleftrightarrow{\partial}_{y^0} \langle \{\alpha\}\, \text{out} | \phi(x) \phi(y) | \beta_1 \cdots \beta_{n-2}\, \text{in}\rangle$$