§2.4 MASSLESS SPIN ONE PARTICLES: PHOTONS

Recall in the introduction that we described a system of noninteracting photons by means of the quantum field four-potential $A^\mu = (\phi, \vec{A})$ with the relativistic invariant Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{2.4.1}$$

where the field strength tensor was defined to be

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.4.2}$$

We subjected the four vector potential to the Coulomb gauge condition $\vec{\bigtriangledown} \cdot \vec{A} = 0$. Consequently, we were able to eliminate two of the four components of A^μ. The remaining two degrees of freedom were subjected to the canonical quantization procedure. More specifically Maxwell's equations implied that

$$\partial_\mu F^{\mu\nu} = 0. \tag{2.4.3}$$

Hence we found $\bigtriangledown^2 \phi = 0$ or $\phi = 0$, and the ETCR were given as

$$[\Pi^i(\vec{x}, t), A^j(\vec{y}, t)] = i\delta^{ij}_{tr}(\vec{x} - \vec{y}) \tag{2.4.4}$$

where the transversality of the physical fields was manifested in the transverse δ-function

$$\delta^{ij}_{tr}(\vec{x} - \vec{y}) = \int \frac{d^3k}{(2\pi)^3} (\delta^{ij} - \frac{k^i k^j}{\vec{k}^2}) e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \tag{2.4.5}$$

and $\vec{\Pi}(x) = -\dot{\vec{A}}$. Further, the Coulomb gauge constraint can be built right into the Fourier transform. Since $\vec{\bigtriangledown} \cdot \vec{A} = 0$ we must have in momentum space

$$\vec{\bigtriangledown} \cdot \vec{A} = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} i\vec{k} \cdot \vec{\tilde{A}}(k) = 0 \tag{2.4.6}$$

implying that

$$\tilde{A}^i(k) = (\delta^{ij} - \frac{k^i k^j}{\vec{k}^2}) B^j(k) \tag{2.4.7}$$

since $\partial_\mu F^{\mu\nu} = 0$ along with $\vec{\bigtriangledown} \cdot \vec{A} = 0$ implies $\phi = 0$, this yields $\partial^2 \vec{A} = 0$,

$$\vec{B}(k) = (2\pi)\delta(k^2)\vec{a}(\vec{k}, k^0) \tag{2.4.8}$$

Hence we obtained with $\vec{a}(\vec{k}, +\omega_k) \equiv \vec{a}(\vec{k})$ and using the hermiticity of $\vec{A}(x)$ so that $\vec{a}(-\vec{k}, -\omega_k) = \vec{a}^\dagger(\vec{k})$

$$
\begin{aligned}
A^i(x) &= \int \frac{d^4k}{(2\pi)^4}(2\pi)\delta(k^2)e^{-ikx}(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2})a^j(\vec{k}, k^0) \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2})[a^j(\vec{k})e^{-ikx} + a^{j\dagger}(\vec{k})e^{+ikx}].
\end{aligned} \tag{2.4.9}
$$

The $(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2})$ projects three-space vectors onto the plane perpendicular to $\vec{k}$, that is, the plane spanned by the polarization vectors $\vec{\epsilon}_r(\vec{k})$. Furthermore, the $\vec{k}$-transverse projector is

$$
P^{ij}_{tr} = \sum_{r=1}^{2} \epsilon^i_r(\vec{k})\epsilon^j_r(\vec{k}) = (\delta^{ij} - \frac{k^i k^j}{\vec{k}^2}), \tag{2.4.10}
$$

the $\vec{\epsilon}_r(\vec{k})$ are complete in this plane. Hence,

$$
A^i(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{r=1}^{2} \epsilon^i_r(\vec{k})[\epsilon^j_r(\vec{k})a^j(\vec{k})e^{-ikx} + \epsilon^j_r(\vec{k})a^{j\dagger}(\vec{k})e^{+ikx}] \tag{2.4.11}
$$

But defining

$$
\begin{aligned}
a_r(\vec{k}) &\equiv \vec{\epsilon}_r(\vec{k}) \cdot \vec{a}(\vec{k}) \\
a^\dagger_r(\vec{k}) &\equiv \vec{\epsilon}_r(\vec{k}) \cdot \vec{a}^\dagger(\vec{k})
\end{aligned} \tag{2.4.12}
$$

we obtained

$$
\vec{A}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{r=1}^{2} \vec{\epsilon}_r(\vec{k})[a_r(\vec{k})e^{-ikx} + a^\dagger_r(\vec{k})e^{+ikx}] \tag{2.4.13}
$$

with

$$
\begin{aligned}
\vec{u}_r(x) &= \vec{\epsilon}_r(\vec{k})e^{-ikx} \\
\vec{v}_r(x) = \vec{u}^*_r(x) &= \vec{\epsilon}_r(\vec{k})^* e^{+ikx},
\end{aligned} \tag{2.4.14}
$$

the positive and negative energy solutions to Maxwell's equations in the Coulomb gauge. The operator $a^\dagger_r(\vec{k})$ creates photons with momentum $\vec{k}$ and polarization (helicity) $\vec{\epsilon}_r(\vec{k})$ while $a_r(\vec{k})$ annihilates such particles. The coordinate space equal time commutation relations led to the momentum space canonical commutation relations.

$$
[a_r(\vec{k}), a^\dagger_s(\vec{l})] = (2\pi)^3 2\omega_k \delta_{rs}\delta^3(\vec{k} - \vec{l})
$$

$$[a_r(\vec{k}), a_s(\vec{l})] = 0$$
$$[a_r^\dagger(\vec{k}), a_s^\dagger(\vec{l})] = 0. \tag{2.4.15}$$

The vacuum was defined to be absent of all photons

$$a_r(\vec{k})|0>= 0 \tag{2.4.16}$$

for all $\vec{k}$ and r. The Hilbert space of photon states was built up from the vacuum state by action of the creation operators

$$|(\vec{k}_1, r_1), \ldots, (\vec{k}_N, r_N)>= a_{r_1}^\dagger(\vec{k}_1)\cdots a_{r_N}^\dagger(\vec{k}_N)|0> . \tag{2.4.17}$$

The Hamiltonian and momentum operators were given as usual by

$$\mathcal{P}^\mu = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{r=1}^{2} k^\mu a_r^\dagger(\vec{k}) a_r(\vec{k}) \tag{2.4.18}$$

with $k^\mu = (\omega_k, \vec{k}) = (|\vec{k}|, \vec{k})$ and $H = \mathcal{P}^0$ having eigenvalues greater than or equal to zero. Further, the angular momentum of the photon was described by the helicity operator

$$\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|} = \frac{1}{2}\mathcal{M}^{jk}\epsilon_{jki}\frac{k^i}{|\vec{k}|} \tag{2.4.19}$$

(recall the time component of the Pauli-Lubanski vector operator, $W^0 = -\vec{\mathcal{J}}\cdot\vec{\mathcal{P}}$, we choose it as one of our CSCO for massless particles rather than W_3) and we found that

$$[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a_\pm^\dagger(\vec{k})] = \pm a_\pm^\dagger(\vec{k}) \tag{2.4.20}$$

where

$$a_\pm^\dagger(\vec{k}) = \pm\frac{1}{\sqrt{2}}(a_1^\dagger(\vec{k}) \pm i a_2^\dagger(\vec{k})) \tag{2.4.21}$$

are the circularly polarized creation operators. Consequently,

$$\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|} = \int \frac{d^3k}{(2\pi)^3 2\omega_k}[a_+^\dagger(\vec{k})a_+(\vec{k}) - a_-^\dagger(\vec{k})a_-(\vec{k})] \tag{2.4.22}$$

which we can derive from a general expression for $\mathcal{M}^{\mu\nu}$ as will be seen later. We may proceed as usual and calculate the all time commutators to find

$$[A^i(x), A^j(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{d^3l}{(2\pi)^3 2\omega_l} \sum_{r=1}^{2} \sum_{s=1}^{2} \epsilon_r^i(\vec{k})\epsilon_s^j(\vec{l})\{[a_r(\vec{k}), a_s^\dagger(\vec{l})]e^{-ikx+ily}$$
$$+ [a_r^\dagger(\vec{k}), a_s(\vec{l})]e^{+ikx-ily}\}$$
$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{r=1}^{2} \epsilon_r^i(\vec{k})\epsilon_r^j(\vec{k})[e^{-ik(x-y)} - e^{+ik(x-y)}]. \tag{2.4.23}$$

But recall $\vec{\epsilon}_r(\vec{k})$ along with $\vec{k}$ form the coordinate system basis shown in Figure 2.4.1.

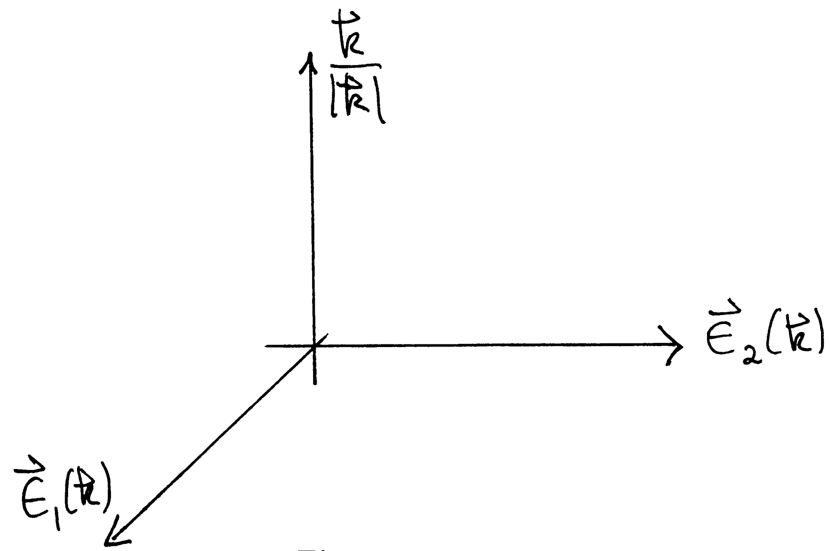

Figure 2.4.1

So $\vec{k} \cdot \vec{\epsilon}_r(\vec{k}) = 0$; $\vec{\epsilon}_r(\vec{k}) \cdot \vec{\epsilon}_s(\vec{k}) = \delta_{rs}$ and thus $\sum_{r=1}^{2} \epsilon_r^i(\vec{k})\epsilon_r^j(\vec{k})$ is a matrix which projects onto the $\vec{\epsilon}_1 - \vec{\epsilon}_2$ plane perpendicular to $\vec{k}$. Hence we can write it as the $\vec{k}$-transverse projector, as already mentioned,

$$P_{tr}^{ij}(\vec{k}) \equiv \sum_{r=1}^{2} \epsilon_r^i(\vec{k})\epsilon_r^j(\vec{k}) = \delta^{ij} - \frac{k^i k^j}{\vec{k}^2}. \tag{2.4.24}$$

So for a vector $\vec{v}$,

$$\mathbf{P}_{tr}(\vec{k})\vec{v} = \vec{v} - \frac{(\vec{v} \cdot \vec{k})}{|\vec{k}|} \frac{\vec{k}}{|\vec{k}|}, \tag{2.4.25}$$

it projects $\vec{v}$ onto the $\vec{\epsilon}_1 - \vec{\epsilon}_2$ plane

$$\mathbf{P}_{tr}\vec{v} = (\vec{v}\cdot\vec{\epsilon}_r)\vec{\epsilon}_r \tag{2.4.26}$$

since using the $(\vec{\epsilon}_1, \vec{\epsilon}_2, \vec{k})$ basis vectors

$$\vec{v} = (\vec{v}\cdot\vec{k})\frac{\vec{k}}{|\vec{k}|^2} + (\vec{v}\cdot\vec{\epsilon}_r)\vec{\epsilon}_r. \tag{2.4.27}$$

Thus

$$\begin{aligned}[A^i(x), A^j(y)] &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2})[e^{-ik(x-y)} - e^{+ik(x-y)}] \\ &= (\delta^{ij} - \frac{\partial_x^i \partial_x^j}{\nabla_x^2})\int \frac{d^3k}{(2\pi)^3 2\omega_k}[e^{-ik(x-y)} - e^{+ik(x-y)}],\end{aligned} \tag{2.4.28}$$

but recall

$$i\Delta(x-y;m) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}[e^{-ik(x-y)} - e^{+ik(x-y)}] \tag{2.4.29}$$

where $\omega_k = \sqrt{|\vec{k}|^2 + m^2}$. For $m = 0$, not to confuse our notation we call this $iD(x-y)$,

$$iD(x-y) \equiv i\Delta(x-y; m=0). \tag{2.4.30}$$

Hence,

$$[A^i(x), A^j(y)] = (\delta^{ij} - \frac{\partial_x^i \partial_x^j}{\nabla_x^2})iD(x-y). \tag{2.4.31}$$

Similarly, we have

$$\begin{aligned}< 0|A^i(x)A^j(y)|0 >&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\sum_{r=1}^{2}\epsilon_r^i(\vec{k})\epsilon_r^j(\vec{k})e^{-ik(x-y)} \\ &= P_{tr}^{ij}(x)iD^+(x-y)\end{aligned} \tag{2.4.32}$$

with

$$P_{tr}^{ij}(x) \equiv (\delta^{ij} - \frac{\partial_x^i \partial_x^j}{\nabla_x^2}) \tag{2.4.33}$$

and

$$iD^{\pm}(x) \equiv i\Delta^{\pm}(x; m=0). \tag{2.4.34}$$

Furthermore, defining the time ordered functions as in the scalar case

$$\begin{aligned}<0|TA^i(x)A^j(y)|0> &= \theta(x^0-y^0) <0|A^i(x)A^j(y)|0> \\ &\quad + \theta(y^0-x^0) <0|A^j(y)A^i(x)|0> \\ &= P^{ij}_{tr}(x)[\theta(x^0-y^0)iD^+(x-y) \\ &\quad - \theta(y^0-x^0)iD^-(x-y)] \\ &= P^{ij}_{tr}(x)D_F(x-y)\end{aligned} \tag{2.4.35}$$

where $D_F(x-y) = \Delta_F(x-y; m=0)$. Hence,

$$\begin{aligned}<0|TA^i(x)A^j(y)|0> &= D^{ij}_{F\,tr}(x-y) = P^{ij}_{tr}(x)D_F(x-y) \\ &= P^{ij}_{tr}(x)\int \frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\frac{i}{k^2+i\epsilon} \\ &= \int \frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}(\delta^{ij}-\frac{k^ik^j}{|\vec{k}|^2})\frac{i}{k^2+i\epsilon}.\end{aligned} \tag{2.4.36}$$

Now we can write this in a more suggestive notation by introducing the four-vector field $A^\mu(x) = (0, \vec{A}(x))$ and a time-like unit vector n^μ. Our results above depended on choosing a frame of reference where the polarization vectors for momentum k^μ were three-vectors, that is, $\epsilon^\mu_r(\vec{k}) = (0, \vec{\epsilon}_r(\vec{k}))$. In this frame, let's introduce another vector perpendicular to $\vec{\epsilon}_{1,2}$ and $\vec{k}$ and call it $n^\mu = (1,0,0,0)$ pointing in the time direction. Hence, we can introduce an orthonormal system of four polarization four-vectors in this frame. We call these vectors $\epsilon^\mu(k,\lambda)$ where $\lambda = 0,1,2,3$ with

$$\epsilon^\mu(k,\lambda) = \begin{cases} n^\mu = (1,0,0,0) & \lambda = 0 \\ (0, \vec{\epsilon}_\lambda(\vec{k})) & \lambda = 1,2 \\ (0, \frac{\vec{k}}{|\vec{k}|}) & \lambda = 3. \end{cases} \tag{2.4.37}$$

These are orthonormal

$$\epsilon^\mu(k,\lambda)\epsilon_\mu(k,\rho) = g_{\lambda\rho} \tag{2.4.38}$$

and complete

$$\sum_{\lambda=0}^{3}\epsilon^\mu(k,\lambda)\epsilon^\nu(k,\lambda)g_{\lambda\lambda} = g^{\mu\nu} \tag{2.4.39}$$

or more directly

$$\sum_{\lambda=0}^{3}\epsilon^\mu(k,\lambda)\epsilon^\nu(k,\lambda) = \delta^{\mu\nu}. \tag{2.4.40}$$

We can write the last polarization vector in terms of k^μ and n^μ as

$$\epsilon^\mu(k,3) = (0, \frac{\vec{k}}{|\vec{k}|}) = \frac{k^\mu - (n\cdot k)n^\mu}{\sqrt{(n\cdot k)^2 - k^2}} \tag{2.4.41}$$

(here we have included a $k^2 \neq 0$ term since later we will want the polarization vectors away from the light-cone. For $k^2 = 0$, this is just $\epsilon^\mu(k,3) = \frac{k^\mu-(n\cdot k)n^\mu}{(n\cdot k)}$). Thus our four polarization vectors can be used as a set of basis vectors $(n^\mu, \epsilon^\mu(k,1), \epsilon^\mu(k,2), \frac{k^\mu-(n\cdot k)n^\mu}{\sqrt{(n\cdot k)^2-k^2)}})$. Using the completeness relation we can rewrite the transverse polarization sum as ($\epsilon^\mu(k,1)$ and $\epsilon^\mu(k,2)$ are called transverse polarizations while $\epsilon^\mu(k,0)$ and $\epsilon^\mu(k,3)$ are called the scalar and longitudinal polarizations, respectively)

$$\begin{aligned}
\sum_{r=1}^{2} \epsilon_r^i(\vec{k})\epsilon_r^j(\vec{k}) g^\mu{}_i g^\nu{}_j &= \sum_{\lambda=1}^{2} \epsilon^\mu(k,\lambda)\epsilon^\nu(k,\lambda) \\
&= -g^{\mu\nu} + \epsilon^\mu(k,0)\epsilon^\nu(k,0) - \epsilon^\mu(k,3)\epsilon^\nu(k,3) \\
&= -g^{\mu\nu} + n^\mu n^\nu - \frac{(k^\mu - (n\cdot k)n^\mu)(k^\nu - (n\cdot k)n^\nu)}{(n\cdot k)^2 - k^2} \\
&= -g^{\mu\nu} - \frac{n^\mu n^\nu k^2 - (n\cdot k)(k^\mu n^\nu + k^\nu n^\mu) + k^\mu k^\nu}{(n\cdot k)^2 - k^2}.
\end{aligned} \tag{2.4.42}$$

Thus, we find

$$\begin{aligned}
D^{\mu\nu}_{F\,tr}(x-y) &\equiv < 0|TA^\mu(x)A^\nu(y)|0> = < 0|TA^i(x)A^j(y)|0> g^\mu{}_i g^\nu{}_j \\
&= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{k^2+i\epsilon}[-g^{\mu\nu} \\
&\quad - \frac{n^\mu n^\nu k^2 - (n\cdot k)(k^\mu n^\nu + k^\nu n^\mu) + k^\mu k^\nu}{(n\cdot k)^2 - k^2}] \\
&= -g^{\mu\nu} D_F(x-y) - \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{k^2+i\epsilon} \\
&\quad \times [\frac{n^\mu n^\nu k^2 - (n\cdot k)(k^\mu n^\nu + k^\nu n^\mu) + k^\mu k^\nu}{(n\cdot k)^2 - k^2}].
\end{aligned} \tag{2.4.43}$$

Similarly with $k^2 = 0$, we have

$$[A^\mu(x), A^\nu(y)]$$

$$= -g^{\mu\nu} iD(x-y) - \frac{\partial_x^\mu \partial_x^\nu}{(n_\lambda \partial_x^\lambda)^2} iD(x-y) + \frac{(n_\lambda \partial_x^\lambda)(n^\mu \partial_x^\nu + n^\nu \partial_x^\mu)}{(n_\lambda \partial_x^\lambda)^2} iD(x-y). \quad (2.4.44)$$

The price of working in the Coulomb gauge now becomes evident. Although our observables were well defined (i.e. $H \geq 0$), and we had a well defined Hilbert space of states, the commutator and propagator are not Lorentz covariant due to their dependence on the preferred vector n^μ. Since our physical operators and their matrix elements are invariant under gauge transformations (i.e. dependent on $F^{\mu\nu}$), we can go to another gauge which manifestly preserves the Lorentz invariance and yields the same physical observables. Unfortunately there is a price to pay for this also and it is that by treating all the components of A_μ equally we will introduce two fictitious photon degrees of freedom, A_0 and A_3 in our special frame tied to k^μ. The space of states will no longer form a Hilbert space since A_0 states will have a negative norm. We will need to eliminate these modes from the physical subspace of states $A_{1,2}$ and show that indeed the matrix elements of the observables are the same as in the Coulomb gauge. Alternatively, we can work in the Coulomb gauge maintaining a good physical interpretation of the states and operators but then we must prove Lorentz covariance of the matrix elements.

The Lorentz invariance will come about in the free case trivially since observables are independent of n^μ. When interactions are considered, for example, when a current appears on the right hand side of the Maxwell field equations

$$\partial_\mu F^{\mu\nu} = -J^\nu, \quad (2.4.45)$$

we no longer can set $\phi = 0$ since the Coulomb gauge implies Gauss' law $\bigtriangledown^2 \phi = J^0$. Thus, the time component can be solved to yield

$$\phi(x) = \int d^3y D_c(\vec{x} - \vec{y}) J^0(y) \quad (2.4.46)$$

where the Coulomb Green function obeys

$$\bigtriangledown_x^2 D_c(\vec{x} - \vec{y}) = \delta^3(\vec{x} - \vec{y}) \quad (2.4.47)$$

and is given by

$$D_c(\vec{x} - \vec{y}) = -\frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{y}|} = \int \frac{d^3k}{(2\pi)^3} e^{+i\vec{k}\cdot(\vec{x}-\vec{y})} \left(-\frac{1}{\vec{k}^2} \right) \quad (2.4.48)$$

or symbolically

$$D_c(\vec{x}-\vec{y}) = \frac{1}{\bigtriangledown_x^2}\delta^3(\vec{x}-\vec{y}). \tag{2.4.49}$$

The Lagrangian for this system is given by

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu \\ &= \frac{1}{2}\dot{\vec{A}}\cdot\dot{\vec{A}} - \frac{1}{4}F^{ij}F^{ij} + \dot{\vec{A}}\cdot\vec{\bigtriangledown}\phi + \frac{1}{2}\vec{\bigtriangledown}\phi\cdot\vec{\bigtriangledown}\phi + \phi J^0 - \vec{A}\cdot\vec{J}\end{aligned} \tag{2.4.50}$$

so that

$$\begin{aligned}L &= \int d^3x\mathcal{L} \\ &= \int d^3x[\frac{1}{2}\dot{\vec{A}}\cdot\dot{\vec{A}} - \frac{1}{4}F^{ij}F^{ij} - \frac{1}{2}\phi\bigtriangledown^2\phi + \phi J^0 - \vec{A}\cdot\vec{J}]\end{aligned} \tag{2.4.51}$$

where $\vec{\bigtriangledown}\cdot\vec{A}=0$ has been used. The interaction Lagrangian is given by the last three terms

$$L_I = \int d^3x[\phi J^0 - \frac{1}{2}\phi\bigtriangledown^2\phi - \vec{A}\cdot\vec{J}] \tag{2.4.52}$$

but $\bigtriangledown^2\phi = J^0$ so

$$L_I = \int d^3x[\frac{1}{2}\phi J^0 - \vec{A}\cdot\vec{J}]. \tag{2.4.53}$$

The interaction Hamiltonian becomes (with $y^0 = x^0$ understood)

$$\begin{aligned}H_I &= -L_I \\ &= \int d^3x\vec{A}\cdot\vec{J} - \frac{1}{2}\int d^3xd^3yJ^0(x)D_c(\vec{x}-\vec{y})J^0(y).\end{aligned} \tag{2.4.54}$$

Hence, the S-operator is given by

$$\begin{aligned}S &= Te^{-i\int dtH_I^{iP}} \\ &= Te^{-i\int d^4x[\vec{A}\cdot\vec{J}-\frac{1}{2}\int d^4y\delta(x^0-y^0)J^0(x)D_C(\vec{x}-\vec{y})J^0(y)]}.\end{aligned} \tag{2.4.55}$$

When we calculate a matrix element of S and apply Wick's theorem to evaluate each term in the interaction picture, we see that it is Lorentz invariant since the n^μ dependent terms in the propagators will cancel the Coulomb interaction terms from H_I. In the Coulomb gauge with $A^\mu = (0, \vec{A})$ we have terms in the S-matrix elements from the $\vec{A}\cdot\vec{J}$ term in H_I of the form

$$\int d^4x_1 d^4x_2 J_\mu(x_1) < 0|TA^\mu(x_1)A^\nu(x_2)|0 > J_\nu(x_2)$$

$$= \int d^4x_1 d^4x_2 J_i(x_1) < 0|TA^i(x_1)A^j(x_2)|0 > J_j(x_2)$$

$$= \int d^4x_1 d^4x_2 (-J_\mu(x_1) g^{\mu\nu} D_F(x_1 - x_2) J_\nu(x_2))$$

$$- \int d^4x_1 d^4x_2 J_\mu(x_1) J_\nu(x_2) [\frac{\partial^2_{x_1} n^\mu n^\nu - (n\cdot\partial_{x_1})(\partial^\mu_{x_1} n^\nu + \partial^\nu_{x_1} n^\mu) + \partial^\mu_{x_1}\partial^\nu_{x_1}}{(n\cdot\partial_{x_1})^2 - \partial^2_{x_1}}]$$
$$\times D_F(x_1 - x_2). \tag{2.4.56}$$

If $\partial_\mu J^\mu = 0$ (i.e. there is gauge invariance of Maxwell's equations, $\partial_\mu F^{\mu\nu} = -J^\nu$, so they imply $-\partial_\mu J^\mu = \partial_\nu \partial_\mu F^{\mu\nu} = 0$), the second and third derivative terms vanish by integration by parts since they always act on one of the conserved currents. So the second term reduces to just the first factor in it

$$- \int d^4x_1 d^4x_2 n^\mu J_\mu(x_1) n^\nu J_\nu(x_2) \frac{\partial^2_{x_1}}{(n\cdot\partial_{x_1})^2 - \partial^2_{x_1}} D_F(x_1 - x_2). \tag{2.4.57}$$

With our choice of $n^\mu = (1, 0, 0, 0)$, this is

$$- \int d^4x_1 d^4x_2 J_0(x_1) (\frac{\partial^2_{x_1}}{(n\cdot\partial_{x_1})^2 - \partial^2_{x_1}} D_F(x_1 - x_2)) J_0(x_2). \tag{2.4.58}$$

Now

$$\begin{aligned}
\frac{\partial^2_{x_1}}{(n\cdot\partial_{x_1})^2 - \partial^2_{x_1}} D_F(x_1 - x_2) &= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x_1-x_2)} \frac{i}{k^2 + i\epsilon} \left(\frac{k^2}{(n\cdot k)^2 - k^2}\right) \\
&= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x_1-x_2)} \frac{i}{\vec{k}^2} \\
&= \delta(x_1^0 - x_2^0) \int \frac{d^3k}{(2\pi)^3} \frac{i}{\vec{k}^2} \\
&= -i\delta(x_1^0 - x_2^0) D_c(\vec{x}_1 - \vec{x}_2).
\end{aligned} \tag{2.4.59}$$

So we get the second term to be

$$+i \int d^4x d^4y J_0(x) D_C(\vec{x} - \vec{y})) J_0(y) \delta(x^0 - y^0) \tag{2.4.60}$$

and thus we find from the $\vec{A}\cdot\vec{J}$ term in H_I terms of the form

$$\frac{(-i)^2}{2}\int d^4xd^4yJ_i(x) < 0|TA^i(x)A^j(y)|0 > J_j(y)$$

$$= \frac{(-i)^2}{2}\int d^4xd^4yJ_\mu(x)(-g^{\mu\nu}D_F(x-y))J_\nu(y)$$

$$-\frac{i}{2}\int d^4xd^4y\delta(x^0-y^0)J_0(x)D_C(\vec{x}-\vec{y})J_0(y). \qquad (2.4.61)$$

The first term here is completely Lorentz invariant because D_F is invariant. The second term exactly cancels the contribution to the S-matrix element that the Coulomb term in H_I^{iP} gives

$$+\frac{i}{2}\int d^4xd^4y\delta(x^0-y^0)J_0(x)D_C(\vec{x}-\vec{y})J_0(y) \qquad (2.4.62)$$

(the combinatorics must be checked more carefully but works out, see Bjorken and Drell chapter 17.9 for a detailed proof). Needless to say this is extremely tedious. Rather than give up manifest Lorentz covariance we face the complications of enlarging the number of photons to include the unphysical degrees of freedom, the longitudinal A_3 and the scalar A_0 photons.

We now reconsider the quantization of photons in a gauge that is manifestly Lorentz invariant. Recall that the Maxwell Lagrangian describing the time evolution of the photon field is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (2.4.63)$$

with the field strength tensor given in terms of the four-vector photon field A_μ

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \qquad (2.4.64)$$

As we have seen, the description of the photon requires only two dynamical degrees of freedom. The introduction of the potential A_μ has four degrees of freedom, two of which are unphysical. This enlarged variable space is reflected in the gauge invariance of the Lagrangian; $\mathcal{L}$ remains unchanged under local gauge transformations

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x)$$

$$F'_{\mu\nu}(x) = \partial_\mu A'_\nu(x) - \partial_\nu A'_\mu(x)$$

$$= \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + \partial_\mu\partial_\nu\Lambda(x) - \partial_\nu\partial_\mu\Lambda(x) = F_{\mu\nu}(x) \tag{2.4.65}$$

and $\mathcal{L}'(x) = \mathcal{L}(x)$. Introducing a unitary operator $U(\Lambda)$ such that

$$U^{-1}(\Lambda)A_\mu(x)U(\Lambda) = A_\mu(x) + \partial_\mu\Lambda(x) \tag{2.4.66}$$

and

$$U^{-1}(\Lambda)F_{\mu\nu}(x)U(\Lambda) = F_{\mu\nu}(x) \tag{2.4.67}$$

so that

$$U^{-1}(\Lambda)\mathcal{L}(x)U(\Lambda) = \mathcal{L}(x). \tag{2.4.68}$$

Since, as seen in the introduction, the physical observables are given in terms of $F_{\mu\nu}$, we have a new phenomenon in that the equivalence class of the field A_μ determines the uniqueness of the physical situation. The members of each equivalence class being related by a gauge transformation. A_μ and $A'_\mu = A_\mu + \partial_\mu\Lambda$ are two members of the same equivalence class. Unfortunately our canonical quantization rules apply to the individual dynamical degrees of freedom not equivalence classes of fields. Thus, we must pick from each equivalence class a representative and apply the quantization rules to it. That is, we must break the gauge invariance of the Lagrangian, this picks a representative of the equivalence class, then we can apply the quantization rules. The physical observables being gauge invariant, we obtain the same values for them, regardless of the gauge we pick in which to quantize.

Of course different gauges have different uses and conveniences. We have studied the Coulomb gauge so far. The condition $\vec{\bigtriangledown} \cdot \vec{A} = 0$ allowed us to eliminate from the start the unphysical degrees of freedom $\phi = 0$, the scalar component and $\vec{\bigtriangledown} \cdot \vec{A} = 0$, the longitudinal component. The price we paid for dealing with transverse physical degrees of freedom alone was to give up manifest Lorentz invariance, what is transverse depends on your frame of reference. We now desire to study a class of gauges that keeps Lorentz invariance manifest. Unfortunately the price we pay for this is that the scalar and longitudinal components must now also become quantized when all components of A_μ are treated on equal footing. The separation of the physical states from all the states in a Lorentz invariant way will be more difficult but, of course, possible. Also at the same time we must use the gauge invariance of the theory to show that we get the same values as in the Coulomb gauge for the physical state matrix elements of the physical observables since now they will also include contributions from the unphysical degrees of freedom.

Now to the point, the gauge invariant Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu \tag{2.4.69}$$

and yields the canonical momentum conjugate to A_μ

$$\Pi^\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0}. \tag{2.4.70}$$

As mentioned above, if we try to apply our canonical quantization rules directly we have

$$\delta(x^0 - y^0)[\Pi^\mu(x), A_\nu(y)] = -i\delta^\mu{}_\nu \delta^4(x-y). \tag{2.4.71}$$

The gauge invariance of the Lagrangian implies $\Pi^0 = F^{00} = 0$ which contradicts the ETCR. This is because F^2 depends only on the equivlence class of A_μ not each A_μ. Thus, to treat each component of A_μ quantum mechanically equivalently, we must choose a gauge, that is, break the gauge invariance of $\mathcal{L}$, now in a way consistent with Lorentz invariance. We will choose the gauge fixing term proposed by Stueckelberg which generalized Fermi's original treatment of QED. The Stueckelberg gauge Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\lambda A^\lambda)^2 \tag{2.4.72}$$

where the real number α is called the gauge parameter ($\alpha = 1$ was Fermi's choice and is called the Feynman gauge). Now since

$$U^{-1}(\Lambda)\partial_\lambda A^\lambda U(\Lambda) = \partial_\lambda A^\lambda + \partial^2 \Lambda \tag{2.4.73}$$

we have

$$U^{-1}(\Lambda)\mathcal{L}(x)U(\Lambda) = \mathcal{L}(x) - \frac{1}{2\alpha}(2(\partial_\lambda A^\lambda)\partial^2\Lambda + (\partial^2\Lambda)^2) \neq \mathcal{L}. \tag{2.4.74}$$

Thus the Stueckelberg gauge fixing term indeed picks out a representative from the equivalence class (modulo $\partial^2\Lambda$ and a specific value for α which we will choose later) and we can now quantize A_μ.

We can motivate the Stueckelberg gauge Lagrangian by considering the following. As stated, in order to apply the canonical quantization procedure we must pick a unique representative from each gauge equivalence class in a Lorentz invariant way. Since the field strength tensor is the curl of the potential its specification leaves free the divergence of the potential, precisely the part of A^μ effected by a gauge transformation. Hence we can pick a representative from the gauge orbit of A^μ by specifying $\partial_\mu A^\mu(x) = f(x)$ with $f(x)$ a given scalar function of x. This is called the gauge condition. That it picks a unique representative A^μ means that using any other field in the orbit of A^μ , $A'^\mu = A^\mu + \partial^\mu\Lambda$, in the gauge condition specifies $\Lambda = 0$. Hence we have $\partial_\mu A'^\mu(x) = \partial_\mu A^\mu(x) + \partial^2\Lambda(x) = f(x)$ which implies $\partial^2\Lambda(x) = 0$ and for appropriately normalized functions, that is since ∂^2 is invertible, yields $\Lambda = 0$. We can then impose the constraint $\partial_\lambda A^\lambda = f$ in the Lagrangian by means of a Lagrange multiplier field D(x),

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D(x)[\partial_\lambda A^\lambda(x) - f(x)]. \tag{2.4.75}$$

The Euler-Lagrange equations imply $\partial_\lambda A^\lambda(x) = f(x)$, as desired, as well as $\partial_\mu F^{\mu\nu} = \partial^\nu D$. We may proceed to canonically quantize this theory now, however in order to obtain the Stueckelberg Lagrangian we choose the arbitrary function $f(x)$ to in fact be proportional to the Lagrange multiplier. Thus letting $f(x) \equiv -\alpha D(x)$, the gauge condition becomes $\partial_\lambda A^\lambda(x) = -\alpha D(x)$ so that the field equation is $\partial_\mu F^{\mu\nu} = -\frac{1}{\alpha}\partial^\nu\partial_\lambda A^\lambda$. This last equation is derivable from the Stueckelberg Lagrangian as Euler-Lagrange equations of motion.

Now the Lagrangian derivative is $\frac{\partial\mathcal{L}}{\partial\partial_\mu A_\nu} = -F^{\mu\nu} - \frac{1}{\alpha}g^{\mu\nu}\partial_\lambda A^\lambda$ so the Euler-Lagrange equations are

$$0 = \frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu\frac{\partial\mathcal{L}}{\partial_\mu A_\nu} = \partial_\mu F^{\mu\nu} + \frac{1}{\alpha}\partial^\nu\partial_\mu A^\mu. \tag{2.4.76}$$

The field equation

$$\partial_\mu F^{\mu\nu} + \frac{1}{\alpha}\partial^\nu\partial_\mu A^\mu = 0 \tag{2.4.77}$$

or

$$\partial_\mu F^{\mu\nu} + \frac{1}{\alpha}\partial^\nu\partial_\mu A^\mu = -J^\nu \tag{2.4.78}$$

in the interacting case, is just

$$\partial^2 A^\nu + \frac{(1-\alpha)}{\alpha}\partial^\nu\partial_\mu A^\mu = -J^\nu. \tag{2.4.79}$$

The utility of Fermi's original choice is now clear, for $\alpha = 1$ this becomes the simple wave equation

$$\partial^2 A^\nu = -J^\nu \tag{2.4.80}$$

or

$$\partial^2 A^\nu = 0 \tag{2.4.81}$$

with no interaction (as is the case here). So we have four fields obeying the massless Klein-Gordon equation, i.e. the wave equation. The Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - \frac{1}{2}\frac{(1-\alpha)}{\alpha}\partial_\mu A_\nu \partial^\nu A^\mu \tag{2.4.82}$$

becomes in the Feynman gauge simply

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu \tag{2.4.83}$$

which was Fermi's starting point. Notice $\partial_\mu \partial_\nu F^{\mu\nu} = 0$ yields

$$\partial^2 \partial_\lambda A^\lambda = 0, \tag{2.4.84}$$

$\partial_\lambda A^\lambda$, called the gauge fixing term, obeys the free wave equation. Note when there is an interaction with current J^μ, described by Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - \frac{1}{2}\frac{(1-\alpha)}{\alpha}\partial_\mu A_\nu \partial^\nu A^\mu + J^\mu A_\mu \tag{2.4.85}$$

the field equation becomes

$$\partial_\mu F^{\mu\nu} + \frac{1}{\alpha}\partial^\nu \partial_\mu A^\mu = -J^\nu. \tag{2.4.86}$$

Taking ∂_ν yields

$$\frac{1}{\alpha}\partial^2 \partial_\mu A^\mu = -\partial_\nu J^\nu. \tag{2.4.87}$$

If J^μ is conserved, as the gauge invariance principle will require,

$$\frac{1}{\alpha}\partial^2 \partial_\mu A^\mu = 0, \tag{2.4.88}$$

$\partial_\mu A^\mu$ still obeys the free wave equation. The momenta canonically conjugate to A_μ are

$$\Pi^\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0} - \frac{1}{\alpha} g^{\mu 0}\partial_\nu A^\nu, \tag{2.4.89}$$

that is

$$\Pi^i = F^{i0} = \partial^i A^0 - \dot{A}^i \tag{2.4.90}$$

and

$$\Pi^0 = -\frac{1}{\alpha}\partial_\nu A^\nu = -\frac{1}{\alpha}(\dot{A}^0 - \partial^i A^i) \tag{2.4.91}$$

and as expected $\Pi^0 \neq 0$ any longer but is given by our gauge fixing term. Thus the ETCR become

$$\begin{aligned} \delta(x^0 - y^0)[\Pi^\mu(x), A_\nu(y)] &= -i\delta^\mu{}_\nu \delta^4(x-y) \\ \delta(x^0 - y^0)[\Pi^\mu(x), \Pi^\nu(y)] &= 0 \\ \delta(x^0 - y^0)[A^\mu(x), A^\nu(y)] &= 0. \end{aligned} \tag{2.4.92}$$

Now taking the spatial derivative of the last commutator, we obtain

$$\delta(x^0 - y^0)[\partial_x^i A^\mu(x), A^\nu(y)] = 0. \tag{2.4.93}$$

Hence the $\Pi - A$ commutator simplifies to become

$$\begin{aligned} \delta(x^0 - y^0)[\dot{A}^i(x), A_\nu(y)] &= +i\delta^i{}_\nu \delta^4(x-y) \\ \delta(x^0 - y^0)[\dot{A}^0(x), A_\nu(y)] &= +i\alpha\delta^0{}_\nu \delta^4(x-y) \end{aligned} \tag{2.4.94}$$

or raising ν, we obtain

$$\begin{aligned} \delta(x^0 - y^0)[\dot{A}^i(x), A^\nu(y)] &= +ig^{i\nu} \delta^4(x-y) \\ \delta(x^0 - y^0)[\dot{A}^0(x), A^\nu(y)] &= +i\alpha g^{0\nu} \delta^4(x-y). \end{aligned} \tag{2.4.95}$$

Thus, we see that we will have four types of "particle" states since all A^μ are involved in the ETCR. Furthermore, we note that A^i has normal quantization rules

$$\delta(x^0 - y^0)[\dot{A}^i(x), A^j(y)] = -i\delta^{ij} \delta^4(x-y) \tag{2.4.96}$$

just like three different scalar boson ETCR, but A^0 obeys a commutation relation with the wrong sign

$$\delta(x^0 - y^0)[\dot{A}^0(x), A^0(y)] = +i\alpha \delta^4(x-y). \tag{2.4.97}$$

If $\alpha = 1$, the Feynman gauge of Fermi, the ETCR are

$$\delta(x^0 - y^0)[\dot{A}^\mu(x), A^\nu(y)] = +ig^{\mu\nu} \delta^4(x-y). \tag{2.4.98}$$

We will see that this wrong sign will lead to states with norm of -1! Not only will we have to eliminate the unphysical states from the physical quantities but the scalar "photon" states have "negative" probability and destroy the usual quantum mechanical interpretation of the inner product in the state space. Further, from the $\Pi^i - \Pi^j$ commutator we have

$$\delta(x^0 - y^0)[\dot{A}^i(x), \dot{A}^j(y)] = 0. \tag{2.4.99}$$

From the $\Pi^0 - \Pi^i$ commutator we find

$$\delta(x^0 - y^0)[\dot{A}^0(x), \dot{A}^i(y)] = +i(1-\alpha)\partial_x^i \delta^4(x-y), \tag{2.4.100}$$

and from the $\Pi^0 - \Pi^0$ commutator we have

$$\delta(x^0 - y^0)[\dot{A}^0(x), \dot{A}^0(y)] = 0. \tag{2.4.101}$$

Thus, we can summarize the ETCR as

$$\begin{aligned}
\delta(x^0 - y^0)[A^\mu(x), A^\nu(y)] &= 0 \\
\delta(x^0 - y^0)[\dot{A}^\mu(x), A^\nu(y)] &= +ig^{\mu\nu}[1 + (\alpha - 1)g^{\mu 0}]\delta^4(x-y) \qquad (\not{\Sigma}\mu) \\
\delta(x^0 - y^0)[\dot{A}^i(x), \dot{A}^j(y)] &= 0 \\
\delta(x^0 - y^0)[\dot{A}^0(x), \dot{A}^0(y)] &= 0 \\
\delta(x^0 - y^0)[\dot{A}^0(x), \dot{A}^i(y)] &= +i(1-\alpha)\partial_x^i \delta^4(x-y).
\end{aligned} \tag{2.4.102}$$

Before Fourier transforming to momentum space to begin our interpretation of the above theory, let's construct the various conserved space-time symmetry currents and charges. The photon gauge field A^μ is a vector field, that is, under Lorentz transformations A_μ's intrinsic variation is that of x^μ. For space-time Poincaré transformations, $x'^\mu = \Lambda^{\mu\nu}x_\nu + a_\nu$, the Poincaré transformation of the fields are implemented by $U(a, \Lambda)$ as usual

$$U^{-1}(a,\Lambda)A^\mu(x')U(a,\Lambda) = \Lambda^{\mu\nu}A_\nu(x), \tag{2.4.103}$$

or multiplying by $U(a,\Lambda)$ and $U^{-1}(a,\Lambda)$ from the left and the right, respectively, we obtain

$$U(a,\Lambda)A^\mu(x)U^{-1}(a,\Lambda) = \Lambda^{-1\mu\nu}A_\nu(\Lambda x + a) = \Lambda^{\nu\mu}A_\nu(\Lambda x + a) \tag{2.4.104}$$

where recall $\Lambda^{-1\mu\nu} = \Lambda^{T\mu\nu} = \Lambda^{\nu\mu}$. As usual we have

$$U(a,\Lambda) = e^{ia_\mu \mathcal{P}^\mu} e^{-\frac{i}{2}\omega_{\mu\nu}(\Lambda)\mathcal{M}^{\mu\nu}} \tag{2.4.105}$$

which for infinitesimal transformations $\Lambda^{\mu\nu} = g^{\mu\nu} + \omega^{\mu\nu}, \omega^{\mu\nu} = -\omega^{\nu\mu}$ and $a^\mu = \epsilon^\mu$, yields the translation and rotation commutation relations

$$[\mathcal{P}^\mu, A^\nu(x)] = -i\partial^\mu A^\nu(x)$$

$$[\mathcal{M}^{\mu\nu}, A^\lambda(x)] = -i[(x^\mu\partial^\nu - x^\nu\partial^\mu)A^\lambda(x) + (g^{\mu\lambda}g^{\nu\rho} - g^{\nu\lambda}g^{\mu\rho})A_\rho(x)] \tag{2.4.106}$$

where $(D^{\mu\nu})^{\lambda\rho} \equiv -(g^{\mu\lambda}g^{\nu\rho} - g^{\nu\lambda}g^{\mu\rho})$, according to the notation of section 1.2, obeys the Lorentz algebra (it is the defining representation or vector representation). Since the Lagrangian is Poincaré invariant,

$$U^{-1}(a,\Lambda)\mathcal{L}(x')U(a,\Lambda) = \mathcal{L}(x), \tag{2.4.107}$$

Noether's theorem implies the conservation of the energy-momentum tensor

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial\partial_\mu A_\lambda}\partial^\nu A_\lambda - g^{\mu\nu}\mathcal{L} \\ &= -F^{\mu\lambda}\partial^\nu A_\lambda - \frac{1}{\alpha}(\partial_\lambda A^\lambda)\partial^\nu A^\mu - g^{\mu\nu}\mathcal{L}. \end{aligned} \tag{2.4.108}$$

Using the field equations (2.4.77), we can check explicitly that this is conserved,

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= -\partial_\mu F^{\mu\lambda}\partial^\nu A_\lambda - F^{\mu\lambda}\partial_\mu\partial^\nu A_\lambda - \frac{1}{\alpha}(\partial_\mu\partial_\lambda A^\lambda)\partial^\nu A^\mu \\ &\quad -\frac{1}{\alpha}(\partial_\lambda A^\lambda)\partial^\nu\partial_\mu A^\mu - \partial_\nu\mathcal{L} \\ &= \frac{1}{\alpha}(\partial^\lambda\partial_\mu A^\mu)\partial^\nu A_\lambda - \frac{1}{2}F^{\mu\lambda}\partial^\nu F_{\mu\lambda} - \frac{1}{\alpha}(\partial^\lambda\partial_\mu A^\mu)\partial^\nu A_\lambda \\ &\quad -\frac{1}{\alpha}(\partial_\lambda A^\lambda)\partial^\nu(\partial_\mu A^\mu) - \partial^\nu\mathcal{L} \\ &= \partial^\nu(-\frac{1}{4}F^{\mu\lambda}F_{\mu\lambda} - \frac{1}{2\alpha}(\partial_\lambda A^\lambda)^2) - \partial^\nu\mathcal{L} \\ &\quad +\frac{1}{4}[\partial^\nu F^{\mu\lambda}, F_{\mu\lambda}] + \frac{1}{2\alpha}[\partial^\nu\partial_\lambda A^\lambda, \partial_\mu A^\mu] \\ &= 0. \end{aligned} \tag{2.4.109}$$

However, $T^{\mu\nu}$ is not symmetric

$$T^{\mu\nu} - T^{\nu\mu} = -F^{\mu\lambda}\partial^\nu A_\lambda + F^{\nu\lambda}\partial^\mu A_\lambda + \frac{1}{\alpha}(\partial_\lambda A^\lambda)F^{\mu\nu} \neq 0 \tag{2.4.110}$$

and $T^{\mu\nu}$ is not traceless

$$T^{\mu}{}_{\mu} = -2\mathcal{L} \neq 0. \tag{2.4.111}$$

According Belinfante's prescription we must calculate $H^{\rho\mu\nu}$ and then $G^{\rho\mu\nu}$ to find the symmetric energy-momentum tensor $\Theta^{\mu\nu} \equiv T^{\mu\nu} - \partial_\rho G^{\rho\mu\nu}$. Recall that, with $\Pi^{\rho\lambda} \equiv \frac{\partial \mathcal{L}}{\partial \partial_\rho A_\lambda}$,

$$\begin{aligned} H^{\rho\mu\nu} &= \Pi^{\rho\lambda}(D^{\mu\nu})_{\lambda\kappa}A^\kappa \\ &= F^{\rho\mu}A^\nu - F^{\rho\nu}A^\mu + \frac{1}{\alpha}(\partial_\lambda A^\lambda)(g^{\rho\mu}A^\nu - g^{\rho\nu}A^\mu). \end{aligned} \tag{2.4.112}$$

Next we need

$$\begin{aligned} G^{\rho\mu\nu} &= \frac{1}{2}(H^{\rho\mu\nu} + H^{\mu\nu\rho} + H^{\nu\mu\rho}) \\ &= F^{\rho\mu}A^\nu + \frac{1}{\alpha}(\partial_\lambda A^\lambda)(g^{\mu\nu}A^\rho - g^{\rho\nu}A^\mu). \end{aligned} \tag{2.4.113}$$

Note $G^{\rho\mu\nu} = -G^{\mu\rho\nu}$ as required. Hence, using the field equation, we find after a little algebra

$$\partial_\rho G^{\rho\mu\nu} = -\frac{1}{\alpha}(\partial^\mu \partial_\lambda A^\lambda)A^\nu - F^{\mu\lambda}\partial_\lambda A^\nu + \frac{1}{\alpha}g^{\mu\nu}\partial_\rho(\partial_\lambda A^\lambda A^\rho) - \frac{1}{\alpha}\partial^\nu(\partial_\lambda A^\lambda A^\mu). \tag{2.4.114}$$

Thus we obtain a conserved, symmetric Belinfante energy-momentum tensor

$$\Theta^{\mu\nu} = F^{\mu\rho}F_\rho{}^\nu - g^{\mu\nu}\mathcal{L} - \frac{1}{\alpha}g^{\mu\nu}\partial_\rho(\partial_\lambda A^\lambda A^\rho)$$

$$+\frac{1}{\alpha}(\partial^\mu \partial_\lambda A^\lambda)A^\nu + \frac{1}{\alpha}(\partial^\nu \partial_\lambda A^\lambda)A^\mu \tag{2.4.115}$$

and

$$\Theta^{\mu\nu} = \Theta^{\nu\mu} \tag{2.4.116}$$

with

$$\partial_\mu \Theta^{\mu\nu} = 0. \tag{2.4.117}$$

Furthermore we find that $\Theta^{\mu\nu}$ is not entirely traceless

$$\Theta^\mu \mu = -\frac{2}{\alpha}\partial_\rho(\partial_\lambda A^\lambda A^\rho). \tag{2.4.118}$$

Of course we could make the energy-momentum tensor traceless but not symmetric, one cannot make the energy-momentum tensor both traceless and symmetric due to the gauge fixing term. Notice also that $\Theta^{\mu\nu}$ and $T^{\mu\nu}$ are not gauge invariant, they depend on the representative not the gauge equivalence class. However, we will see that physical photon matrix elements of $\Theta^{\mu\nu}$ are gauge invariant. Hence, Noether's theorem tells us how to construct the energy-momentum operator in terms of the dynamical variables

$$
\begin{aligned}
\mathcal{P}^\mu \equiv \int d^x \Theta^{0\mu} &= \int d^3x T^{0\mu} \\
&= \int d^3x[-F^{0\lambda}\partial^\mu A_\lambda - \frac{1}{\alpha}(\partial_\lambda A^\lambda)\partial^\mu A^0 - g^{0\mu}\mathcal{L}] \\
&= \int d^3x[(F^{\lambda 0} - \frac{1}{\alpha}g^{\lambda 0}\partial_\rho A^\lambda)\partial^\mu A_\lambda - g^{0\mu}\mathcal{L}] \\
&= \int d^3x[\Pi^\lambda \partial^\mu A_\lambda - g^{\mu 0}\frac{1}{2}\Pi^{\lambda\rho}\partial_\lambda A_\rho] \qquad (2.4.119)
\end{aligned}
$$

where recall that $\Pi^\lambda = \Pi^{0\lambda}$ with $\Pi^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu A_\nu}$ and the Lagrangian can be written as $\mathcal{L} = \frac{1}{2}\Pi^{\mu\nu}\partial_\mu A_\nu$ while the Euler-Lagrange equations are $\partial_\mu \Pi^{\mu\nu} = 0$. Thus, the Hamiltonian is

$$
\begin{aligned}
H = \mathcal{P}^0 &= \int d^3x[\Pi^\lambda \partial^0 A_\lambda - \frac{1}{2}\Pi^{\lambda\rho}\partial_\lambda A_\rho] \\
&= \int d^3x[\Pi^\lambda \partial^0 A_\lambda - \frac{1}{2}\Pi^{0\rho}\partial^0 A_\rho - \frac{1}{2}\Pi^{i\rho}\partial_i A_\rho] \\
&= \int d^3x[\frac{1}{2}\Pi^\lambda \partial^0 A_\lambda + \frac{1}{2}\partial_i \Pi^{i\rho} A_\rho - \frac{1}{2}\partial_i(\Pi^{i\rho}A_\rho)] \\
&= \int d^3x[\frac{1}{2}\Pi^\lambda \overset{\leftrightarrow}{\partial}{}^0 A_\lambda + \frac{1}{2}\partial_\mu \Pi^{\mu\rho} A_\rho] \\
&= \int d^3x[\frac{1}{2}\Pi^\lambda \overset{\leftrightarrow}{\partial}{}^0 A_\lambda]. \qquad (2.4.120)
\end{aligned}
$$

Hence, we have the Heisenberg equations of motion

$$
\begin{aligned}
[\mathcal{P}^0, A^\mu(x)] &= \int_{x^0=y^0} d^3y \frac{1}{2}[\Pi^\lambda(y), A^\mu(x)]\overset{\leftrightarrow}{\partial}{}^0_y A_\lambda(y) \\
&= \frac{1}{2}\int d^3y[-i\delta^3(\vec{y}-\vec{x})]\overset{\leftrightarrow}{\partial}{}^0_y A^\mu(y) \\
&= -i\partial^0 A^\mu(x). \qquad (2.4.121)
\end{aligned}
$$

Similarly, for the momentum operator

$$\vec{\mathcal{P}} = -\int d^3x \Pi^\lambda \vec{\bigtriangledown} A_\lambda, \tag{2.4.122}$$

we find

$$\begin{aligned}[\vec{\mathcal{P}}, A^\mu(x)] &= \int_{x^0=y^0} i\delta^3(\vec{y}-\vec{x})\vec{\bigtriangledown}_y A^\mu(y) \\ &= -i\vec{\bigtriangledown} A^\mu(x).\end{aligned} \tag{2.4.123}$$

Hence, we checked that indeed $\mathcal{P}^\mu$ so constructed obeys

$$[\mathcal{P}^\mu, A^\nu(x)] = -i\partial^\mu A^\nu(x). \tag{2.4.124}$$

Additionally, according to Belinfante's procedure the angular momentum tensor is given by $\mathcal{M}^{\mu\nu\rho} = x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu}$ where $\partial_\mu \mathcal{M}^{\mu\nu\rho} = 0$ and the angular momentum operator is

$$\begin{aligned}\mathcal{M}^{\mu\nu} &= \int d^3x \mathcal{M}^{0\mu\nu} \\ &= \int d^3x (x^\mu \Theta^{0\nu} - x^\nu \Theta^{0\mu}).\end{aligned} \tag{2.4.125}$$

Its explicit form in terms of A^μ is left as an exercise. Also we can verify that this operator $\mathcal{M}^{\mu\nu}$ obeys

$$[\mathcal{M}^{\mu\nu}, A^\lambda(x)] = -i[(x^\mu\partial^\nu - x^\nu\partial^\mu)A^\lambda(x) + (g^{\mu\lambda}g^{\nu\rho} - g^{\nu\lambda}g^{\mu\rho})A_\rho(x)] \tag{2.4.126}$$

as required.

We are now in a position to begin interpreting this quantum field theory in terms of creation and annihilation operators for physical as well as unphysical degrees of freedom. Towards this end we would like to expand $A^\mu(x)$ in terms of plane wave solutions to the Euler-Lagrange equations

$$\partial^2 A^\nu(x) + \frac{(1-\alpha)}{\alpha}\partial^\nu \partial_\mu A^\mu(x) = 0 \tag{2.4.127}$$

and its divergence,

$$\frac{1}{\alpha}\partial^2 \partial_\lambda A^\lambda(x) = 0. \tag{2.4.128}$$

Expanding $A^\mu(x)$

$$A^\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{A}^\mu(k) \tag{2.4.129}$$

we have that

$$0 = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} (k^2 g^{\mu\nu} + \frac{(1-\alpha)}{\alpha} k^\mu k^\nu) \tilde{A}_\nu(k). \tag{2.4.130}$$

Notice if $\alpha = 1$, we immediately have that $\tilde{A}^\mu = (2\pi)\delta(k^2) a^\mu(\vec{k}, k^0)$. However, for $\alpha \neq 1$, $\tilde{A}^\mu(k)$ cannot be simply proportional to $\delta(k^2)$ due to the $k^\mu k^\nu$ term in the field equation. Viewing this as a four-by-four matrix equation, if there are to be nontrivial solutions for $\tilde{A}^\mu$ the determinant of $\det(k^2 g^{\mu\nu} + \frac{(1-\alpha)}{\alpha} k^\mu k^\nu)$ must vanish. But,

$$\det(k^2 g^{\mu\nu} + \frac{(1-\alpha)}{\alpha} k^\mu k^\nu) = -\frac{1}{\alpha}(k^2)^4 \tag{2.4.131}$$

so we still have $\tilde{A}^\mu(k)$ with support on the light cone $k^2 = 0$. This is possible by $\tilde{A}^\mu(k)$ having a $\delta(k^2)$ as well as a $\delta'(k^2)$ term. Note that

$$\int d^4k \delta'(k^2) k^2 f(k^\mu) = -\int d^4k \delta(k^2) f(k^\mu) \tag{2.4.132}$$

where the $\partial/\partial k^2$ must operate on the k^2 to get a nontrivial answer. Thus, we try

$$\tilde{A}_\nu(k) = (2\pi)\delta(k^2) a_\nu(\vec{k}, k^0) + (2\pi)\,\delta'(k^2) k_\nu b(\vec{k}, k^0). \tag{2.4.133}$$

The field equation becomes

$$\begin{aligned} 0 = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} [&(2\pi)\delta'(k^2) k^2 k^\mu b(\vec{k}, k^0) \\ &+ (2\pi)\delta(k^2) \frac{(1-\alpha)}{\alpha} k^\mu k^\nu a_\nu(\vec{k}, k^0) + (2\pi)\delta'(k^2) \frac{(1-\alpha)}{\alpha} k^2 k^\mu b(\vec{k}, k^0)] \\ = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} (2\pi)\delta(k^2) [&-k^\mu b(\vec{k}, k^0) + \frac{(1-\alpha)}{\alpha} k^\mu k^\nu a_\nu(\vec{k}, k^0) \\ &- \frac{(1-\alpha)}{\alpha} k^\mu b(\vec{k}, k^0)]. \end{aligned} \tag{2.4.134}$$

implying

$$\frac{1}{\alpha} b(\vec{k}, k^0) = \frac{(1-\alpha)}{\alpha} k^\nu a_\nu(\vec{k}, k^0) \tag{2.4.135}$$

where $k^0 = +\omega_k = +|\vec{k}|$. So we find

$$\tilde{A}_\nu(k) = (2\pi)\delta(k^2) a_\nu(\vec{k}, k^0) + (2\pi)\,\delta'(k^2)(1-\alpha) k_\nu k_\rho a^\rho(\vec{k}, k^0) \tag{2.4.136}$$

and the Fourier transform becomes

$$A^\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx}[(2\pi)\delta(k^2)a^\mu(\vec{k},k^0) + (1-\alpha)(2\pi)\,\delta'(k^2)k^\mu k_\rho a^\rho(\vec{k},k^0)]. \tag{2.4.137}$$

Note that the Minkowski longitudinal component is simply

$$\begin{aligned} i\partial_\mu A^\mu(x) &= \int \frac{d^4k}{(2\pi)^4} e^{-ikx}[(2\pi)\delta(k^2)k_\mu a^\mu(\vec{k},k^0) \\ &+(1-\alpha)(2\pi)\delta'(k^2)k^2 k_\mu a^\mu(\vec{k},k^0)] \\ &= \alpha \int \frac{d^4k}{(2\pi)^4} e^{-ikx}(2\pi)\delta(k^2)k_\mu a^\mu(\vec{k},k^0) \\ &= \alpha \int \frac{d^3k}{(2\pi)^3 2\omega_k}[k_\mu a^\mu(\vec{k})e^{-ikx} - k_\mu a^{\mu\dagger}(\vec{k})e^{+ikx}] \end{aligned} \tag{2.4.138}$$

where $a^\mu(\vec{k}) \equiv a^\mu(\vec{k}, +\omega_k)$ and $a_\mu(-\vec{k}, -\omega_k) = a^\dagger_\mu(\vec{k})$ from the hermiticity of $A_\mu(x)$. Note that this is consistent with $\partial^2\partial_\mu A^\mu = 0$ and $k^0 = \omega_k = |\vec{k}|$. Continuing we have

$$\begin{aligned} A^\mu(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}[a^\mu(\vec{k})e^{-ikx} + a^{\mu\dagger}(\vec{k})e^{+ikx}] \\ &+(1-\alpha)\int \frac{d^4k}{(2\pi)^4}(2\pi)\delta'(k^2)k^\mu k^\nu a_\nu(\vec{k},k^0)e^{-ikx}. \end{aligned} \tag{2.4.139}$$

The second integral is rather messy but note we can always exploit the residual gauge invariance of $\mathcal{L}$ to simplify it. Recall

$$U^{-1}(\Lambda)\mathcal{L}(x)U(\Lambda) = \mathcal{L}(x) - \frac{1}{2\alpha}(2(\partial_\lambda A^\lambda)\partial^2\Lambda + (\partial^2\Lambda)^2) \tag{2.4.140}$$

so if $\partial^2\Lambda = 0$ then $U^{-1}(\Lambda)\mathcal{L}(x)U(\Lambda) = \mathcal{L}(x)$. Now for $\partial^2\Lambda = 0$ we see that we can add to A^μ a term of the form $\partial^\mu\Lambda$. Thus we find

$$\begin{aligned} \Lambda(x) &= \int \frac{d^4k}{(2\pi)^4} e^{-ikx}(2\pi)\delta(k^2)\lambda(\vec{k},k^0) \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}[e^{-ikx}\lambda(\vec{k}) + e^{+ikx}\lambda^\dagger(\vec{k})] \end{aligned} \tag{2.4.141}$$

so that

$$\partial^\mu\Lambda(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx}(2\pi)\delta(k^2)(-ik^\mu\lambda(\vec{k},k^0)) \tag{2.4.142}$$

and we see that the Euler-Lagrange equation (i.e. the Lagrangian is invariant) are satisfied by $\partial^\mu\Lambda(x)$

$$[\partial^2 g^{\mu\nu} + \left(\frac{1-\alpha}{\alpha}\right)\partial^\mu\partial^\nu]\partial_\nu\Lambda = \frac{1}{\alpha}\partial^\mu\partial^2\Lambda = 0. \tag{2.4.143}$$

If we let $-i\lambda(\vec{k},k^0) = +\frac{(1-\alpha)}{\alpha}\partial_k^\lambda a_\lambda(\vec{k},k^0)$, that is, we choose this gauge condition, we find

$$\begin{aligned}
A'^\mu(x) &= A^\mu(x) + \partial^\mu\Lambda(x) \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}[a^\mu(\vec{k})e^{-ikx} + a^{\mu\dagger}(\vec{k})e^{+ikx}] \\
&\quad + \frac{(1-\alpha)}{2}\int \frac{d^4k}{(2\pi)^4}[k^\mu 2k^\nu(2\pi)\frac{\partial}{\partial k^2}\delta(k^2)a_\nu(\vec{k},k^0)e^{-ikx} \\
&\quad + k^\mu(2\pi)\delta(k^2)\partial_k^\nu a_\nu(\vec{k},k^0)e^{-ikx}] \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}[a^\mu(\vec{k})e^{-ikx} + a^{\mu\dagger}(\vec{k})e^{+ikx}] \\
&\quad + \frac{(1-\alpha)}{2}\int \frac{d^4k}{(2\pi)^4}[k^\mu e^{-ikx}][\partial_k^\nu(2\pi\delta(k^2)a_\nu(\vec{k},k^0))]
\end{aligned} \tag{2.4.144}$$

which by using integration by parts in the last term becomes

$$\begin{aligned}
A'^\mu(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}[a^\mu(\vec{k})e^{-ikx} + a^{\mu\dagger}(\vec{k})e^{-ikx}] \\
&\quad - i\partial_x^\mu\frac{(1-\alpha)}{2}\int \frac{d^4k}{(2\pi)^4}(\partial_k^\nu e^{-ikx})2\pi\delta(k^2)a_\nu(\vec{k},k^0) \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}[a^\mu(\vec{k})e^{-ikx} + a^{\mu\dagger}e^{+kx}] \\
&\quad - \partial_x^\mu[x^\nu\frac{(1-\alpha)}{2}\int \frac{d^4k}{(2\pi)^4}e^{-ikx}(2\pi)\delta(k^2)a_\nu(\vec{k},k^0)].
\end{aligned} \tag{2.4.145}$$

Thus in coordinate space-time we have

$$\begin{aligned}
A'^\mu(x) &= \left[g^{\mu\nu} + \frac{(\alpha-1)}{2}\partial^\mu x^\nu\right]a_\nu(x) \\
&= a^\mu(x) + \left[\frac{\alpha-1}{2}\right]\partial^\mu(x^\nu a_\nu(x))
\end{aligned} \tag{2.4.146}$$

where

$$
\begin{aligned}
a^\mu(x) &= \int \frac{d^4k}{(2\pi)^4} e^{-ikx}(2\pi)\delta(k^2)a^\mu(\vec{k},k^0) \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}[e^{-ikx}a^\mu(\vec{k}) + e^{+ikx}a^{\mu\dagger}(\vec{k})].
\end{aligned}
\tag{2.4.147}
$$

Now let's reverse the line of reasoning so that we may check our calculation. We start with $A^\mu(x) = A'^\mu(x) - \partial^\mu\Lambda(x)$ and choose $A'^\mu(x)$ so that it is a solution to the Maxwell equations in the Stueckelberg gauge. Thus we let

$$
\begin{aligned}
A^\mu(x) &= a^\mu(x) + \frac{(\alpha-1)}{2}\partial^\mu(x^\lambda a_\lambda(x)) - \partial^\mu\Lambda(x) \\
&= \frac{(1+\alpha)}{2}a^\mu(x) + \frac{(\alpha-1)}{2}x^\lambda\partial^\mu a_\lambda(x) - \partial^\mu\Lambda(x) \\
&= \left(\frac{1+\alpha}{2}\right)\left[a^\mu(x) + \frac{(\alpha-1)}{(\alpha+1)}x^\lambda\partial^\mu a_\lambda(x)\right] - \partial^\mu\Lambda(x)
\end{aligned}
\tag{2.4.148}
$$

where $\partial^2 a^\mu(x) = 0 = \partial^2\Lambda(x)$. Noting that $(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)\partial_\nu = 0$, we find indeed that the field equations are obeyed

$$
\begin{aligned}
&(\partial^2 g^{\mu\nu} + \left(\frac{1-\alpha}{\alpha}\right)\partial^\mu\partial^\nu)A_\nu(x) \\
&= \left(\frac{\alpha-1}{2}\right)\partial^\mu\partial^2(x^\lambda a_\lambda) + \left(\frac{1-\alpha}{\alpha}\right)\partial^\mu\partial^\lambda a_\lambda + \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha-1}{2}\right)\partial^\mu\partial^2(x^\lambda a_\lambda) \\
&= \left(\frac{1}{\alpha}\right)\left(\frac{\alpha-1}{2}\right)\partial^\mu\partial^2(x^\lambda a_\lambda) + \left(\frac{1-\alpha}{\alpha}\right)\partial^\mu\partial^\lambda a_\lambda \\
&= \left(\frac{\alpha-1}{\alpha}\right)\partial^\mu\partial^\lambda a_\lambda + \left(\frac{1-\alpha}{\alpha}\right)\partial^\mu\partial^\lambda a_\lambda \\
&= 0.
\end{aligned}
\tag{2.4.149}
$$

So we are adding two Minkowski longitudinal pieces to A^μ; $\partial^\mu(x^\rho a_\rho)$ and $\partial^\mu\Lambda$, with Λ being completely arbitrary subject only to $\partial^2\Lambda = 0$ and thus reflecting the residual gauge invariance of $\mathcal{L}$.

Now we may apply our Fourier analysis to these terms. Since $\partial^2 a^\mu(x) = 0$, we have immediately

$$
\begin{aligned}
a^\mu(x) &= \int \frac{d^4k}{(2\pi)^4} e^{-ikx}(2\pi)\delta(k^2)a^\mu(\vec{k},k^0) \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}[e^{-ikx}a^\mu(\vec{k}) + e^{+ikx}a^{\mu\dagger}(\vec{k})].
\end{aligned}
\tag{2.4.150}
$$

Now we need the Fourier transform of $\partial^\mu(x^\lambda a_\lambda)$

$$\begin{aligned}\partial^\mu(x^\lambda a_\lambda(x)) &= \partial_x^\mu x^\lambda \int \frac{d^4k}{(2\pi)^4} e^{-ikx}(2\pi)\delta(k^2) a_\lambda(\vec{k}, k^0) \\ &= \partial_x^\mu \int \frac{d^4k}{(2\pi)^4} \left(\frac{i\partial}{\partial k_\lambda} e^{-ikx}\right)(2\pi)\delta(k^2) a_\lambda(\vec{k}, k^0).\end{aligned} \tag{2.4.151}$$

Integrating by parts we obtain

$$\partial^\mu(x^\lambda a_\lambda(x)) = \partial_x^\mu \int \frac{d^4k}{(2\pi)^4} e^{-ikx}(-i)\frac{\partial}{\partial k_\lambda}[(2\pi)\delta(k^2) a_\lambda(\vec{k}, k^0)]. \tag{2.4.152}$$

Since $\partial_k^\lambda \delta(k^2) = 2k^\lambda \frac{\partial}{\partial k^2}\delta(k^2) = 2k^\lambda \delta'(k^2)$, it follows that

$$\begin{aligned}\partial^\mu(x^\lambda a_\lambda(x)) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx}[&-2k^\mu k^\lambda (2\pi)\delta'(k^2) a_\lambda(\vec{k}, k^0) \\ &-k^\mu (2\pi)\delta(k^2)\partial_k^\lambda a_\lambda(\vec{k}, k^0)].\end{aligned} \tag{2.4.153}$$

Thus the Fourier transformation becomes

$$\begin{aligned}A^\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}[e^{-ikx} a^\mu(\vec{k}) + e^{+ikx} a^{\mu\dagger}(\vec{k})] \\ + \int \frac{d^4k}{(2\pi)^4} e^{-ikx}[(1-\alpha)k^\mu k^\lambda (2\pi)\delta'(k^2) a_\lambda(\vec{k}, k^0) + \frac{(1-\alpha)}{2}(2\pi)\delta(k^2) k^\mu \partial_k^\lambda a_\lambda(\vec{k}, k^0)] \\ -\partial^\mu \Lambda(x).\end{aligned} \tag{2.4.154}$$

Now Λ may also be expanded

$$\Lambda(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx}(2\pi)\delta(k^2)\lambda(\vec{k}, k^0) \tag{2.4.155}$$

so that

$$\partial^\mu \Lambda(x) = -i \int \frac{d^4k}{(2\pi)^4} e^{-ikx}(2\pi)\delta(k^2) k^\mu \lambda(\vec{k}, k^0). \tag{2.4.156}$$

Letting $-i\lambda(\vec{k}, k^0) = +\frac{(1-\alpha)}{2}\partial_k^\lambda a_\lambda(\vec{k}, k^0)$, if we wish, the last two terms in (2.4.154) cancel and $A^\mu(x)$ is equivalent to

$$\begin{aligned}A^\mu(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}[e^{-ikx} a^\mu(\vec{k}) + e^{+ikx} a^{\mu\dagger}(\vec{k})] \\ &+ (1-\alpha)\int \frac{d^4k}{(2\pi)^4} e^{-ikx} k^\mu k^\nu (2\pi)\delta'(k^2) a_\lambda(\vec{k}, k^0),\end{aligned} \tag{2.4.157}$$

which was just equation (2.4.139).

Thus, the general expression for $A^\mu(x)$ in the Stueckelberg α-gauge is

$$A^\mu(x) = a^\mu(x) + \frac{(\alpha-1)}{2}\partial^\mu(x^\lambda a_\lambda) - \partial^\mu\Lambda(x). \tag{2.4.158}$$

In order to specify the gauge completely we must choose Λ and α. Since the algebra for $\alpha \neq 1$ is messy and masks the interpretation, let's for convenience choose the Feynman gauge $\alpha = 1$ and postpone fixing Λ for now (that is let $\Lambda = 0$ first). Consequently, in this gauge $A^\mu(x) = a^\mu(x)$ and hence

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}[e^{-ikx}a^\mu(\vec{k}) + e^{+ikx}a^{\mu\dagger}(\vec{k})] \tag{2.4.159}$$

and the Euler-Lagrange equations reduce to simply

$$\partial^2 A^\mu(x) = 0. \tag{2.4.160}$$

Since $a^\mu(\vec{k})$ is a four-vector we can expand it in terms of our complete orthonormal basis of polarization vectors associated with the k^μ-frame. Hence,

$$a^\mu(\vec{k}) = \sum_{\lambda=0}^{3} \epsilon^\mu(k,\lambda)\epsilon^\nu(k,\lambda)g_{\lambda\lambda}a_\mu(\vec{k}). \tag{2.4.161}$$

We now define

$$a_{(\lambda)}(\vec{k}) \equiv \epsilon^\mu(k,\lambda)g_{\lambda\lambda}a_\mu(\vec{k}) \tag{2.4.162}$$

and

$$a_{(\lambda)}{}^\dagger(\vec{k}) \equiv \epsilon^\mu(k,\lambda)g_{\lambda\lambda}a_\mu{}^\dagger(\vec{k}) \tag{2.4.163}$$

for $\lambda = 0, 1, 2, 3$ with no summation over λ. Note that we can invert this also

$$a^\mu(\vec{k}) = \sum_{\lambda=0}^{3} \epsilon^\mu(k,\lambda)a_{(\lambda)}(\vec{k}) \tag{2.4.164}$$

$$a^{\mu\dagger}(\vec{k}) = \sum_{\lambda=0}^{3} \epsilon^\mu(k,\lambda)a^\dagger_{(\lambda)}(\vec{k}). \tag{2.4.165}$$

Thus we can write

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\sum_{\lambda=0}^{3}[\epsilon^\mu(k,\lambda)a_{(\lambda)}(\vec{k})e^{-ikx} + \epsilon^\mu(k,\lambda)a_{(\lambda)}{}^\dagger(\vec{k})e^{+ikx}]. \tag{2.4.166}$$

The inverse of these Fourier transforms is as usual

$$a^\mu(\vec{k}) = i\int d^3x e^{+ikx}\overleftrightarrow{\partial}_0 A^\mu(x) = \int d^3x e^{+ikx}[\omega_k A^\mu(x) + i\dot{A}^\mu(x)]$$

$$a^{\mu\dagger}(\vec{k}) = i\int d^3x A^\mu(x)\overleftrightarrow{\partial}_0 e^{-ikx} = \int d^3x e^{-ikx}[\omega_k A^\mu(x) - i\dot{A}^\mu(x)] \tag{2.4.167}$$

but we can invert the polarization sum using the above definition

$$a_{(\lambda)}(\vec{k}) = \epsilon^\mu(k,\lambda) g_{\lambda\lambda} a_\mu(\vec{k}), \tag{2.4.168}$$

so we have

$$\begin{aligned} a^{(\lambda)}(\vec{k}) &= i\int d^3x \epsilon^\mu(k,\lambda) g_{\lambda\lambda} e^{+ikx}\overleftrightarrow{\partial}_0 A_\mu(x) \\ a^{(\lambda)\dagger}(\vec{k}) &= i\int d^3x A^\mu(x)\overleftrightarrow{\partial}_0 \epsilon_\mu(k,\lambda) g_{\lambda\lambda} e^{-ikx}. \end{aligned} \tag{2.4.169}$$

We can call

$$U^\mu_{k,\lambda}(x) \equiv \epsilon^\mu(k,\lambda) e^{-ikx} \tag{2.4.170}$$

the positive energy plane wave solution of the wave equation with polarization $\epsilon^\mu(k,\lambda)$,

$$\partial^2 U^\mu_{k,\lambda}(x) = 0, \tag{2.4.171}$$

and

$$V^\mu_{k,\lambda}(x) \equiv U^\mu_{k,\lambda}(x)^* = \epsilon^\mu(k,\lambda) e^{+ikx} \tag{2.4.172}$$

the negative energy plane wave solution of the wave equation with polarization $\epsilon^\mu(k,\lambda)^* = \epsilon^\mu(k,\lambda)$,

$$\partial^2 V^\mu_{k,\lambda}(x) = 0. \tag{2.4.173}$$

Hence the field expansion in terms of plane wave solutions has the usual form

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=0}^{3} [U^\mu_{k,\lambda}(x) a_{(\lambda)}(\vec{k}) + V^\mu_{k,\lambda}(x) a_{(\lambda)}{}^\dagger(\vec{k})] \tag{2.4.174}$$

with inverses

$$\begin{aligned} a^{(\lambda)}(\vec{k}) &= i\int d^3x U^\mu_{k,\lambda}(x)^* g_{\lambda\lambda} e^{+ikx}\overleftrightarrow{\partial}_0 A_\mu(x) \\ a^{(\lambda)\dagger}(\vec{k}) &= i\int d^3x A_\mu(x)\overleftrightarrow{\partial}_0 V^\mu_{k,\lambda}(x)^*. \end{aligned} \tag{2.4.175}$$

We can use these expansions to find the commutation relations obeyed by a and $a^\dagger$ as in the scalar case,

$$[a^\mu(\vec{k}), a^{\nu\dagger}(\vec{l})] = \int_{x^0=y^0} d^3x d^3y e^{ikx-ily}\{[\dot{A}^\mu(x), A^\nu(y)]i\omega_l$$

$$-i\omega_k[A^\mu(x), \dot{A}^\nu(y)] + [\dot{A}^\mu(x), \dot{A}^\nu(y)]$$

$$+\omega_k\omega_l[A^\mu(x), A^\nu(y)]\}. \quad (2.4.176)$$

Recalling equations (2.4.102) the ETCR with $\alpha = 1$ simply reduce to

$$\delta(x^0 - y^0)[\dot{A}^\mu(x), \dot{A}^\nu(x)] = ig^{\mu\nu}\delta^4(x-y) \quad (2.4.177)$$

with all others vanishing. Hence

$$\begin{aligned}[a^\mu(\vec{k}), a^{\nu\dagger}(\vec{l})] &= \int_{x^0=y^0} d^3x d^3y e^{ikx-ily} ig^{\mu\nu}\delta^3(\vec{x}-\vec{y})[i\omega_l + i\omega_k] \\ &= -\int d^3x e^{-i(\vec{k}-\vec{l})\cdot\vec{x}} g^{\mu\nu}(\omega_l + \omega_k) \\ &= -(2\pi)^3 2\omega_k g^{\mu\nu}\delta^3(\vec{k}-\vec{l}).\end{aligned} \quad (2.4.178)$$

Thus the CCR for the Fourier transform operators are

$$[a^\mu(\vec{k}), a^{\nu\dagger}(\vec{l})] = -g^{\mu\nu}(2\pi)^3 2\omega_k\delta^3(\vec{k}-\vec{l})$$

$$[a^{\mu\dagger}(\vec{k}), a^{\nu\dagger}(\vec{l})] = 0$$

$$[a^\mu(\vec{k}), a^\nu(\vec{l})] = 0. \quad (2.4.179)$$

We obtain the harmonic oscillator creation and annihilation operator CCR but with a minus sign for a^0! We could further expand $a^{\mu\dagger}$ and a^μ in terms of the polarization creation and annihilation operators to obtain

$$\epsilon_\mu(k,\lambda)g_{\lambda\lambda}[a^\mu(\vec{k}), a^{\nu\dagger}(\vec{l})]\epsilon_\nu(l,\rho)g_{\rho\rho} = [a_{(\lambda)}(\vec{k}), a_{(\rho)}{}^\dagger(\vec{l})]$$

$$= -g_{\lambda\lambda}g_{\rho\rho}g^{\mu\nu}(2\pi)^3 2\omega_k\delta^3(\vec{k}-\vec{l})\epsilon_\mu(k,\lambda)\epsilon_\nu(k,\rho)$$

$$= -(2\pi)^3 2\omega_k\delta^3(\vec{k}-\vec{l})g_{\lambda\rho}g_{\lambda\lambda}g_{\rho\rho}. \quad (2.4.180)$$

Since λ must equal ρ, $g_{\lambda\lambda}g_{\rho\rho} = +1$. Thus, the polarization creation and annihilation operator CCR are

$$[a_{(\lambda)}(\vec{k}), a_{(\rho)}{}^{\dagger}(\vec{l})] = -g_{\lambda\rho}(2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{l})$$
$$[a_{(\lambda)}{}^{\dagger}(\vec{k}), a^{\dagger}_{(\rho)}(\vec{l})] = 0$$
$$[a_{(\lambda)}(\vec{k}), a_{(\rho)}(\vec{l})] = 0. \tag{2.4.181}$$

Again the scalar photon created by $a^{\dagger}_{(0)}(\vec{k})$ has the wrong sign CCR.

We are next ready to expand the energy-momentum (and angular momentum) operators in terms of the momentum space creation and annihilation operators. Recall the expressions for the Hamiltonian and momentum

$$H = \mathcal{P}^0 = \int d^3x [\frac{1}{2}\Pi^{\lambda} \overleftrightarrow{\partial}^0 A_{\lambda}]$$
$$\vec{\mathcal{P}} = \int d^3x \Pi^{\lambda} \vec{\bigtriangledown} A_{\lambda}. \tag{2.4.182}$$

For $\alpha = 1$ the momenta are

$$\begin{aligned} \Pi^{\mu} &= F^{\mu 0} - \frac{1}{\alpha} g^{\mu 0}\partial_{\lambda}A^{\lambda} \\ &= \partial^{\mu}A^0 - \partial^0 A^{\mu} - g^{\mu 0}\partial_{\lambda}A^{\lambda}, \end{aligned} \tag{2.4.183}$$

hence

$$\Pi^0 = -\partial_{\lambda}A^{\lambda} = -\dot{A}^0 + \partial^i A^i$$
$$\Pi^i = -\dot{A}^i + \partial^i A^0. \tag{2.4.184}$$

So the Hamiltonian becomes

$$\begin{aligned} H &= \int d^3x \frac{1}{2}[(-\dot{A}^0 + \partial^i A^i)\overleftrightarrow{\partial}^0 A^0 - (-\dot{A}^i + \partial^i A^0)\overleftrightarrow{\partial}^0 A^i] \\ &= \int d^3x \frac{1}{2}[\ddot{A}^0 A^0 - \dot{A}^0\dot{A}^0 + \partial^i A^i \overleftrightarrow{\partial}^0 A^0 + \dot{A}^i\dot{A}^i - \ddot{A}^i A^i - \partial^i A^0 \overleftrightarrow{\partial}^0 A^i], \end{aligned} \tag{2.4.185}$$

where by parts the two terms indicated cancel yielding

$$H = \int d^3x \frac{1}{2}[\dot{A}^i \overleftrightarrow{\partial}^0 A^i - \dot{A}^0 \overleftrightarrow{\partial}^0 A^0]. \tag{2.4.186}$$

Plugging in the momentum expansion for A^μ and performing the algebra (or using the scalar case as a guide) we find

$$\begin{aligned} H &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{1}{2}\omega_k \sum_{\lambda=0}^{3} (-g_{\lambda\lambda})[a^{\lambda\dagger}(\vec{k})a^\lambda(\vec{k}) + a^\lambda(\vec{k})a^{\lambda\dagger}(\vec{k})] \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=0}^{3} \omega_k[-g_{\lambda\lambda}a^{\lambda\dagger}(\vec{k})a^\lambda(\vec{k})] + E_0 \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=0}^{3} \omega_k[-a^{\lambda\dagger}(\vec{k})a_\lambda(\vec{k})] + E_0, \end{aligned} \tag{2.4.187}$$

with

$$E_0 = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=0}^{3} \frac{1}{2}\omega_k[(2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{k})]. \tag{2.4.188}$$

As usual E_0 is the infinite zero point energy for the four harmonic oscillator modes which is eliminated by normal ordering the Lagrangian and the energy-momentum tensor. In a similar manner we obtain the momentum operator

$$\begin{aligned} \mathcal{P} &= -\int d^3x \Pi^\lambda \vec{\bigtriangledown} A_\lambda \\ &= -\int d^3x[(-\dot{A}^0 + \partial^i A^i)\vec{\bigtriangledown} A^0 - (-\dot{A}^i + \partial^i A^0)\vec{\bigtriangledown} A^i] \\ &= -\int d^3x[-\dot{A}^0\vec{\bigtriangledown} A^0 + \dot{A}^i\vec{\bigtriangledown} A^i + \partial^i A^i\vec{\bigtriangledown} A^0 - \partial^i A^0\vec{\bigtriangledown} A^i] \\ &= \int d^3x[+\dot{A}^0\vec{\bigtriangledown} A^0 - \dot{A}^i\vec{\bigtriangledown} A^i] \end{aligned} \tag{2.4.189}$$

where integration by parts was used. Fourier expanding the fields and applying the commutation relations we obtain

$$\begin{aligned} \vec{\mathcal{P}} &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{1}{2}\vec{k}[-g_{\lambda\lambda} \sum_{\lambda=0}^{3} (a^{\lambda\dagger}(\vec{k})a^\lambda(\vec{k}) + a^\lambda(\vec{k})a^{\lambda\dagger}(\vec{k}))] \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=0}^{3} \vec{k}[-g_{\lambda\lambda}a^{\lambda\dagger}(\vec{k})a^\lambda(\vec{k})] \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=0}^{3} \vec{k}[-a^{\lambda\dagger}(\vec{k})a_\lambda(\vec{k})]. \end{aligned} \tag{2.4.190}$$

Thus

$$\mathcal{P}^\mu = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=0}^{3} k^\mu [-g_{\lambda\lambda} a^{\lambda\dagger}(\vec{k}) a^\lambda(\vec{k})]$$
$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=0}^{3} k^\mu [-a^{\lambda\dagger}(\vec{k}) a_\lambda(\vec{k})] \tag{2.4.191}$$

where we have normal ordered $\mathcal{L}$ and $T^{\mu\nu}$ to eliminate E_0 as usual for bosonic operators, that is,

$$\hat{\mathcal{L}} = N[\mathcal{L}]$$
$$\hat{T}^{\mu\nu} = N[T^{\mu\nu}]$$
$$\hat{\Theta}^{\mu\nu} = N[\Theta^{\mu\nu}] \tag{2.4.192}$$
$$\hat{M}^{\mu\nu\rho} = N[M^{\mu\nu\rho}]$$

with $\mathcal{P}^\mu$ and $\mathcal{M}^{\mu\nu}$ now given by

$$\mathcal{P}^\mu = \int d^3x N[\Theta^{0\mu}]$$

$$\mathcal{M}^{\mu\nu} = \int d^3x N[M^{0\mu\nu}]. \tag{2.4.193}$$

The Euler-Lagrange equations, canonical momenta, and the ETCR remain unchanged.

Thus, we have the Hamiltonian

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k [-a^{0\dagger}(\vec{k}) a^0(\vec{k}) + a^{i\dagger}(\vec{k}) a^i(\vec{k})]. \tag{2.4.194}$$

We would like to interpret a^μ as annihilation operators and $a^{\mu\dagger}$ as creation operators. (Note that associating $a^{0\dagger}$ as an annihilation operator and a^0 as a creation operator leads to further interpretive difficulty. In particular, the energy becomes unbounded below, i.e.,$|\vec{k}, 0 >= a^0(\vec{k})|0>$ and $H|\vec{k}, 0 >= -\omega_k|\vec{k}, 0>$, hence there is no stable ground state in such an interpretation.) At first it might appear that H is not positive and thus there is no ground state, but consider an eigenstate of H denoted by $|E>$, $H|E>= E|E>$, then

$$H a^\mu(\vec{k})^\dagger |E> = [H, a^\mu(\vec{k})^\dagger]|E> + a^\mu(\vec{k})^\dagger H|E>$$
$$= (E + \omega_k) a^\mu(\vec{k})^\dagger |E> . \tag{2.4.195}$$

Because of the presence of $(-g^{\mu\nu})$ in the CCR, the minus sign for the $a^{0\dagger}$ particle cancels to give a positive energy. As usual, we have a state of lowest energy which we define to have zero energy and no particles. Hence the ground state, $|0>$, is defined by

$$a^\mu(\vec{k})|0>= 0 \quad \text{for all } \vec{\text{k}} \text{ and } \mu. \tag{2.4.196}$$

Thus $|0>$ has zero energy, momentum and angular momentum

$$\mathcal{P}^\mu|0>= 0$$

$$\mathcal{M}^{\mu\nu}|0>= 0. \tag{2.4.197}$$

As usual we find that $a^\mu(\vec{k})^\dagger$ creates states (particles) with momentum $\vec{k}$ while $a^\mu(\vec{k})$ annihilates the same. To clarify this and the angular momentum of the states further let's find the commutator of $\mathcal{P}^\mu$ and $\mathcal{M}^{\mu\nu}$ with a^μ and $a^{\mu\dagger}$. Starting with

$$[\mathcal{P}^\mu, A^\lambda(x)] = -i\partial^\mu A^\lambda(x)$$

$$[\mathcal{M}^{\mu\nu}, A^\lambda(x)] = -i[(x^\mu\partial^\nu - x^\nu\partial^\mu)A^\lambda(x) + (g^{\mu\lambda}g^{\nu\rho} - g^{\nu\lambda}g^{\mu\rho})A_\rho(x)] \tag{2.4.198}$$

in coordinate space we obtain for the creation and annihilation operators

$$[\mathcal{P}^\mu, a^\lambda(\vec{k})] = -k^\mu a^\lambda(\vec{k})$$

$$[\mathcal{P}^\mu, a^{\lambda\dagger}(\vec{k})] = +k^\mu a^{\lambda\dagger}(\vec{k})$$

$$[\mathcal{M}^{\mu\nu}, a^\lambda(\vec{k})] = i\int d^3x e^{+ikx}\overleftrightarrow{\partial}_0[-i(x^\mu\partial^\nu - x^\nu\partial^\mu)A^\lambda(x)]$$

$$-i(g^{\mu\lambda}g^{\nu\rho} - g^{\nu\lambda}g^{\mu\rho})a_\rho(\vec{k})$$

$$[\mathcal{M}^{\mu\nu}, a^{\lambda\dagger}(\vec{k})] = i\int d^3x[-i(x^\mu\partial^\nu - x^\nu\partial^\mu)A^\lambda(x)\overleftrightarrow{\partial}_0 e^{+ikx}]$$

$$-i(g^{\mu\lambda}g^{\nu\rho} - g^{\nu\lambda}g^{\mu\rho}){a_\rho}^\dagger(\vec{k}). \tag{2.4.199}$$

We must further consider which operators among $\mathcal{P}^\mu$ and $\mathcal{M}^{\mu\nu}$ commute and hence, have eigenvalues which label the states. As in the fermi case, we begin by choosing our states to be translationally invariant states, that is, the eigenvalues of $\mathcal{P}^\mu$ and $\mathcal{P}^2$ will label the states. For photons the $\mathcal{P}^2$ eigenvalue of 0 labels the single particle states; so we can label the one particle states by the $\vec{\mathcal{P}}$ eigenvalue $|\vec{k}>, \vec{\mathcal{P}}|k>= \vec{k}|\vec{k}>$

; $\mathcal{P}^0|\vec{k}>=|\vec{k}||\vec{k}>$ since $\mathcal{P}^2|\vec{k}>=0$. In addition, we will use the Pauli-Lubanski vector W^μ to further label the single photon states. Since W^μ and $\mathcal{P}^\mu$ are both vectors we can consider going to a frame in which $k^\mu=(k,0,0,k)$ for the one particle states. Since $W^\mu\mathcal{P}_\mu=0$, we have $W^3=W^0$. Using the $[W^\mu,W^\nu]=-i\epsilon^{\mu\nu\rho\sigma}W_\rho P_\sigma$ commutator applied to a one photon state in this frame we find

$$
\begin{aligned}
[W^1,W^2]&=0\\
[W^3,W^1]&=-ikW^2\\
[W^3,W^2]&=+ikW^1.
\end{aligned}
\tag{2.4.200}
$$

These are the commutation relations for rotations in a plane with generator W^3 and translations in the plane generated by (W^1,W^2); this is the Euclidean group of motion in a plane called E_2. Thus, W^1 and W^2 can take on any value, they have continuous eigenvalues. It then follows that $W^2=W^\mu W_\mu$ is not quantized and its eigenvalues are zero and $W^2<0$. $W^2<0$ corresponds to states with continuous spin not photons (such single particle states are not observed in nature) the photon states have $W^2=0$, that is, $W^1=W^2=0$ eigenvalues. So since we already have $W^0=W^3$ in this frame, $k^0=k^3$; that is $k^\mu=(k,0,0,k)$, we have that $W^\mu|\vec{k}>\propto\mathcal{P}^\mu|\vec{k}>$ for the single photon states. Since W^μ and $\mathcal{P}^\mu$ transform the same way under Lorentz transformations, they are four-vectors, proportionality in one frame is the same proportionality in all frames, hence, $W^\mu=-\lambda\mathcal{P}^\mu$ for massless single photon states (note that $W^2=\lambda^2\mathcal{P}^2=0$ as required) where λ is the constant of proportionality and is Lorentz invariant. Furthermore,

$$
-\lambda=\frac{W^0}{\mathcal{P}^0}=-\frac{\vec{\mathcal{J}}\cdot\vec{\mathcal{P}}}{|\vec{\mathcal{P}}|}
\tag{2.4.201}
$$

or on the $|\vec{k}>$ state

$$
\lambda=\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}
\tag{2.4.202}
$$

the projection of the angular momentum along the direction of motion, the helicity. In the special frame $\lambda=\mathcal{J}_3$ the third component of the angular momentum, hence in general, λ has eigenvalues $\lambda=0,\pm\frac{1}{2},\pm1,\cdots$. For the single photon states we see that $\lambda=\pm1$ for circularly polarized right-handed and left-handed photons and because we have two unphysical states, $\lambda=0$ has a multiplicity of two, one for the longitudinal photon, partner of the $\lambda=\pm1$ photons, and the scalar photon. Thus as we discussed in the introduction the states are labeled by the complete set of commuting observables ($\mathcal{P}^2 = 0, W^2 = 0, \vec{\mathcal{P}}, \lambda = \frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}$) as well in our case the fact that they are longitudinal, scalar, or transverse photons, that is, the other observable in our non-interacting case is the number operator for each type of photon

$$N_{(\lambda)} \equiv -\int \frac{d^3k}{(2\pi)^3 2\omega_k} g_{\lambda\lambda} a^\dagger_{(\lambda)}(\vec{k}) a_{(\lambda)}(\vec{k}). \tag{2.4.203}$$

So we are as usual interested in not all of the components of $\mathcal{M}^{\mu\nu}$ but in $\mathcal{J}^i = \frac{1}{2}\epsilon_{ijk}\mathcal{M}^{ij}$, thus we find

$$[\mathcal{M}^{jl}, a^\lambda(\vec{k})] = -i(k^j\partial^l_k - k^l\partial^j_k)a^\lambda(\vec{k}) - i(g^{j\lambda}g^{l\rho} - g^{l\lambda}g^{j\rho})a_\rho(\vec{k})$$

$$[\mathcal{M}^{jl}, {a^\lambda}^\dagger(\vec{k})] = -i(k^j\partial^l_k - k^l\partial^j_k){a^\lambda}^\dagger(\vec{k}) - i(g^{j\lambda}g^{l\rho} - g^{l\lambda}g^{j\rho}){a_\rho}^\dagger(\vec{k}). \tag{2.4.204}$$

This yields

$$[\mathcal{J}^i, a^\lambda(\vec{k})] = -i\epsilon^{ijl}k^i\partial^l_k a^\lambda(\vec{k}) - i\epsilon^{ijl}g^{j\lambda}a^l(\vec{k})$$

$$[\mathcal{J}^i, {a^\lambda}^\dagger(\vec{k})] = -i\epsilon^{ijl}k^i\partial^l_k {a^\lambda}^\dagger(\vec{k}) - i\epsilon^{ijl}g^{j\lambda}{a^l}^\dagger(\vec{k}). \tag{2.4.205}$$

So the helicity operator $\lambda = \frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}$ yields

$$[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a^\lambda(\vec{k})] = -i\epsilon^{ijl}g^{j\lambda}\frac{k^i}{|\vec{k}|}a^l(\vec{k})$$

$$[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, {a^\lambda}^\dagger(\vec{k})] = -i\epsilon^{ijl}g^{j\lambda}\frac{k^i}{|\vec{k}|}{a^l}^\dagger(\vec{k}). \tag{2.4.206}$$

Thus the helicity commutators become

$$[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a^0(\vec{k})] = 0$$

$$[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, {a^0}^\dagger(\vec{k})] = 0$$

$$[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a^l(\vec{k})] = -i\epsilon^{ilj}\frac{k^i}{|\vec{k}|}a^j(\vec{k})$$

$$[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, {a^l}^\dagger(\vec{k})] = -i\epsilon^{ilj}\frac{k^i}{|\vec{k}|}{a^j}^\dagger(\vec{k}). \tag{2.4.207}$$

Now we see the meaning of expanding a^{λ} and $a^{\lambda\dagger}$ in terms of polarization vectors

$$\begin{aligned}
[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a_{(\lambda)}(\vec{k})] &= \epsilon^{\mu}(k,\lambda)g_{\lambda\lambda}[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a_{\mu}(\vec{k})] \\
&= -i\epsilon^{ijl}g^{j}_{\mu}\frac{k^i}{|\vec{k}|}\epsilon^{\mu}(k,\lambda)g_{\lambda\lambda}\sum_{\rho=0}^{3}\epsilon^{l}(k,\rho)a_{(\rho)}(\vec{k}) \\
&= -i\frac{k^i}{|\vec{k}|}\epsilon^{ijl}\sum_{\rho=0}^{3}\epsilon^{j}(k,\lambda)g_{\lambda\lambda}\epsilon^{l}(k,\rho)a_{(\rho)}(\vec{k}) \\
&= -i\sum_{\rho=0}^{3}g_{\lambda\lambda}\frac{\vec{k}}{|\vec{k}|}\cdot\left(\vec{\epsilon}(k,\lambda)\times\vec{\epsilon}(k,\rho)\right)a_{(\rho)}(\vec{k}).
\end{aligned} \tag{2.4.208}$$

But recall for $\lambda = 0$ that $\vec{\epsilon}(k,0) = 0$ and for $\lambda = 3$ that $\vec{\epsilon}(k,3) = \vec{k}/|\vec{k}|$, thus

$$\frac{\vec{k}}{|\vec{k}|}\cdot\left(\vec{\epsilon}(k,\lambda)\times\vec{\epsilon}(k,\rho)\right) = \begin{cases} 0 & \text{for } \lambda = 0, \\ \epsilon^{\lambda\rho 3} & \text{for } \lambda = 1,\ 2\ , \\ 0 & \text{for } \lambda = 3. \end{cases} \tag{2.4.209}$$

Hence we find

$$\begin{aligned}
[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a_{(0)}(\vec{k})] &= 0 \\
[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a_{(3)}(\vec{k})] &= 0 \\
[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a_{(1)}(\vec{k})] &= +ia_{(2)}(\vec{k}) \\
[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a_{(2)}(\vec{k})] &= -ia_{(1)}(\vec{k})
\end{aligned} \tag{2.4.210}$$

and similarly

$$\begin{aligned}
[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a^{\dagger}_{(0)}(\vec{k})] &= 0 \\
[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a^{\dagger}_{(3)}(\vec{k})] &= 0 \\
[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a^{\dagger}_{(1)}(\vec{k})] &= +ia^{\dagger}_{(2)}(\vec{k}) \\
[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a^{\dagger}_{(2)}(\vec{k})] &= -ia^{\dagger}_{(1)}(\vec{k}).
\end{aligned} \tag{2.4.211}$$

As usual we go over to circularly polarized photons, the helicity eigenstates,

$$a_\pm(\vec{k}) \equiv \pm\frac{1}{\sqrt{2}}[a_{(1)}(\vec{k}) \mp i a_{(2)}(\vec{k})]$$

$$a^\dagger_\pm \equiv \pm\frac{1}{\sqrt{2}}[a^\dagger_{(1)}(\vec{k}) \pm i a^\dagger_{(2)}(\vec{k})]. \tag{2.4.212}$$

It then follows that

$$[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a_{(0)}(\vec{k})] = 0 = [\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a_{(3)}(\vec{k})]$$

$$[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a^\dagger_{(0)}(\vec{k})] = 0 = [\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a^\dagger_{(3)}(\vec{k})]$$

$$[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a_\pm(\vec{k})] = \mp a_\pm(\vec{k})$$

$$[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a^\dagger_\pm(\vec{k})] = \pm a^\dagger_\pm(\vec{k}). \tag{2.4.213}$$

So finally we have the commutators of the creation and annihilation operators with the commuting observables;

$$[\mathcal{P}^\mu, a_{(\lambda)}(\vec{k})] = -k^\mu a_{(\lambda)}(\vec{k})$$

$$[\mathcal{P}^\mu, a^\dagger_{(\lambda)}(\vec{k})] = +k^\mu a^\dagger_{(\lambda)}(\vec{k})$$

$$[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a_{(0)}(\vec{k})] = 0 = [\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a_{(3)}(\vec{k})]$$

$$[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a^\dagger_{(0)}(\vec{k})] = 0 = [\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a^\dagger_{(3)}(\vec{k})]$$

$$[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a_\pm(\vec{k})] = \mp a_\pm(\vec{k})$$

$$[\frac{\vec{\mathcal{J}}\cdot\vec{k}}{|\vec{k}|}, a^\dagger_\pm(\vec{k})] = \pm a^\dagger_\pm(\vec{k}). \tag{2.4.214}$$

As we have seen earlier, the lowest energy state is normalized to zero energy by definition; $a_{(\lambda)}(\vec{k})|0>=0$ for all $\vec{k}$ and λ, so that $\mathcal{P}^\mu|0>=0$ and $\mathcal{M}^{\mu\nu}|0>=0$. The single particle states are given by their momentum $\vec{k}$ and helicity $0, 0, \pm 1$ and whether it is a longitudinal, scalar, or transverse photon. For the transverse photons

$$|\vec{k}, \pm > \equiv a^\dagger_\pm(\vec{k})|0> \tag{2.4.215}$$

while for the scalar photon

$$|\vec{k}, 0 > \equiv a^\dagger_{(0)}(\vec{k})|0> \tag{2.4.216}$$

and the longitudinal photon

$$|\vec{k}, 3 > \equiv a^\dagger_{(3)}(\vec{k})|0> . \tag{2.4.217}$$

In general we have

$$\begin{aligned} \vec{\mathcal{P}}|\vec{k}, \lambda > &= \vec{k}|\vec{k}, \lambda >, \quad \lambda = 0, 3, \pm \\ \frac{\vec{\mathcal{J}} \cdot \vec{k}}{|\vec{k}|}|\vec{k}, \lambda > &= \pm|\vec{k}, \lambda >, \quad \lambda = \pm \\ \frac{\vec{\mathcal{J}} \cdot \vec{k}}{|\vec{k}|}|\vec{k}, \lambda > &= 0, \quad \lambda = 0, 3. \end{aligned} \tag{2.4.218}$$

Thus, $a^{(\lambda)\dagger}(\vec{k})$ creates photons with momentum $\vec{k}$ and helicity ± 1 for $\lambda = \pm$ and 0 for $\lambda = 0, 3$. Likewise, $a_{(\lambda)}(\vec{k})$ annihilates the same photons.

The inner product of the one photon states is given by the commutator of a and $a^\dagger$

$$[a_{(\lambda)}(\vec{k}), a^\dagger_{(\rho)}(\vec{l})] = -g_{\lambda\rho}(2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{l}). \tag{2.4.219}$$

Thus

$$< \vec{k}, \lambda|\vec{l}, \rho > = < 0|a_{(\lambda)}(\vec{k})a^\dagger_{(\rho)}(\vec{l})|0 > = -g_{\lambda\rho}(2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{l}). \tag{2.4.220}$$

We now see our problem of negative norm appearing for the scalar photon

$$< \vec{k}, 0|\vec{l}, 0 > = -(2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{l}), \tag{2.4.221}$$

we cannot apply the usual quantum mechanical probabilistic interpretation to these states. The state space is no longer a Hilbert space (only a Fock space) since the "inner product" does not have the correct inner product properties. However, this does not overly disturb us since we already know that the scalar and longitudinal photons are unphysical degrees of freedom, only the transverse photons are physical and only these states need to form a Hilbert space. More generally speaking, the multi-particle states can be made by repeat application of the creation operators

$$|(\vec{k}_1,\lambda_1),\cdots,(\vec{k}_n,\lambda_n)>=a^\dagger_{(\lambda_1)}(\vec{k}_1)\cdots a^\dagger_{(\lambda_n)}(\vec{k}_n)|0> . \tag{2.4.222}$$

These states have the energy-momentum

$$\mathcal{P}^\mu|(\vec{k}_1,\lambda_1),\ldots,(\vec{k}_n,\lambda_n) = (\sum_{i=1}^n k_i^\mu)|(\vec{k}_1,\lambda_1),\ldots,(\vec{k}_n,\lambda_n)> . \tag{2.4.223}$$

With the number density operator for photons with momentum $\vec{k}$ and helicity λ given by

$$N_{(\lambda)}(\vec{k})\equiv\frac{1}{(2\pi)^3 2\omega_k}(-g_{\lambda\lambda})a^\dagger_{(\lambda)}(\vec{k})a_{(\lambda)}(\vec{k}) \tag{2.4.224}$$

we have

$$N_{(\lambda)}(\vec{k})|(\vec{k}_1,\lambda_1),\ldots,(\vec{k}_n,\lambda_n)>=(\sum_{i=1}^n\delta^3(\vec{k}-\vec{k}_i)\delta_{\lambda\lambda_i})|(\vec{k}_1,\lambda_1),\ldots,(\vec{k}_n,\lambda_n)> . \tag{2.4.225}$$

The n-particle state has norm

$$<(\vec{k}_1,\lambda_1),\ldots,(\vec{k}_n,\lambda_n)|(\vec{l}_1,\rho_1),\ldots,(\vec{l}_n,\rho_n)>$$

$$=\delta_{mn}\sum_{(1,\ldots,n)\to(i_1,\ldots,i_n)}\prod_{a=1}^n[-g_{\lambda_a\rho_{i_a}}(2\pi)^3(2\omega_{k_a})\delta^3(\vec{k}_a-\vec{l}_{i_a})]. \tag{2.4.226}$$

Hence, we see that states with an odd number of scalar photons have negative norm. As usual our full Fock space $\mathcal{V}$ is made as a direct sum of these n-particle spaces

$$\mathcal{V}=|0>\oplus|\vec{k},\lambda>\oplus\Big(|(\vec{k}_1,\lambda_1)>\otimes|\vec{k}_2,\lambda_2>\Big)\oplus\cdots. \tag{2.4.227}$$

¿From this space we would like to pick out in a Lorentz covariant way a physical subspace $\mathcal{H}_{Phys}\subset\mathcal{V}$ which has positive norm and whose states describe the physical photon states. We also must have the observable operators when projected on the physical subspace yield the same matrix elements as we found in the Coulomb gauge. Clearly the physical space is spanned by the states that involve only the transverse photons. The means to define the states in this space in a Lorentz covariant way was found by Gupta and Bleuler in 1950. Since the matrix elements of the vector potential go over for states with a large number of photons to the classical Maxwell value of the field, they should be solutions to the Maxwell equations. But since we added a gauge fixing term to the Lagrangian this is spoiled unless

$$< \Psi|\partial_\mu A^\mu(x)|\Phi >= 0 \tag{2.4.228}$$

for $|\Psi >$ and $|\Phi >$ physical states. Since $\partial_\mu A^\mu(x)$ always obeys a free wave equation even in the presence of a conserved current, we can always expand $\partial_\mu A^\mu(x)$ in terms of positive and negative frequency fields in a Lorentz invariant way. Gupta and Bleuler were able to define the physical subspace by those states for which

$$\partial_\mu A^{\mu +}(x)|\Phi >= 0. \tag{2.4.229}$$

If $\partial_\mu A^{\mu +}(x)|\Phi >= 0$, then $< \Phi|\partial_\mu A^{\mu -}(x) = 0$, so that $< \Psi|\partial_\mu A^\mu(x)|\Phi >= 0$ and the Maxwell equations are recovered for the matrix elements.

Let's study the structure of this $\mathcal{H}_{Phys}$ more carefully. Recall the creation and annihilation operators involved in $\partial_\mu A^\mu(x)$

$$\begin{aligned} i\partial_\mu A^\mu(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}[k_\mu a^\mu(\vec{k})e^{-ikx} - k_\mu a^{\mu\dagger}(\vec{k})e^{+ikx}] \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\sum_{\lambda=0}^{3} k_\mu \epsilon^\mu(k,\lambda)[a_{(\lambda)}(\vec{k})e^{-ikx} - a^\dagger_{(\lambda)}(\vec{k})e^{+ikx}]. \end{aligned} \tag{2.4.230}$$

Now for our $\epsilon^\mu(k,\lambda)$ polarization vectors

$$k_\mu \epsilon^\mu(k,\lambda) = \begin{cases} k^0 & \text{for } \lambda = 0, \\ 0 & \text{for } \lambda = 1,\ 2\ , \\ -|\vec{k}| & \text{for } \lambda = 3 \end{cases} \tag{2.4.231}$$

hence we have

$$i\partial_\mu A^\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k[(a_{(0)}(\vec{k}) - (a_{(3)}(\vec{k}))e^{-ikx} - (a^\dagger_{(0)}(\vec{k}) - a^\dagger_{(3)}(\vec{k}))e^{+ikx}]. \tag{2.4.232}$$

Thus, we see that

$$\partial_\mu A^{\mu +}(x)|\Phi >= \int \frac{d^3k}{(2\pi)^3 2\omega_k}e^{-ikx}\sum_{\lambda=0}^{3}\omega_k[a_{(0)}(\vec{k}) - a_{(3)}(\vec{k})]|\Phi > \tag{2.4.233}$$

and that $\partial_\mu A^{\mu+}(x)|\Phi>=0$ implies

$$[a_{(0)}(\vec{k}) - a_{(3)}(\vec{k})]|\Phi>=0. \tag{2.4.234}$$

Hence, if $|\Phi>$ is an element of $\mathcal{H}_{Phys}$ then

$$a_{(0)}(\vec{k})|\Phi>=a_{(3)}(\vec{k})|\Phi> . \tag{2.4.235}$$

Clearly, $|\Phi>$ consisting of only transverse modes is a solution to this, but it is not the most general solution. Note in general

$$|\Phi>=|\Phi_{tr}>|\hat{\Phi}> \tag{2.4.236}$$

where $|\Phi_{tr}>$ is made from transverse photons only and $|\hat{\Phi}>$ is such that

$$[a_{(0)}(\vec{k}) - a_{(3)}(\vec{k})]|\hat{\Phi}>=0 \tag{2.4.237}$$

and contains no transverse photons. Note that because of the vanishing commutator

$$\begin{aligned}
&[(a_{(0)}(\vec{k}) - a_{(3)}(\vec{k})), (a^\dagger_{(0)}(\vec{l}) - a^\dagger_{(3)}(\vec{l}))] \\
&= [a_{(0)}(\vec{k}), a^\dagger_{(0)}(\vec{l})] + [a_{(3)}(\vec{k}), a^\dagger_{(3)}(\vec{l})] \\
&= -(2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{l}) + (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{l}) \\
&= 0
\end{aligned} \tag{2.4.238}$$

or, calling $(a_{(0)}(\vec{k}) - a_{(3)}(\vec{k})) \equiv \mathcal{G}(\vec{k})$, equation (2.4.238) becomes

$$[\mathcal{G}(\vec{k}), \mathcal{G}^\dagger(\vec{l})] = 0, \tag{2.4.239}$$

we have

$$\mathcal{G}(\vec{k})\mathcal{G}^\dagger(\vec{k}_1)\cdots\mathcal{G}^\dagger(\vec{k}_n)|0>=0. \tag{2.4.240}$$

The most general state $|\hat{\Phi}>$ has definite numbers of scalar and longitudinal photons and is given by

$$\begin{aligned}
|\hat{\Phi}>=[1 + \int \frac{d^3k}{(2\pi)^3 2\omega_k} c(\vec{k})\mathcal{G}^\dagger(\vec{k}) \\
+\cdots+ \int \frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}} \cdots \int \frac{d^3k_n}{(2\pi)^3 2\omega_{k_n}} c(\vec{k}_1,\ldots,\vec{k}_n)\mathcal{G}^\dagger(\vec{k}_1)\cdots\mathcal{G}^\dagger(\vec{k}_n)+\cdots]|0>
\end{aligned} \tag{2.4.241}$$

where $|0>$ is the $a_\mu(\vec{k})|0>=0$ state. Note that these $|\hat{\Phi}>$ states have a unit norm since

$$< 0|\mathcal{G}(\vec{k}_1)\cdots\mathcal{G}(\vec{k}_n)|\mathcal{G}^\dagger(\vec{l}_1)\cdots\mathcal{G}^\dagger(\vec{l}_m)|0>=0. \tag{2.4.242}$$

These are zero norm states except for $|0>$, so $<\hat{\Phi}|\hat{\Phi}>=<0|0>=1$. Hence, any physical state $|\Phi>=|\Phi_{tr}>|\hat{\Phi}>$ and $|\Psi>=|\Psi_{tr}>|\hat{\Psi}>$ has an inner product

$$<\Phi|\Psi>=<\Phi_{tr}|\Psi_{tr}>. \tag{2.4.243}$$

So allowed admixtures of scalar and longitudinal states do not affect the norm of the transverse state vectors. Since these are positive this is a positive norm or scalar product.

Furthermore, when we calculate the physical state matrix elements of products of vector potential operators these will only depend on the transverse parts of these fields. In particular, consider the matrix elements of the energy- momentum operator

$$\begin{aligned}
\mathcal{P}^\mu &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=0}^{3} k^\mu[-a^{\lambda\dagger}(\vec{k})a_\lambda(\vec{k})] \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}(-k^\mu) \sum_{\lambda=0}^{3}\sum_{\rho=0}^{3} \epsilon^\lambda(k,\rho) \sum_{\kappa=0}^{3} \epsilon_\lambda(k,\kappa) a^\dagger_{(\rho)}(\vec{k})a_{(\kappa)}(\vec{k}) \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\rho=0}^{3}\sum_{\kappa=0}^{3}(-k^\mu) g_{\rho\kappa} a^\dagger_{(\rho)}(\vec{k})a_{(\kappa)}(\vec{k}) \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\rho=0}^{3} k^\mu[-g_{\rho\rho} a^\dagger_{(\rho)}(\vec{k})a_{(\rho)}(\vec{k})].
\end{aligned} \tag{2.4.244}$$

Thus, $\mathcal{P}^\mu = \mathcal{P}^\mu_{tr} + \hat{\mathcal{P}}^\mu$ with

$$\mathcal{P}^\mu_{tr} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\rho=1}^{2} k^\mu a^\dagger_{(\rho)}(\vec{k})a_{(\rho)}(\vec{k})$$

$$\begin{aligned}
\hat{\mathcal{P}}^\mu &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} k^\mu[a^\dagger_{(3)}(\vec{k})a_{(3)}(\vec{k}) - a^\dagger_{(0)}(\vec{k})a_{(0)}(\vec{k})] \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}(-k^\mu)[a^\dagger_{(3)}(\vec{k})\mathcal{G}(\vec{k}) + \mathcal{G}^\dagger(\vec{k})a_{(0)}(\vec{k})].
\end{aligned} \tag{2.4.245}$$

Now the physical state matrix elements of $\mathcal{P}^\mu$ are

$$\begin{aligned} < \Psi|\mathcal{P}^\mu|\Phi > &=< \Psi|\mathcal{P}^\mu_{tr}|\Phi > + < \Psi|\hat{\mathcal{P}}^\mu|\Phi > \\ &=< \Psi_{tr}|\mathcal{P}^\mu_{tr}|\Phi_{tr} >< \hat{\Psi}|\hat{\Phi} > + < \Psi_{tr}|\Phi_{tr} >< \hat{\Psi}|\hat{\mathcal{P}}^\mu|\hat{\Phi} > \end{aligned} \tag{2.4.246}$$

but

$$\begin{aligned} &< \hat{\Psi}|(a^\dagger_{(3)}(\vec{k})a_{(3)}(\vec{k}) - a^\dagger_{(0)}(\vec{k})a_{(0)}(\vec{k}))|\hat{\Phi} > \\ &=< \hat{\Psi}|(a^\dagger_{(3)}(\vec{k}) - a^\dagger_{(0)}(\vec{k}))a_{(3)}(\vec{k})|\hat{\Phi} > \\ &= 0 \end{aligned} \tag{2.4.247}$$

by means of the subsidiary condition $\partial_\mu A^{\mu +}(x)|\hat{\Phi} >= 0$. Consequently, since $< \hat{\Psi}|(a^\dagger_{(3)}(\vec{k}) - a^\dagger_{(0)}(\vec{k})) = 0$,

$$< \hat{\Psi}|\hat{\mathcal{P}}^\mu|\hat{\Phi} >= 0. \tag{2.4.248}$$

The scalar and longitudinal photon contribution to $\mathcal{P}^\mu$ cancels for the physical states. Since $< \hat{\Psi}|\hat{\Phi} >= 1$,

$$< \Psi|\mathcal{P}^\mu|\Phi >=< \Psi_{tr}|\mathcal{P}^\mu_{tr}|\Phi_{tr} > . \tag{2.4.249}$$

Thus the energy-momentum vector has the same matrix elements as we found in the Coulomb gauge and they are positive for $\mathcal{P}^0 = H$. That is the physical states, by the subsidiary condition, contain the same number of longitudinal as scalar photons for each momentum $\vec{k}$. The number operator for each momentum being $a^\dagger_{(0)}(\vec{k})a_{(0)}(\vec{k})$ for scalar photons and $a^\dagger_{(3)}(\vec{k})a_{(3)}(\vec{k})$ for longitudinal photons. Between physical states

$$< \Psi|a^\dagger_{(0)}(\vec{k})a_{(0)}(\vec{k})|\Phi >=< \Psi|a^\dagger_{(3)}(\vec{k})a_{(3)}(\vec{k})|\Phi > . \tag{2.4.250}$$

In general we can show that for $|\Psi >$ and $|\Phi >$ elements of $\mathcal{H}_{Phys}$

$$< \Psi|A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)|\Phi >=< \Psi_{tr}|A^{tr}_{\mu_1}(x_1)\cdots A^{tr}_{\mu_n}(x_n)|\Phi_{tr} > \tag{2.4.251}$$

where

$$A^{tr}_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=1}^{2} \epsilon_\mu(k,\lambda)[a_{(\lambda)}(\vec{k})e^{-ikx} + a^\dagger_{(\lambda)}(\vec{k})e^{+ikx}]. \tag{2.4.252}$$

Notice that

$$
\begin{aligned}
A_\mu(x) = A_\mu^{tr}(x) + \int \frac{d^3k}{(2\pi)^3 2\omega_k} &[(n_\mu a_{(0)}(\vec{k}) + \frac{k_\mu}{\omega_k} a_{(3)}(\vec{k}) - n_\mu a_{(3)}(\vec{k}))e^{-ikx} \\
&+ (n_\mu a^\dagger_{(0)}(\vec{k}) + \frac{k_\mu}{\omega_k} a^\dagger_{(3)}(\vec{k}) - n_\mu a^\dagger_{(3)}(\vec{k}))e^{+ikx}] \\
= A_\mu^{tr}(x) + n_\mu &\int \frac{d^3k}{(2\pi)^3 2\omega_k} [\mathcal{G}(\vec{k})e^{-ikx} + \mathcal{G}^\dagger(\vec{k})e^{+ikx}] \\
&+ \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{k_\mu}{\omega_k} [a_{(3)}(\vec{k})e^{-ikx} + a^\dagger_{(3)}(\vec{k})e^{+ikx}].
\end{aligned}
\tag{2.4.253}
$$

Hence for physical states we find

$$
< \Psi|A_\mu(x)|\Phi > = < \Psi_{tr}|A_\mu^{tr}(x)|\Phi_{tr} >
$$

$$
+ < \Psi_{tr}|\Phi_{tr} >< \hat{\Psi}| \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{k_\mu}{\omega_k} [a_{(3)}(\vec{k})e^{-ikx} + a^\dagger_{(3)}(\vec{k})e^{+ikx}]|\hat{\Phi} > . \tag{2.4.254}
$$

But defining the function

$$
\Lambda(x) \equiv < \hat{\Psi}| \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{i}{\omega_k} [a_{(3)}(\vec{k})e^{-ikx} - a^\dagger_{(3)}(\vec{k})e^{+ikx}]|\hat{\Phi} > \tag{2.4.255}
$$

we note that $\partial^2 \Lambda = 0$ and we have

$$
< \Psi|A_\mu(x)|\Phi > = < \Psi_{tr}|A_\mu^{tr}(x)|\Phi_{tr} > + \partial_\mu \Lambda(x). \tag{2.4.256}
$$

Said differently we can always choose a gauge so that

$$
< \Psi|A'_\mu(x)|\Phi > = < \Psi_{tr}|A_\mu^{tr}(x)|\Phi_{tr} > \tag{2.4.257}
$$

with $A'_\mu = A_\mu + \partial_\mu \Lambda$ and Λ an operator now. From this, and the fact that A_μ^{tr} and $\mathcal{G}$ commute, we can show that with such a gauge choice

$$
< \Psi|A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n)|\Phi > = < \Psi_{tr}|A^{tr}_{\mu_1}(x_1) \cdots A^{tr}_{\mu_n}(x_n)|\Phi_{tr} >, \tag{2.4.258}
$$

as stated above.

To summarize: The indefinite metric space, $\mathcal{V}$, is constructed from its vacuum state $|0>$, $a_{(\lambda)}(\vec{k})|0> = 0$ for all $\vec{k}$ and λ by the action of creation operators $a^\dagger_{(\lambda)}(\vec{k})$. This space has negative norm states in it corresponding to odd numbers of scalar photons.

As well it contains unphysical longitudinal photon states. The physical subspace of states in $\mathcal{V}$, $\mathcal{H}_{Phys}$, is obtained by considering states that obey the Gupta-Bleuler subsidiary condition $\partial_\mu A^{\mu +}(x)|\Phi >= 0$. The set of all such states forms a pre-Hilbert space. By separating out the zero norm subspace of states $\{(\mathcal{G}^\dagger(\vec{k}_1)\cdots\mathcal{G}^\dagger(\vec{k}_n)|\Phi >)\}$, that is by considering equivalence classes of states, we find the states in a positive norm Hilbert space $|\Phi >= |\Phi_{tr}> |\hat{\Phi}>$ where $|\Phi_{tr}>$ is made from transverse photons only and

$$|\hat{\Phi} >= [1 + \int \frac{d^3k}{(2\pi)^3 2\omega_k} c(\vec{k})\mathcal{G}^\dagger(\vec{k}) + \cdots$$

$$+ \int \frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}} \cdots \int \frac{d^3k_n}{(2\pi)^3 2\omega_{k_n}} c(\vec{k}_1, \ldots, \vec{k}_n)\mathcal{G}^\dagger(\vec{k}_1)\cdots\mathcal{G}^\dagger(\vec{k}_n) + \cdots]|0> . \quad (2.4.259)$$

Then we have

$$< \Psi|A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)|\Phi >=< \Psi_{tr}|A^{tr}_{\mu_1}(x_1)\cdots A^{tr}_{\mu_n}(x_n)|\Phi_{tr}> \quad (2.4.260)$$

and in particular $< \Psi|\mathcal{P}^\mu|\Phi >=< \Psi_{tr}|\mathcal{P}^\mu_{tr}|\Phi_{tr}>$ as in the Coulomb gauge. The subsidiary condition guarantees that the scalar and longitudinal photons are unobservable. The states $|\Phi>$ and $|\Phi_{tr}>$ are physically indistinguishable. Mathematically they correspond to different choices of gauge Λ with $\partial^2\Lambda = 0$. Thus, we see the price we must pay for keeping Lorentz invariance manifest. The vector field A^μ creates states which are unphysical as well as physical. The Gupta- Bleuler subsidiary condition defines in a covariant manner the physical subspace of states. As we have seen it is more convenient in intermediate steps to calculate in a Lorentz invariant manner. Hence, we will only work in such gauges from now on, the physical quantities, i.e. the physical state matrix elements of observables, will be independent of gauge as guaranteed by the Gupta-Bleuler procedure.

We are now in a position to calculate the covariant commutators, Wightman functions and time ordered products. In the Feynman gauge

$$[A^\mu(x), A^\nu(y)] = [A^{\mu +}(x), A^{\nu -}(y)] + [A^{\mu -}(x), A^{\nu +}(y)]. \quad (2.4.261)$$

As usual we find

$$\begin{aligned}[A^{\mu +}(x), A^{\nu -}(y)] &= -g^{\mu\nu} i\Delta^+(x-y)\\ [A^{\mu +}(x), A^{\nu -}(y)] &= -g^{\mu\nu} i\Delta^-(x-y)\end{aligned} \quad (2.4.262)$$

so that

$$[A^\mu(x), A^\nu(y)] = -g^{\mu\nu} i\Delta(x-y), \quad (2.4.263)$$

which is clearly covariant. Similarly we find the Wightman two point function

$$\begin{aligned} < 0|A^\mu(x)A^\nu(y)|0> &=< 0|A^{\mu+}(x)A^{\nu-}(y)|0> \\ &= -g^{\mu\nu} i\Delta^+(x-y) \end{aligned} \tag{2.4.264}$$

and we have the general result

$$< 0|A^\mu(x)A^{\mu_1}(x_1)\cdots A^{\mu_n}(x_n)|0>$$

$$= \sum_{i=1}^{n} -ig^{\mu\mu_i} i\Delta^+(x-x_i) < 0|A^{\mu_1}(x_1)\cdots \not{A}^{\mu_i}(x_i)\cdots A^{\mu_n}(x_n)|0> . \tag{2.4.265}$$

Hence we have that

$$< 0|A^{\mu_1}(x_1)\cdots A^{\mu_n}(x_n)|0>$$

$$= \delta_{n,\ even} \sum_P < 0|A^{\mu_{i_1}}(x_{i_1})A^{\mu_{j_1}}(x_{j_1})|0> \cdots < 0|A^{\mu_{i_{\frac{n}{2}}}}(x_{i_{\frac{n}{2}}})A^{\mu_{j_{\frac{n}{2}}}}(x_{j_{\frac{n}{2}}})|0>, \tag{2.4.266}$$

where the sum is over all permutations P of $(1,\cdots,n)$ into $\frac{n}{2}$ ordered sets $(i_1,j_1),\cdots,(i_{\frac{n}{2}},j_{\frac{n}{2}})$ with the ordering such that $i_a < j_a$ and $i_1 < i_2 < \cdots < i_{\frac{n}{2}}$. Also we have defined

$$\delta_{n,\ even} = \begin{cases} 0 & \text{if } n = odd \\ 1 & \text{if } n = even. \end{cases}$$

Finally, we may evaluate the time ordered functions. Immediately we have that

$$< 0|TA^\mu(x)A^\nu(y)|0 >= -g^{\mu\nu}\Delta_F(x-y) \tag{2.4.267}$$

and in general that

$$< 0|TA^{\mu_1}(x_1)\cdots A^{\mu_n}(x_n)|0>$$

$$= \delta_{n,\ even} \sum_P < 0|TA^{\mu_{i_1}}(x_{i_1})A^{\mu_{j_1}}(x_{j_1})|0> \cdots < 0|TA^{\mu_{i_{n/2}}}(x_{i_{n/2}})A^{\mu_{j_{n/2}}}(x_{j_{n/2}})|0> \tag{2.4.268}$$

where the sum is over the same permutations P of $(1,\cdots,n)$ as above. Quite generally we can find the time ordered functions in other than the Feynman gauge, $\alpha = 1$. Recall that A^μ obeys the general Euler-Lagrange equation

$$\left(\partial^2 g^{\mu\nu} + \left(\frac{1-\alpha}{\alpha}\right)\partial^\mu\partial^\nu\right)A_\nu(x) = 0. \tag{2.4.269}$$

So applying this to the time ordered functions, the time derivatives of $\theta(x^0 - y^0)$ and $\theta(y^0 - x^0)$ give the equal time commutation relations

$$
\begin{aligned}
&(\partial_x^2 g^{\mu\nu} + \left(\frac{1-\alpha}{\alpha}\right)\partial_x^\mu \partial_x^\nu) T A_\nu(x) A^\lambda(y) \\
&= \delta(x^0 - y^0)[\dot{A}^\mu(x), A^\lambda(y)] + \left(\frac{1-\alpha}{\alpha}\right) g^{\mu 0}\delta(x^0 - y^0)[\dot{A}^0(x), A^\lambda(y)] \\
&= i g^{\mu\lambda}[1 + (\alpha - 1) g^{\mu 0}]\delta^4(x-y) + \left(\frac{1-\alpha}{\alpha}\right) g^{\mu 0} i g^{0\lambda}\alpha\delta^4(x-y) \\
&= i g^{\mu\lambda}\delta^4(x-y).
\end{aligned}
\tag{2.4.270}
$$

Thus we secure

$$
(\partial_x^2 g^{\mu\nu} + \left(\frac{1-\alpha}{\alpha}\right)\partial_x^\mu \partial_x^\nu) < 0|T A_\nu(x) A^\lambda(y)|0 >= i g^{\mu\lambda}\delta^4(x-y). \tag{2.4.271}
$$

Now Fourier transforming with the definition

$$
< 0|T A_\nu(x) A^\lambda(y)|0 >= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \tilde{\Delta}_{F\nu}{}^\lambda(k), \tag{2.4.272}
$$

since $< 0|T A_\nu(x) A^\lambda(y)|0 >=< 0|T A_\nu(x-y) A^\lambda(0)|0 >$ by translation invariance. So the differential equation implies

$$
(-k^2 g^{\mu\nu} - \left(\frac{1-\alpha}{\alpha}\right) k^\mu k^\nu)\tilde{\Delta}_{F\nu}{}^\lambda(k) = i g^{\mu\lambda}. \tag{2.4.273}
$$

The four-by-four matrix on the left hand side can be easily inverted since we can write it as projectors. We define the (Minkowski) transverse projector as

$$
P_T^{\mu\nu} \equiv g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}
$$

and the (Minkowski) longitudinal projector as

$$
P_L^{\mu\nu} \equiv \frac{k^\mu k^\nu}{k^2}. \tag{2.4.274}
$$

These are projectors since

$$
P_T^{\mu\nu} + P_L^{\mu\nu} = g^{\mu\nu}
$$

$$P_T^{\mu\nu} P_{T\nu\lambda} = P_{T\lambda}^{\mu}$$

$$P_L^{\mu\nu} P_{L\nu\lambda} = P_{L\lambda}^{\mu}$$

$$P_T^{\mu\nu} P_{L\nu\lambda} = 0. \tag{2.4.275}$$

The propagator equation is then

$$(-k^2 P_T^{\mu\nu} - k^2 \frac{1}{\alpha} P_L^{\mu\nu}) \tilde{\Delta}_{F\nu}{}^{\lambda}(k) = i g^{\mu\lambda}. \tag{2.4.276}$$

The beauty of projectors is that any matrix written as a sum over them

$$M^{\mu\nu} = a P_T^{\mu\nu} + b P_L^{\mu\nu} \tag{2.4.277}$$

has an inverse, $M^{-1\mu\nu} M_{\nu}{}^{\lambda} = g^{\mu\lambda}$, given by

$$M^{-1\mu\nu} = \frac{1}{a} P_T^{\mu\nu} + \frac{1}{b} P_L^{\mu\nu}. \tag{2.4.278}$$

Hence, we see that the propagator in momentum space becomes

$$\tilde{\Delta}_F^{\mu\nu}(k) = -\frac{i}{k^2} P_T^{\mu\nu} - \frac{i\alpha}{k^2} P_L^{\mu\nu}. \tag{2.4.279}$$

We see the difficulty of not adding the gauge fixing Stueckelberg term $\alpha \to \infty$. We cannot invert the field equations to find the propagator. Thus we have

$$\tilde{\Delta}_F^{\mu\nu}(k) = -\frac{i}{k^2} g^{\mu\nu} - \frac{i}{k^2}(\alpha - 1)\frac{k^\mu k^\nu}{k^2}. \tag{2.4.280}$$

Of course we must have the correct boundary conditions for the propagator, that is, we must go around the poles at $k^0 = \pm\omega_k$ in the usual way. Consequently,

$$< 0|TA^\mu(x) A^\nu(y)|0 >= \Delta_F^{\mu\nu}(x-y)$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} [\frac{-i}{k^2 + i\epsilon} g^{\mu\nu} - \frac{i}{k^2 + i\epsilon}(\alpha - 1)\frac{k^\mu k^\nu}{k^2 + i\epsilon}]. \tag{2.4.281}$$

Notice at $\alpha = 1$ we recover the familiar Feynman gauge results

$$< 0|TA^\mu(x) A^\nu(y)|0 > |_{\alpha=1} = -g^{\mu\nu} \Delta_F^{\mu\nu}(x-y) = -g^{\mu\nu} D_F(x-y). \tag{2.4.282}$$

(The fact that the $(\alpha-1)$ term has $1/(k^2+i\epsilon)^2$ is not a trivial matter to check.) Again the general results are obtained for the n-point time ordered functions

$$< 0|TA^{\mu}(x)A^{\mu_1}(x_1)\cdots A^{\mu_n}(x_n)|0>$$

$$= \sum_{i=1}^{n} < 0|TA^{\mu}(x)A^{\mu_i}(x_i)|0 >< 0|TA^{\mu_1}(x_1)\cdots \not{A}^{\mu_i}(x_i)\cdots A^{\mu_n}(x_n)|0 > . \qquad (2.4.283)$$

Hence, Wick's theorem has the usual form

$$< 0|TA^{\mu_1}(x_1)\cdots A^{\mu_n}(x_n)|0>$$

$$= \delta_{n.\ even} \sum_{P} < 0|TA^{\mu_{i_1}}(x_{i_1})A^{\mu_{j_1}}(x_{j_1})|0 > \cdots < 0|TA^{\mu_{i_{n/2}}}(x_{i_{n/2}})A^{\mu_{j_{n/2}}}(x_{j_{n/2}})|0 > \qquad (2.4.284)$$

where again the sum is over the permutations P of $(1,\cdots,n)$ defined above. The field equations become the time ordered functions equations of motion

$$(\partial_x^2 g^{\mu\nu} + \left(\frac{1-\alpha}{\alpha}\right)\partial_x^{\mu}\partial_x^{\nu}) < 0|TA_{\nu}(x)A^{\mu_1}(x_1)\cdots A^{\mu_n}(x_n)|0>$$

$$= \sum_{i=1}^{n} ig^{\mu\mu_i}\delta^4(x-x_i) < 0|TA^{\mu_1}(x_1)\cdots \not{A}^{\mu_i}(x_i)\cdots A^{\mu_n}(x_n)|0 > . \qquad (2.4.285)$$

We are now in a position to introduce interactions amongst our particles.