So to summarize

$$G^{(n)}(x_1,\dots,x_n) = \frac{\langle 0\,\text{out}|T\phi(x_1)\cdots\phi(x_n)|0\,\text{in}\rangle}{\langle 0\,\text{out}|0\,\text{in}\rangle}$$

$$= \sum_{\Gamma\in G^n}' \langle 0\,\text{out}|T\phi(x_1)\cdots\phi(x_n)|0\,\text{in}\rangle_\Gamma$$

$$= \int \frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}\, e^{-i\sum_{i=1}^n p_i x_i} \times$$

$$\times \sum_{\Gamma\in G^n}' \alpha(\Gamma)\,(2\pi)^4\delta^4(p_{11}+\cdots+p_{a1})\cdots(2\pi)^4\delta^4(p_{1n_p}+\cdots+p_{zn_p})$$

$$\times \int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_{m(\Gamma)}}{(2\pi)^4}\, I_\Gamma(p_1,\dots,p_n,k_1,\dots,k_{m(\Gamma)})$$

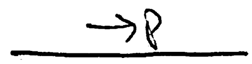
where 1) G^n = Set of all topologically distinct Feynman diagrams (excluding vacuum bubbles) with n external lines carrying momenta $p_1,\dots,p_n$ into the graph. Each diagram has vertices and internal lines joining vertices such that there are $m(\Gamma)$ independent loops.

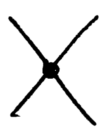
2) The momentum is routed through the diagram so that each connected subdiagram has overall E-M conservation and there is E-M conservation at each vertex. Further the external momenta can flow through the diagram in an arbitrary manner, subject only to E-M conservation at each vertex. As well the independent loop momenta flowing through each line are independently conserved at each vertex.

3) $\alpha(\Gamma)$ = symmetry number of Γ calculated from Wick's theorem and the GML expansion.

4) Separate E-M delta function for each connected subdiagram $1, \ldots, n_p$ of them

5) I_Γ = Feynman integrand corresponding to the graph Γ. It consists ~~of~~ ~~factors~~ a propagator factor for each line and a coupling strength for each vertex

$\rightarrow p$  $\longleftrightarrow$ $\dfrac{i}{p^2-m^2+i\epsilon}$

 $\longleftrightarrow$ $-i\lambda$

The $(p_1 \cdots p_n)$ are the external momenta and $(k_1, \dots, k_{m(r)})$ the internal momenta.

(Aside:
Again from the topology of the graph we have that first the number of independent loops is equal to the number of momentum integrals left over,

$$m = L - V + 1$$

L = # of internal lines since $\int d^4k$ for each
V = # of vertices since $\int d^4y$ for each

Since one overall E-M δ-function results we have one ~~set~~ of momentum integrals left.

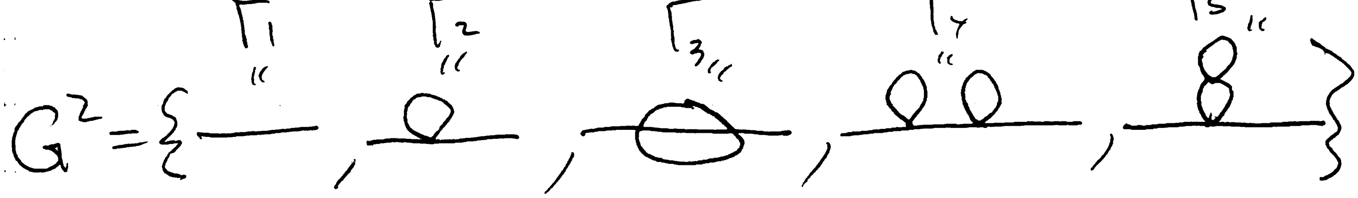
we had that $4V = 2L + n$ from the graph topology

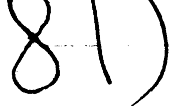
So $2m = 2V + 2 - n$; $\underline{m = V + 1 - \frac{n}{2}}$

if we are working in V^{th} order in the coupling λ this is equivalent to m loops in the graph.

Hence we can view the # of loops as our perturbation parameter since it is equiv. to powers of λ.)

<u>For example</u> Find $G^{(2)}(x_1, x_2)$ through second order in λ.

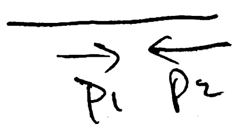
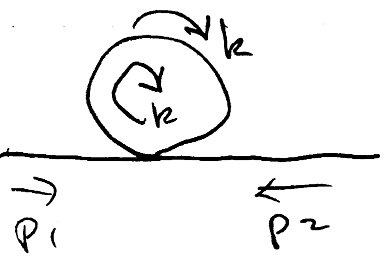
$G^2 = \{$  $\}$

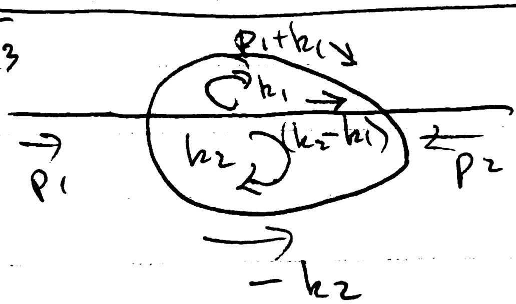
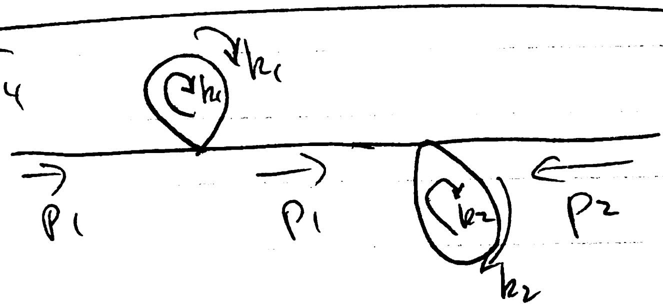
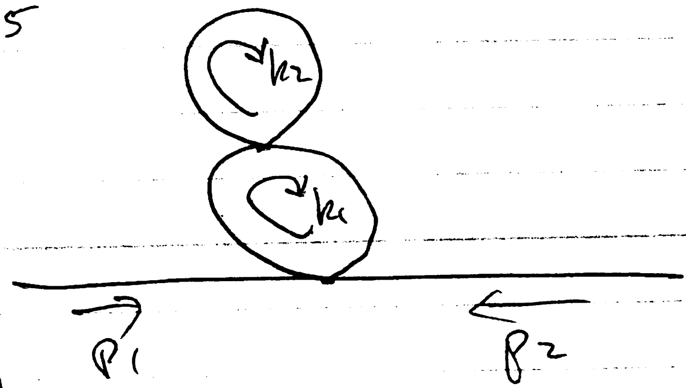
(no vacuum bubbles i.e. )

So

$$\frac{\langle 0\,\text{out} | T\,\phi(x_1)\,\phi(x_2) | 0\,\text{in}\rangle}{\langle 0\,\text{out} | 0\,\text{in}\rangle}$$

$$= \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\, e^{-ip_1x_1}\, e^{-ip_2x_2}\,(2\pi)^4\delta(p_1+p_2)\,\times$$
$$\times \sum_{\Gamma\in G^2}{}' \alpha(\Gamma)\int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_{m(\Gamma)}}{(2\pi)^4}\, I_\Gamma(p,k)$$

$\alpha(\Gamma)$	Γ	I_Γ
1	$\Gamma_1 =$ 	$\dfrac{i}{p_1^2-m^2+i\epsilon}$
$\frac{1}{2}$	Γ_2 	$\dfrac{i}{p_1^2-m^2}\,\dfrac{i}{p_2^2-m^2}\,(-i\lambda)\,\dfrac{i}{k^2-m^2}$

$\alpha(\Gamma)$	Γ	I_Γ
$\frac{1}{6}$	Γ_3 	$\frac{i}{p_1^2-m^2}\frac{i}{p_2^2-m^2}\frac{i}{k_2^2-m^2}\frac{i}{(p_1+k_1)^2-m^2}\frac{i}{(k_2-k_1)^2-m^2}(-i\lambda)^2$
$\left(\frac{1}{2}\right)^2$	Γ_4 	$\frac{i}{p_1^2-m^2}\frac{i}{p_1^2-m^2}\frac{i}{p_2^2-m^2}\frac{i}{k_1^2-m^2}\frac{i}{k_2^2-m^2}(-i\lambda)^2$
$\left(\frac{1}{2}\right)^2$	Γ_5 	$\frac{i}{p_1^2-m^2}\frac{i}{p_2^2-m^2}\frac{i}{k_1^2-m^2}\frac{i}{k_2^2-m^2}(-i\lambda)^2$

So

$$\frac{\langle 0\,\mathrm{out}|T\tilde{\phi}(p)\phi(0)|0\,\mathrm{in}\rangle}{\langle 0\,\mathrm{out}|0\,\mathrm{in}\rangle}$$

$$\equiv \int d^4p\, e^{+ipx_1}\, G^{(2)}(x_1,0)$$

$$= \sum_{\Gamma\in G_2^2} \alpha(\Gamma) \int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_{m(\Gamma)}}{(2\pi)^4}\, I_\Gamma(p,k)$$

$$= \frac{i}{p^2-m^2+i\epsilon} + \frac{(-i\lambda)}{2}\left[\frac{i}{p^2-m^2+i\epsilon}\right]^2 \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\epsilon}$$

$$+ \frac{(-i\lambda)^2}{6}\left[\frac{i}{p^2-m^2+i\epsilon}\right]^2 \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\left[\frac{i}{(p+k_1)^2-m^2+i\epsilon}\right]\left[\frac{i}{(k_1-k_2)^2-m^2+i\epsilon}\right]\left[\frac{i}{k_2^2-m^2+i\epsilon}\right]$$

$$+ \frac{(-i\lambda)^2}{4}\left[\frac{i}{p^2-m^2+i\epsilon}\right]^3\left[\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\epsilon}\right]^2$$

$$+ \frac{(-i\lambda)^2}{4}\left[\frac{i}{p^2-m^2+i\epsilon}\right]^2\left[\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\epsilon}\right]^2 .$$

The relation of the time ordered functions to the S-matrix elements can easily be seen by ~~R~~ recalling our perturbative expansion for S_{fi}

$$S_{fi} = \frac{\langle f\,in | T e^{+i\int_{-\infty}^{+\infty} d^4x \mathcal{L}_{I\,in}(x)} | i\,in\rangle}{\langle 0\,in | T e^{+i\int_{-\infty}^{+\infty} d^4x \mathcal{L}_I^{in}(x)} | 0\,in\rangle}$$

and the ~~fact~~ fact that the in-states are made by the repeated action of the in-field creation operators on the in-vacuum.

$$|i\,in\rangle = a^\dagger_{in}(\vec{p}_1)\cdots a^\dagger_{in}(\vec{p}_n)|0\,in\rangle$$

$$\equiv |\vec{p}_1,\ldots,\vec{p}_n\ in\rangle$$

$$|f\,in\rangle = a^\dagger_{in}(\vec{q}_1)\cdots a^\dagger_{in}(\vec{q}_m)|0\,in\rangle$$

$$= |\vec{q}_1,\ldots,\vec{q}_m\ in\rangle$$

So

$$S_{fi} = \frac{\langle 0_{in} | a_{in}(\vec{q}_1) \cdots a_{in}(\vec{q}_m) \left(T e^{-\frac{i\lambda}{4!}\int d^4y\, \phi^4_{in}(y)} \right) \times a^\dagger_{in}(\vec{p}_1) \cdots a^\dagger_{in}(\vec{p}_n) | 0_{in} \rangle}{\langle 0_{in} | T e^{+i\int d^4x\, \mathcal{L}_I^{in}(x)} | 0_{in} \rangle}$$

Hence; when we expand the exponential and apply Wick's Theorem to evaluate the S_{fi} we find exactly the same Feynman rules for the Feynman diagram expansion

except the incoming and outgoing lines are no longer external propagators $\langle 0_{in} | T\, \phi_{in}(x_i)\, \phi_{in}(y) | 0_{in} \rangle = \Delta_F(x_i - y)$

as in the Green function case

but are

simply momentum space wavefunctions

<u>incoming</u> $\langle 0in | \phi_{in}(y) a^{+}_{in}(\vec{p}) | 0in \rangle$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-iky} \underbrace{\langle 0in | a_{in}(\vec{k}) a^{+}_{in}(\vec{p}) | 0in \rangle}_{= (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{p})}$$

$$= e^{-ipy} \qquad (p^0 = \omega_p)$$

<u>outgoing</u> $\langle 0in | a_{in}(\vec{q}) \phi_{in}(y) | 0in \rangle$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{+iky} \underbrace{\langle 0in | a_{in}(\vec{q}) a^{+}_{in}(\vec{k}) | 0in \rangle}_{(2\pi)^3 2\omega_k \delta^3(\vec{q}-\vec{k})}$$

$$= e^{+iqy} \qquad (q^0 = \omega_q)$$

(*) Hence, when the vertex integrals $\int d^4y$ are done to go to momentum space

(*) (the exponentials become part of the E&M conserving δ-functions and we are left with just 1)

Not external

$$\langle 0\,in| T\,\phi_{in}(y)\,\phi_{in}(x)|0\,in\rangle$$

$$= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{i}{p^2-m^2+i\epsilon}$$

Or if they do not attach to a vertex but just go through we have

$$\langle 0\,in|\, a_{in}(\vec{q})\, a^{\dagger}_{in}(\vec{p})\,|0\,in\rangle$$

$$= (2\pi)^3\, 2\omega_p\, \delta^3(\vec{p}-\vec{q})$$

Not

$$\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\, e^{-ipx}\, e^{-iqy}\, \frac{i}{p^2-m^2+i\epsilon}\,(2\pi)^4\delta^4(p+q)$$

$$= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{i}{p^2-m^2+i\epsilon}$$

The incoming and outgoing lines just have their momentum space wavefunction = 1 left. So the S-matrix Rules are the same as the time ordered function Feynman rules with the substitution

1) associate a factor 1 with each incoming line

Not $\int \frac{d^4 p_i}{(2\pi)^4} e^{-ip_i x_i} \frac{i}{p_i^2 - m^2 + i\epsilon}$

2) associate a factor 1 with each outgoing line

not $\int \frac{d^4 p_i}{(2\pi)^4} e^{-ip_i x_i'} \frac{i}{p_i^2 - m^2 + i\epsilon}$

3) Straight through line is

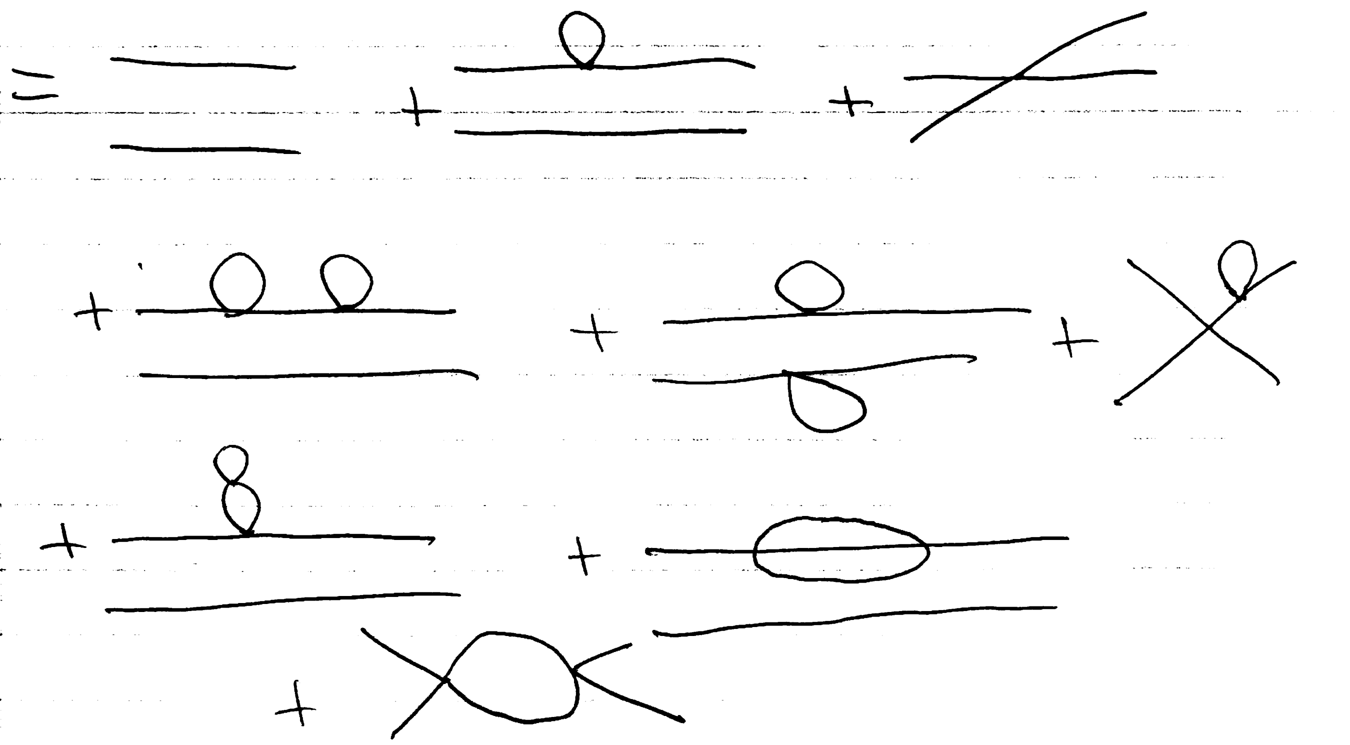
$(2\pi)^3 2\omega_p \delta^3(\vec{p} - \vec{q})$

not $\frac{i}{p_1^2 - m^2 + i\epsilon} \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} e^{-ip_1(x_1 - x_2)} (2\pi)^4 \delta^4(p_1 + p_2)$

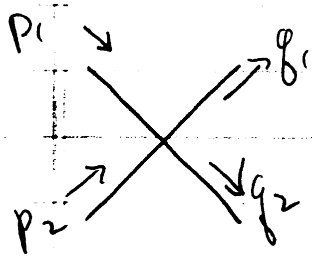
For instance the graphical expansion for 2 scalar particles scattering into 2 scalar particles gives

$$S_{fi} = \langle \vec{q}_1, \vec{q}_2 \text{ out} | \vec{p}_1, \vec{p}_2 \text{ in} \rangle_{NVB}$$

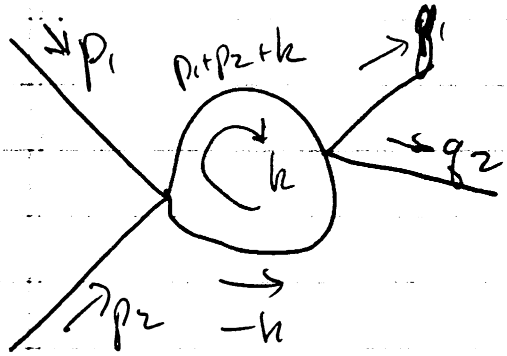
$$= \frac{\langle 0\,\text{in} | a_{in}(\vec{q}_1)\, a_{in}(\vec{q}_2) \left(T e^{-\frac{i\lambda}{4!}\int d^4y\, \phi_{in}^4(y)} \right) \times a_{in}^\dagger(\vec{p}_1)\, a_{in}^\dagger(\vec{p}_2) | 0\,\text{in} \rangle}{\langle 0_{in} | T e^{-\frac{i\lambda}{4!}\int d^4y\, \phi_{in}^4(y)} | 0\,\text{in} \rangle}$$



For example evaluating the 3 ad last graphs



$$= -i\lambda (2\pi)^4 \delta^4(p_1+p_2-q_1-q_2)$$



$$= \frac{(-i\lambda)^2}{2} (2\pi)^4 \delta^4(p_1+p_2-q_1-q_2) \int \frac{d^4k}{(2\pi)^4} \frac{i}{[(p_1+p_2+k)^2-m^2+i\epsilon]} \frac{i}{k^2-m^2+i\epsilon}$$

+ perm. of $(p_1 p_2 q_1 q_2)$

Thus we see how to Recover S-matrix elements from Time ordered functions

For the terms in S_{fi} that involve scattering of all the particles in the in and out states. That is not including any terms with straight through external lines

To consider these type of terms we can choose none of the initial states to be the same as the final states hence any δ-function, straight through unscattered terms are ruled out. (Of course we can relate the S-matrix to the τ-functions for the case of one or more straight through lines with the others interacting since we can pull out the needed δ-function terms explicitly, then analyze the remaining interacting terms just as we do now)

Hence we can relate the S-matrix elements to the Green functions by Fourier transforming the time ordered functions and multiplying it by the inverse propagator for each external line

i.e. $-i(p_i^2-m^2)$; then after this setting

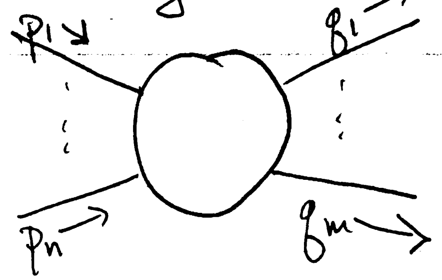
$p_i^2=m^2$ since we have $p_i^0=\omega_{p_i}$ for the

S-matrix in- and out-states

Thus we have for n incoming particles scattering into m outgoing particles, with none of the n in-particles the same as any of the m-out particles

$$S_{fi} = \frac{\langle f\,\text{out} | i\,\text{in} \rangle}{\langle 0\,\text{out} | 0\,\text{in} \rangle}$$

(figuratively



)

$$= \left[-i(p_1^2 - m^2)\right] \cdots \left[-i(p_n^2 - m^2)\right]\left[-i(q_1^2 - m^2)\right] \cdots \left[-i(q_m^2 - m^2)\right]$$

$$\times \int d^4x_1 \cdots d^4x_n\, d^4y_1 \cdots d^4y_m\, e^{+ip_1x_1} \cdots e^{+ip_nx_n}$$

$$\times e^{-iq_1y_1} \cdots e^{-iq_my_m} \times$$

$$\times \left.\frac{\langle 0\,\text{out} | T\,\phi(x_1) \cdots \phi(x_n)\phi(y_1) \cdots \phi(y_m) | 0\,\text{in} \rangle}{\langle 0\,\text{out} | 0\,\text{in} \rangle}\right|_{\substack{\text{all } p_i^2 = m^2 \\ q_j^2 = m^2}}$$

We notice that the inverse propagators can be brought through the integrals as the Klein-Gordon operator! So we obtain

$S_{fi} =$

$$+ i^{m+n} \int d^4x_1 \cdots d^4y_m \, (\partial_{x_1}^2 + m^2) e^{ip_1 x_1} \cdots (\partial_{y_m}^2 + m^2) e^{-iq_m y_m} \left. \frac{\langle 0\,\mathrm{out} | T\phi(x_1)\cdots\phi(y_m) | 0\,\mathrm{in}\rangle}{\langle 0\,\mathrm{out} | 0\,\mathrm{in}\rangle} \right|_{\text{mass shell}}$$

Now we integrate by parts to throw the Klein Gordon operator over to the Green function. Here we must be careful about surface terms. We should be using wavepacket states to guarantee they vanish, but for now we find

$$
\boxed{\begin{aligned}
S_{fi} = {} & \\
& + i^{m+n} \int d^4x_1 \cdots d^4y_m \, e^{ip_i x_i} \, e^{-iq_j y_j} \times \\
& \times (\partial_{x_1}^2 + m^2) \cdots (\partial_{y_m}^2 + m^2) \left. \frac{\langle 0\,out | T\phi(x_1)\cdots\phi(y_m) | 0\,in \rangle}{\langle 0\,out | 0\,in \rangle} \right|_{\substack{mass\\shell}}
\end{aligned}}
$$

where $\Big|_{\substack{mass\\shell}}$ means after all operators are performed we set $p_i^2 = m^2$; $q_j^2 = m^2$. This formula is known as the <u>LSZ reduction formula.</u> It relates the Time ordered functions to the S-matrix elements. Recall we have $p_i \neq q_j$ for any i & j above.

So we finally obtain our seminal relation amongst Heisenberg picture quantities.

In general the matrix elements of any observable can be related to time ordered functions in a similar manner so that for any operators $A_1(z_1)\cdots A_p(z_p)$ we have that the LSZ reduction formula becomes

$$\frac{\langle f\,out|T\,A_1(z_1)\cdots A_p(z_p)|\,i\,in\rangle}{\langle 0\,out|0\,in\rangle}$$

$$= i^{m+n}\int d^4x_1\cdots d^4y_m\, e^{ip_ix_i}\, e^{-iq_jy_j}\times$$

$$\times(\partial_{x_1}^2+m^2)\cdots(\partial_{y_m}^2+m^2)\left.\frac{\langle 0\,out|T\,A_1(z_1)\cdots A_p(z_p)\phi(x_1)\cdots\phi(y_m)|0\,in\rangle}{\langle 0\,out|0\,in\rangle}\right|_{\text{mass shell}}$$

Thus once the singularities of the Green functions are known the observables of the quantum theory can be calculated.

As a final point before reformulating quantum field theory from an axiomatic point of view let's consider the $\langle 0\,out | 0\,in \rangle$ matrix element further, or as it is known the vacuum persistence amplitude.

Intuitively we realize that in the absence of external fields, the vacuum ~~should~~ be a unique state of lowest energy, momentum etc. after all no particles is no particles and we should have that $|0\,in\rangle = |0\,out\rangle \equiv |0\rangle$ a unique lowest energy state, "The Vacuum". Of course when we couple to external fields they could cause the creation of particles so that $|0\,in\rangle$ is no longer $|0\,out\rangle$ and we should be able, at least in principle, to calculate the transition amplitude.

In more mathematical detail let's consider again these two states

$$C_{in} |0\,in\rangle \equiv \frac{U(0,-\infty)|0\rangle}{(0|U(0,-\infty)|0)}$$

$$C_{out} |0\,out\rangle \equiv \frac{U(0,+\infty)|0)}{(0|U(0,+\infty)|0)}$$

Hence

$$\langle 0_{out}|0_{in}\rangle = \frac{\langle 0|U^{-1}(0,+\infty)\,U(0,-\infty)|0\rangle}{\langle 0|U^{-1}(0,+\infty)|0\rangle\,\langle 0|U(0,-\infty)|0\rangle\, C_{in}C^{*}_{out}}$$

$$= \frac{\langle 0|U(+\infty,-\infty)|0\rangle}{\langle 0|U(+\infty,0)|0\rangle\langle 0|U(0,-\infty)|0\rangle C_{in}C^{*}_{out}}$$

Now we would like to again, from a different perspective, show that this is 1 in the absence of external fields. To show this we would like to prove that in fact $|0\rangle$ is an eigenstate of $U(\pm\infty)$. Hence it cannot connect up to other $n \geq 1$ particle states. So consider the arbitrary state $|k,\alpha\rangle$ with a particle with momentum k and arbitrary other particles denoted cryptically by α &

$$\langle k,\alpha|U^{-1}(0,t)|0\rangle = \langle\alpha|a^{ip}(k)\,U^{-1}(0,t)|0\rangle$$

$$= i\int d^3y\, e^{iky}\overleftrightarrow{\partial}_{y_0}\langle\alpha|\phi^{ip}(y)\,U^{-1}(0,t)|0\rangle$$

where the inverse F.T. relation was used. Now recall that

$$\phi^{ip}(y) = U(y^0,0)\,\phi(y)\,U^{-1}(y^0,0)$$

So

$$= i\int d^3y\, e^{iky} \overleftrightarrow{\partial}_{y_0} (\alpha| U(y^0,0)\,\phi(y)\, U^{-1}(y^0,0)\, U^{-1}(0,t)|0)$$

So for $y_0 = t \to +\infty$, we find (recall $a^{ip}(k)$ is indep. of time)

$$= (\alpha| \dot{U}^{-1}(0,+\infty)\, a(k, y^0 \to +\infty)|0)$$

$$+ \lim_{y_0 = t\to+\infty} i\int d^3y\, e^{iky} (\alpha| \dot{U}(t,0)\,\phi(y) + U(t,0)\,\phi(y)\,\dot{U}^{-1}(t,0)$$

$$U^{-1}(0,t)|0)$$

Now as $y^0 \to +\infty$ the $H_I^{ip}(t) \to 0$ so that $a(k, y^0\to+\infty) \longrightarrow a^{ip}(k)$ so the first term vanishes.

The second set of terms can be analyzed by considering with $y^0 = t$ understood

$$\dot{U}(t,0)\,\phi(y) + U(t,0)\,\phi(y)\,\dot{U}^{-1}(t,0)\; U(t,0)$$

$$= \dot{U}(t,0)\,U^{-1}(t,0)\,\phi^{ip}(y)\,U(t,0) + U(t,0)\,\phi(y)\,(-U^{-1}(t,0)\,\dot{U}(t,0))$$

$$= \dot{U}(t,0)\,U^{-1}(t,0)\,\phi^{ip}(y)\,U(t,0)$$

$$- \phi^{ip}(y)\,\dot{U}(t,0)\,U^{-1}(t,0)\,U(t,0)$$

$$= [\dot{U}(t,0)U^{-1}(t,0), \phi^{ip}(y)]U(t,0)$$

But
$$\dot{U}(t,0)U^{-1}(t,0) = -iH_I^{ip}(t)$$

So the second set of terms becomes

$$= -i[H_I^{ip}(t), \phi^{ip}(\vec{y},t)]U(t,0)$$

Now if we assume $H_I = H_I(\phi)$ only no derivative coupling then this equal Time Commutator vanishes

$$= 0.$$

Thus we recover

$$(\alpha,\vec{k}|U^{-1}(0,+\infty)|0) = 0$$

Hence $(\mathcal{Y}|U(+\infty,0)|0) = 0$ for $(\mathcal{Y}|$ any state but the vacuum.

Thus we have that in the absence of external sources $U(+\infty, 0)$ connects the vacuum to only the vacuum; similarly for $U(0,-\infty)$.

Of course our "proof" is only formal (not-rigorous) in fact no proof for this property can be given; it is a basic assumption of field theory called the stability of the vacuum.

Even above in our heuristic proof it is unclear where the lack of external sources was assumed. By braking the F.T. integral up into a sum of 2 sets of terms we assume convergence was uniform, this is not true if external source is present, as we will see.

Proceeding anyway

So for No external fields we have that

$$\langle n | U(0,-\infty) | 0\rangle = 0$$

for $|n\rangle \neq |0\rangle$ the vacuum state.

hence, since $|n\rangle$ for $n \geq 0$ are complete we have that

$$U(0,-\infty)|0\rangle \propto |0\rangle \quad \text{again}$$

Thus $U(0,-\infty)|0\rangle = \lambda_- |0\rangle$ with

λ_- a phase, $\langle 0|U(0,-\infty)|0\rangle = \lambda_- \langle 0|0\rangle = \lambda_- \equiv \omega_-$

Similarly $U(0,+\infty)|0\rangle = \omega_+ |0\rangle$

with ω_+ a phase

and $\langle 0|U(+\infty,-\infty)|0\rangle = \sum_n \langle 0|U(+\infty,0)|n\rangle\langle n| U(0,-\infty)|0\rangle$

Thus $= \langle 0|U(+\infty,0)|0\rangle\langle 0|U(0,-\infty)|0\rangle = \omega_+^* \omega_-$

$$\langle 0_{out}|0_{in}\rangle = \frac{\omega_+^*\omega_-}{\omega_+^*\omega_-} = 1,$$

in the absence of external sources (with $c_{in}c_{out}^* = e^{i(\varphi_{in}-\varphi_{out})} \equiv 1$).

Hence in this case our formulae simplify to yield, for No external sources

$$S_{fi} = \langle f\,out | i\,in \rangle \quad \text{with}$$

$|0\,out\rangle = |0\,in\rangle \equiv |0\rangle$, the unique stable vacuum state.

The Green functions are defined as

$$G^{(n)}(x_1, \ldots, x_n) = \langle 0 | T\,\phi(x_1)\cdots\phi(x_n) | 0 \rangle$$

$$= \frac{\langle 0 | T\,\phi_{in}(x_1)\cdots\phi_{in}(x_n)\, e^{-i\int_{-\infty}^{+\infty} dt\, H_{I in}(t)} | 0 \rangle}{\langle 0 | T\, e^{-i\int_{-\infty}^{+\infty} dt\, H_{I in}(t)} | 0 \rangle}$$

and so

$$S_{fi} = i^{m+n} \int d^4x_1 \cdots d^4y_m\, e^{ip_i x_i}\, e^{-iq_j y_j} \times$$

$$\times (\partial_{x_1}^2 + m^2) \cdots (\partial_{y_m}^2 + m^2) \langle 0 | T\,\phi(x_1)\cdots\phi(x_m) | 0 \rangle \Big|_{\text{mass shell}}$$

In the case that external sources do not vanish $|0_{in}\rangle \neq |0_{out}\rangle$ and we can calculate the transition amplitude $\langle 0_{out}|0_{in}\rangle$.

For example suppose that we consider the hermitian scalar field coupled to a prescribed classical source $J(x)$ so that

$$(\partial^2 + m^2)\phi = J .$$

The Lagrangian is given by

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 + J\phi .$$

As usual we treat $+J\phi$ as the interaction Lagrangian $\mathcal{L}_I = J\phi = -\mathcal{H}_I$ and the free Lagrangian as usual

$$\mathcal{L}_0 = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 .$$

Since our only interaction is with an external field we do not need to worry about the infinite phase factor in the definition of the in- and out-states, it is finite. Hence we will choose the normalization constants C_{in} and C_{out} to precisely cancel this factor. Thus we choose here

$$C_{in} \equiv \frac{1}{\langle 0|U(0,-\infty)|0\rangle}$$

$$C_{out} \equiv \frac{1}{\langle 0|U(0,+\infty)|0\rangle}$$

So that

$$|\Psi_{in}\rangle \equiv U(0,-\infty)|\Psi\rangle$$

$$|\Psi_{out}\rangle \equiv U(0,+\infty)|\Psi\rangle .$$

Hence

$$\langle f_{out}|i_{in}\rangle = \langle f|U(+\infty,-\infty)|0\rangle^{i}$$

Now

$$|i_{in}\rangle = U(0,-\infty)U^{-1}(0,+\infty)|i_{out}\rangle$$

$$\equiv S|i_{out}\rangle$$

Thus $S = U(0,-\infty)U^{-1}(0,+\infty)$ here.

$$U(0,-\infty)U^{-1}(0,+\infty) = U(0,-\infty)U(+\infty,-\infty)U^{-1}(0,+\infty) = U^{in}(+\infty,-\infty)$$

From our previous analysis we have (p. -95-) So that the S-operator is given by

$$S \equiv T e^{i\int d^4x \mathcal{L}_{I\,in}}$$

So

$$S = T e^{i\int d^4x\, J(x)\,\phi_{in}(x)}.$$

We can apply Wick's theorem to this Time ordered product so that it can be written as a Normal product wrt the $|0in\rangle$ state.

$$T\,\phi_{in}(x_1)\cdots\phi_{in}(x_n) = N[\phi_{in}(x_1)\cdots\phi_{in}(x_n)]$$

$$+\sum_{1\,pair} \frac{\langle 0in|T\,\phi_{in}(x_i)\phi_{in}(x_j)|0in\rangle}{\langle 0in|0in\rangle} \times$$

$$\times N[\phi_{in}(x_1)\cdots \cancel{i} \cdots \cancel{j} \cdots \phi_{in}(x_n)]$$

$$+\sum_{2\,pairs} \cdots + \frac{\langle 0in|T\,\phi_{in}(x_1)\ \phi_{in}(x_2)|0in\rangle}{\langle 0in|0in\rangle} \times \cdots \frac{\langle 0in|T\,\phi_{in}(x_{n/2})\phi_{in}(x_{n/2})|0in\rangle}{\langle 0in|0in\rangle}$$

We can now multiply by $J(x_1)\cdots J(x_n)$ and integrate over $\int d^4x_1\cdots d^4x_n$

Relabelling the dummy integration variables we find

$$\int dx_1 \cdots dx_n \, T\,\phi_{in}(x_1) \cdots \phi_{in}(x_n) \; J(x_1) \cdots J(x_n)$$

$$= \int dx_1 \cdots dx_n \, J(x_1) \cdots J(x_n) \, N[\phi_{in}(x_1) \cdots \phi_{in}(x_n)]$$

$$+ \frac{n(n-1)}{2} \int dy_1 dy_2 \, J(y_1) \Delta_F(y_1 - y_2) J(y_2)$$

$$\int dx_1 \cdots dx_{n-2} \, J(x_1) \cdots J(x_{n-2}) \, N[\phi_{in}(x_1) \cdots \phi_{in}(x_{n-2})]$$

$$+ \cdots + \frac{n(n-1)\cdots(n-2l)}{2^l \, (l)!} \int dy_1 \cdots dy_{2l} \, J(y_1) \Delta_F(y_1 - y_2) J(y_2)$$

$$J(y_3) \Delta_F(y_3 - y_4) J(y_4)$$

$$\cdots J(y_{2l-1}) \Delta_F(y_{2l-1} - y_{2l}) J(y_{2l})$$

$$\int dx_1 \cdots dx_{n-2l} \, J(x_1) \cdots J(x_{n-2l}) \, N[\phi_{in}(x_1) \cdots \phi_{in}(x_{n-2l})]$$

$$+ \cdots$$

$$= \sum_{l=0}^{n/2} \frac{n!}{2^l (n-2l)!\, l!} \left[\int dy_1 dy_2\, J(y_1) \Delta_F(y_1-y_2) J(y_2)\right]^l$$

$$\times \int dx_1 \cdots dx_{n-2l}\, N[\phi_{in}(x_1) \cdots \phi_{in}(x_{n-2l})] \times$$

$$\times J(x_1) \cdots J(x_{n-2l})$$

Thus we find

$$S = T e^{i\int d^4x\, J(x) \phi_{in}(x)}$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{n/2} \frac{i^n\, n!}{n!\, l!\, (n-2l)!} \left[\frac{1}{2}\int dy_1 dy_2\, J(y_1) \Delta_F(y_1-y_2) J(y_2)\right]^l$$

$$\times \int dx_1 \cdots dx_{n-2l}\, J(x_1) \cdots J(x_{n-2l}) N[\phi_{in}(x_1) \cdots \phi_{in}(x_{n-2l})]$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{l!} \left[\frac{i^2}{2}\int dy_1 dy_2\, J(y_1) \Delta_F(y_1-y_2) J(y_2)\right]^l$$

$$\times \frac{i^n}{n!} \int dx_1 \cdots dx_n\, J(x_1) \cdots J(x_n) N[\phi_{in}(x_1) \cdots \phi_{in}(x_n)]$$

Thus

$$\boxed{\begin{aligned} S &= T e^{i\int d^4x\, J(x)\,\phi_{in}(x)} \\ &= \left[e^{-\frac{1}{2}\int dy_1^4 dy_2^4\, J(y_1)\,\Delta_F(y_1-y_2)\,J(y_2)} \right] \times \\ &\quad \times N\left[e^{i\int d^4x\, J(x)\,\phi_{in}(x)} \right] \end{aligned}}$$

where $\Delta_F(x-y) = \dfrac{\langle 0\,in|\,T\phi_{in}(x)\,\phi_{in}(y)\,|0\,in\rangle}{\langle 0\,in|\,0\,in\rangle}$

So we find that

$$\boxed{\langle 0\,in|\,S\,|\,0\,in\rangle = e^{-\frac{1}{2}\int dx\,dy\; J(x)\,\Delta_F(x-y)\,J(y)}}$$

$$\boxed{= \langle 0\,out|\,0\,in\rangle}$$

Along similar lines we have that

$$S^{iP} = T e^{i\int dx\, J(x)\, \phi^{iP}(x)}$$

$$= \left[e^{-\frac{1}{2}\int dy_1 dy_2\, J(y_1)\, \Delta_F(y_1-y_2)\, J(y_2)} \right] \times$$

$$\times N\left[e^{i\int dx\, J(x)\, \phi^{iP}(x)} \right]$$ where N is defined wrt $|0\rangle$ now.

So $$\boxed{\langle 0| S^{iP} |0\rangle = e^{-\frac{1}{2}\int dx\, dy\, J(x)\, \Delta_F(x-y)\, J(y)}}$$

Similarly

$$U(t,t') = T e^{i\int_{t'}^{t} dx\, J(x)\, \phi^{iP}(x)}$$

$$= \left[e^{-\frac{1}{2}\int_{t'}^{t} dy_1 dy_2\, J(y_1)\, \Delta_F(y_1-y_2)\, J(y_2)} \right]$$

$$N\left[e^{i\int_{t'}^{t} dx\, J(x)\, \phi^{iP}(x)} \right]$$

Thus

$$\boxed{\langle 0_{out} | 0_{in} \rangle = e^{-\frac{1}{2}\int d^4x\, d^4y\, J(x)\Delta_F(x-y)J(y)} \neq 1.}$$

We can further analyze this amplitude by expressing the above in terms of Fourier transformed quantities

$$\tilde{J}(p) \equiv \int d^4x\, e^{+ipx} J(x)$$

$$J(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \tilde{J}(p)$$

So

$$\int d^4x\, d^4y\, J(x)\Delta_F(x-y)J(y)$$

$$= \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} \int d^4x\, d^4y\, \tilde{J}(p) e^{-ipx} \frac{e^{-ik(x-y)}\, i}{k^2-m^2+i\epsilon} \times$$

$$\times \tilde{J}(q) e^{-iqy}$$

$$= \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} \tilde{J}(p)\tilde{J}(q) \frac{i}{k^2-m^2+i\epsilon} (2\pi)^4\delta^4(p+k) \times$$

$$\times (2\pi)^4 \delta^4(q-k)$$

$$= \int \frac{d^4k}{(2\pi)^4} \tilde{J}(-k)\tilde{J}(k) \frac{i}{k_0^2 - \omega_k^2 + i\epsilon}$$

$$= \int \frac{d^3k}{(2\pi)^3} \int \frac{dk^0}{(2\pi)} \tilde{J}(-k)\tilde{J}(k) \frac{i}{(k_0 - \omega_k + i\epsilon)(k_0 + \omega_k - i\epsilon)}$$

Closing the contour either above or below the real k^0-line we find

$$= \int \frac{d^3k}{(2\pi)^3} \left[\frac{\tilde{J}(-k)\tilde{J}(k)\, i}{k_0 + \omega_k} \frac{(-2\pi i)}{(2\pi)} \right] \Bigg|_{k^0 = +\omega_k}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \tilde{J}(\vec{k}, \omega_k)\tilde{J}(-\vec{k}, -\omega_k)$$

But $J(x) = J(x)^*$ is real $\Rightarrow J(-k) = J^*(k)$

So

$$\int d^4x\, d^4y\, J(x)\, \Delta_F(x-y)\, J(y)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} |\tilde{J}(\vec{k}, \omega_k)|^2$$

Thus

$$\boxed{\langle 0\,out | 0\,in \rangle = e^{-\frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega_k} |\tilde{J}(\vec{k},\omega_k)|^2}}$$

$$\neq 1.$$

Further we can consider the transition of $|0\,in\rangle$ into an arbitrary N particle state $|k_1,\ldots,k_N\,out\rangle$. ~~As~~ usual now the transition amplitude is calculated relative to the $|0\,in\rangle$ to $|0\,out\rangle$ amplitud

$$S_{N0} = \frac{\langle k_1,\ldots,k_N\,out | 0\,in\rangle}{\langle 0\,out | 0\,in\rangle}$$

$$= \frac{\langle 0\,out | a_{out}(\vec{k}_1)\cdots a_{out}(\vec{k}_N) S | 0\,out\rangle}{\langle 0\,out | 0\,in\rangle}$$

but

$$S = T\, e^{i\int d^4x\, J(x)\phi_{out}(x)}$$

$$= \left[e^{-\frac{1}{2}\int d^4y_1 d^4y_2\, J(y_1)\Delta_F(y_1-y_2)J(y_2)}\right] \times$$

$$\times N\left[e^{i\int d^4x\, J(x)\phi_{out}(x)}\right]$$

So
$$S = \langle 0_{out} | 0_{in} \rangle N\left[e^{i\int d^4x\, J(x)\, \phi_{out}(x)} \right]$$

thus

$$S_{N0} = \langle 0_{out} | a_{out}(\vec{k}_1) \cdots a_{out}(\vec{k}_N) \times$$
$$\times N\left[e^{i\int d^4x\, J(x)\, \phi_{out}(x)} \right] | 0_{out} \rangle$$

Only the N^{th} order term in the exponential contributes to the vacuum expectation value and only the creation operator part of ϕ_{out} does not annihilate the $|0_{out}\rangle$ vacuum state, hence

$$S_{N0} = \frac{i^N}{N!} \int dx_1 \cdots dx_N\, J(x_1) \cdots J(x_N)$$
$$\underbrace{\langle 0_{out} | a_{out}(\vec{k}_1) \cdots a_{out}(\vec{k}_N)\, \phi^-_{out}(x_1) \cdots \phi^-_{out}(x_N) | 0_{out} \rangle}_{\text{as usual this is just a series of commutators; if fact } N! \text{ of them}}$$

$$= i^N \int dx_1 \cdots dx_N\, J(x_1) \cdots J(x_N) \left[a_{out}(\vec{k}_1), \phi^-_{out}(x_1) \right] \cdots$$
$$\cdots \left[a_{out}(\vec{k}_N), \phi^-_{out}(x_N) \right]$$

where we have re-labelled the indices on the $\int dx\, J(x)\, \phi_{out}(x)$ terms to match the $a_{out}(\vec{k}_i)$ it commutes with.

Now

$$\int d^4x\, J(x)\, [a_{out}(\vec{k}), \phi^-_{out}(x)]$$

$$= \int d^4x\, J(x) \int \frac{d^3p}{(2\pi)^3 2\omega_p} e^{+ipx} \underbrace{[a_{out}(\vec{k}), a^\dagger_{out}(\vec{p})]}_{(2\pi)^3 2\omega_p \delta^3(\vec{k}-\vec{p})}$$

$$= \int d^4x\, J(x)\, e^{+ikx} \qquad \text{with } k^0 = \omega_k$$

$$= \tilde{J}(\vec{k}, \omega_k)$$

Thus we get

$$\boxed{S_{N0} = i\tilde{J}(\vec{k}_1, \omega_{k_1}) \cdots i\tilde{J}(\vec{k}_N, \omega_{k_N})}$$

Hence the probability to emit any number N of bosons with all momenta relative to the vacuum to vacuum transition is

$$P_N = \int \frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}} \cdots \frac{d^3k_N}{(2\pi)^3 2\omega_{k_N}} \frac{1}{N!} |S_{N0}|^2$$

where the $\frac{1}{N!}$ is due to the $\langle k_1, \ldots, k_N \text{ out} | l_1, \ldots, l_N \text{ out} \rangle$

$$= \sum_{\substack{P \\ (1\cdots N) \to (i_1 \cdots i_N)}} (2\pi)^3 2\omega_{k_{i_1}} \delta^3(\vec{k}_1 - \vec{l}_{i_1}) \cdots (2\pi)^3 2\omega_{k_{i_N}} \delta^3(\vec{k}_N - \vec{l}_{i_N})$$

$N!$ terms in the inner product normalization of our states.

So
$$\boxed{P_N = \frac{1}{N!} \left(\int \frac{d^3k}{(2\pi)^3 2\omega_k} |\tilde{J}(\vec{k}, \omega_k)|^2 \right)^N}$$

Defining
$$\bar{n} \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} |\tilde{J}(\vec{k}, \omega_k)|^2$$

we have the probability to emit N-bosons normalized to $\langle 0 \text{ out} | 0 \text{ in} \rangle$ is

$$\boxed{P_N = \frac{1}{N!} \bar{n}^N}.$$

Since the sum over probabilities to emit N-bosons must add up to one, we can eliminate the $\langle 0_{out}|0_{in}\rangle$ normalization directly by requiring the physical or normalized probability p_N to add up to 1

Let $\quad p_N \equiv C P_N \quad$ with $P_0 = 1$

then

$$1 = \sum_{N=0}^{\infty} p_N = C \sum_{N=0}^{\infty} P_N = C \sum_{N=0}^{\infty} \frac{\bar{n}^N}{N!}$$

$$= C e^{+\bar{n}}$$

$$\Rightarrow \boxed{C = e^{-\bar{n}}}$$

and

$$p_N = \frac{1}{N!} \frac{\bar{n}^N}{e^{+\bar{n}}}$$

Thus the normalized probability for the in-vacuum to decay into N-bosons is

$$\boxed{P_N = e^{-\bar{n}} \frac{\bar{n}^N}{N!}} \quad \text{, The Poisson distribution}$$

The average number of emitted bosons is just

$$\langle N \rangle = \frac{\sum_{N=0}^{\infty} P_N N}{\sum_{N=0}^{\infty} P_N} = \bar{n}$$

$$\left(\text{i.e.} \quad = \frac{\sum_{N=0}^{\infty} \frac{1}{N!} N \bar{n}^N}{\sum_{N=0}^{\infty} \frac{\bar{n}^N}{N!}} = \frac{\bar{n} \sum_{N=1}^{\infty} \frac{\bar{n}^{N-1}}{(N-1)!}}{\sum_{N=0}^{\infty} \frac{\bar{n}^N}{N!}} = \bar{n} \right)$$

The transition probability for $|0\text{in}\rangle \to |0\text{out}\rangle$ is simply

$$|\langle 0\text{out}|0\text{in}\rangle|^2 = p_0 = e^{-\bar{n}}$$

Thus $|\langle 0\text{out}|0\text{in}\rangle| = e^{-\bar{n}/2}$

Now as the average # of emitted bosons $\bar{n} \to \infty$ $|\langle 0\text{out}|0\text{in}\rangle| \to 0$!!!!

All in-state & out-state matrix elements vanish. We cannot construct the out states from the in-states and the S-operator. Hence for systems with ∞ number of degrees of freedom under certain conditions like $\overline{n} \to \infty$, inequivalent representations of the canonical commutation relations may exist. Here the final states actually have ∞ # of bosons in them and cannot be described by finite # of particle out-states.

Of course in physical situations we only observe a finite range of momentum-energy etc. so the integrals & sums are cutoff and we can avoid these problems.

Hence, we have re-expressed our fundamental field theoretic formulae in terms of Heisenberg picture states and operators and explored the principles these formulae are based upon to some degree. We can now turn the tables and ask how far can we go by stating certain fundamental axioms whose validity seem physically reasonable from our above discussions. We will find that we can obtain the LSZ reduction formulae without a lot of input; but to go further we will need more direct dynamical assumptions.

Hence we start by considering the implications of the LSZ axioms.