CHAPTER 2.
LAGRANGIAN QUANTUM FIELD THEORY

§2.1 GENERAL FORMALISM

In quantum field theory we will consider systems with an infinite number of quantum mechanical dynamical variables. As a motivation and guide for the many definitions and procedures to be discussed, we will use concepts and techniques from classical field theory, the classical mechanics of infinitely many degress of freedom. The quantum fields will in addition become operators in Hilbert space and, as we will see, not every "classical" manipulation will make sense. We will have to discover the modifications to our theory needed to define a consistent quantum field theory in a sort of give and take process. We begin by recalling the basic tennants of classical field theory. In general we will consider a continuous system described by several classical fields $\phi_r(x), r = 1, 2, ...N$. (In general we will denote classical fields by lower case letters and quantum fields by the upper case of the same letters.) The index r can label the different components of the same function such as the components of the vector potential $\vec{A}(x)$ or the index can label two or more sets of completely independent fields like the components of the vector potential and the components of the gravitational field $g_{\mu\nu}(x)$. Also $\phi_r(x)$ can be complex, in which case, ϕ_r and ϕ_r^* can be considered independent or the complex fields can be written in terms of real and imaginary parts which then can be treated as independent. The dynamical equations for the time evolution of the fields, the so called field equations or equations of motion, will be assumed to be derivable from Hamilton's variational principle for the action

$$S(\Omega) = \int_\Omega d^4x \mathcal{L}(\phi, \partial_\mu \phi_r) \tag{2.1.1}$$

where Ω is an arbitrary volume in space-time and $\mathcal{L}$ is the Langrangian density which is assumed to depend on the fields and their first dervatives $\partial_\mu \phi_r$. Hamilton's principle states that S is stationary

$$\delta S(\Omega) = 0 \tag{2.1.2}$$

under variations in the fields

$$\phi_r(x) \to \phi_r(x) + \delta\phi_r(x) = \phi_r'(x) \tag{2.1.3}$$

which vanish on the boundary $\partial\Omega$ of the volume Ω,

$$\delta\phi_r(x) = 0 \quad \text{on} \quad \partial\Omega. \tag{2.1.4}$$

The physical field configuration in the space-time volume is such that the action S remains invariant under small variations in the fields for fixed boundary conditions. The calculation of the variation of the action yields the Euler- Lagrange equations of motion for the fields

$$\begin{aligned} \delta S(\Omega) &= \int_\Omega d^4x \delta\mathcal{L} \\ &= \int_\Omega d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi_r}\delta\phi_r + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_r}\delta\partial_\mu\phi_r \right) \end{aligned}. \tag{2.1.5}$$

But

$$\begin{aligned} \delta\partial_\mu\phi_r &= \partial_\mu\phi'_r - \partial_\mu\phi_r \\ &= \partial_\mu(\phi'_r - \phi_r) = \partial_\mu\delta\phi_r. \end{aligned} \tag{2.1.6}$$

Thus, performing an integration by parts we have

$$\begin{aligned} \delta S(\Omega) &= \int_\Omega d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_r} \right) \delta\phi_r \\ &+ \int_\Omega \left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_r}(\partial_\mu\delta\phi_r) + \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_r}\delta\phi_r \right] \\ &= \int_\Omega d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_r} \right] \delta\phi_r + \int_\Omega d^4x \partial_\mu [\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_r}\delta\phi_r]. \end{aligned} \tag{2.1.7}$$

The last integral yields a surface integral over $\partial\Omega$ by Gauss' divergence theorem

$$\int_\Omega d^4x \partial_\mu F^\mu = \int_{\partial\Omega} d^3\sigma_\mu F^\mu \tag{2.1.8}$$

but $\delta\phi_r = 0$ on $\partial\Omega$ hence the integral is zero. So

$$\delta S(\Omega) = \int_\Omega d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_r} \right] \delta\phi_r = 0 \tag{2.1.9}$$

by Hamilton's principle. Since $\delta\phi_r$ is arbitrary inside Ω, the integrand vanishes

$$\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_r} = 0 \qquad for \quad r = 1, ..., N. \tag{2.1.10}$$

These Euler-Lagrange equations are the equations of motion for the fields ϕ_r.

According to the canonical quantization procedure to be developed, we would like to deal with generalized coordinates and their canonically conjugate momenta so that we may impose the quantum mechanical commutation relations between them. Hence we would like to Legendre transform our Lagrangian system to a Hamiltonian formulation. We can see how to introduce the appropriate dynamical variables for this transformation by exhibiting the classical mechanical or particle analogue for our classical field theory. This can be done a few ways, in the introduction we introduced discrete conjugate variables by Fourier transforming the space dependence of the fields into momentum space then breaking momentum space into cells. We could also expand in terms of a complete set of momentum space functions to achieve the same result. Here let's be more direct and work in space-time. For each point in space the fields are considered independent dynamical variables with a given time dependence. We imagine approaching this continuum limit by first dividing up three-space into cells of volume $\delta\vec{x}_i$, $i = 1, 2, \cdots$. Then we approximate the values of the field in each cell by its average over the cell

$$\bar{\phi}_r(\vec{x}_i, t) = \frac{1}{\delta\vec{x}_i} \int_{\delta\vec{x}_i} d^3x \phi_r(\vec{x}, t) \approx \phi_r(\vec{x}_i, t). \tag{2.1.11}$$

This is roughly the value of $\phi_r(x, t)$ say at the center of the cell $\vec{x} = \vec{x}_i$.

Our field system is now is described by a discrete set of generalized coordinates,

$$q_{ri}(t) = \bar{\phi}_r(\vec{x}_i, t) \approx \phi_r(\vec{x}_i, t), \tag{2.1.12}$$

the field variables evaluated at the lattice sites, and their generalized velocities

$$\dot{q}_{ri}(t) = \dot{\bar{\phi}}_r(\vec{x}_i, t) \approx \dot{\phi}_r(\vec{x}_i, t). \tag{2.1.13}$$

Since $\mathcal{L}$ depends on $\vec{\bigtriangledown}\phi_r$ also, we define this as the difference in the field values at neighboring sites. Thus, $\mathcal{L}(\vec{x}, t)$ in the i^{th} cell, denoted by $\mathcal{L}_i$, is a function of q_{ri}, $\dot{q}_{ri}$ and q'_{ri} the coordinates of the nearest neighbors,

$$\mathcal{L}_i = \mathcal{L}_i(q_{ri}, \dot{q}_{ri}, q'_{ri}). \tag{2.1.14}$$

Hence, the Lagrangian is the spatial integral of the Langrangian density

$$\mathrm{L}(t) = \int d^3x \mathcal{L} = \sum_i \delta\vec{x}_i \mathcal{L}_i(q_{ri}, \dot{q}_{ri}, q'_{ri}). \tag{2.1.15}$$

and we have a mechanical system with a countable infinity of generalized coordinates.

We can now introduce in the usual way the momenta p_{ri} canonically conjugate to the coordinates q_{ri}

$$p_{ri}(t) = \frac{\partial L}{\partial \dot{q}_{ri}(t)} = \sum_j \delta\vec{x}_j \frac{\partial \mathcal{L}_j(t)}{\partial \dot{q}_{ri}(t)} = \delta\vec{x}_i \frac{\partial \mathcal{L}_i(t)}{\partial \dot{q}_{ri}(t)} \tag{2.1.16}$$

and the Legendre transformation to the Hamiltonian is

$$\begin{aligned} H(q_{ri}, p_{ri}) = H &= \sum p_{ri}\dot{q}_{ri} - L \\ &= \sum_i \delta\vec{x}_i \left[\frac{\partial \mathcal{L}_i(t)}{\partial \dot{q}_{ri}(t)} \dot{q}_{ri}(t) - \mathcal{L}_i \right]. \end{aligned} \tag{2.1.17}$$

The Euler-Lagrange equations are now replaced by Hamilton's equations

$$\frac{\partial H}{\partial q_{ri}} = -\dot{p}_{ri} \quad , \qquad \frac{\partial H}{\partial p_{ri}} = \dot{q}_{ri} \quad . \tag{2.1.18}$$

We define the momentum field canonically conjugate to the field coordinate by

$$\pi_r(\vec{x}_i, t) = \frac{\partial \mathcal{L}_i(t)}{\partial \dot{q}_{ri}(t)} = \frac{\partial \mathcal{L}_i}{\partial \dot{\phi}_r(\vec{x}_i, t)}. \tag{2.1.19}$$

Then

$$\begin{aligned} p_{ri}(t) &= \pi_r(\vec{x}_i, t)\delta\vec{x}_i \\ H &= \sum_i \delta\vec{x}_i [\pi_r(\vec{x}_i, t)\dot{\phi}_r(\vec{x}_i, t) - \mathcal{L}_i]. \end{aligned} \tag{2.1.20}$$

Going over to the continuum limit $\delta\vec{x}_i \to 0$ the fields go over to the continuum values $\phi_r(\vec{x}, t)$and the conjugate momentum fields to theirs $\pi_r(\vec{x}, t)$ (recall π_r is a function of $\phi_r, \partial_\mu\phi_r$) and $\mathcal{L}_i \to \mathcal{L}$. So that

$$\pi_r(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r(x)}. \tag{2.1.21}$$

The Hamiltonian corresponding to the Lagrangian L is

$$H = \int d^3x \mathcal{H} \tag{2.1.22}$$

with Hamiltonian density

$$\mathcal{H} = \pi_r(x)\dot{\phi}_r(x) - \mathcal{L}(\phi_r, \partial_\mu \phi_r) \tag{2.1.23}$$

and Hamilton's equations are

$$\frac{\delta \mathrm{H}}{\delta \phi_r(x)} = -i\dot{\pi}_r, \qquad \frac{\delta \mathrm{H}}{\delta \pi_r(x)} = \dot{\phi}_r(x). \tag{2.1.24}$$

where $\frac{\delta}{\delta \phi_r(x)}$ and $\frac{\delta}{\delta \pi_r(x)}$ denote functional derivatives defined by the continuum limit of, for example,

$$\frac{1}{\delta \vec{x}_i} \frac{\partial \mathrm{H}}{\partial \phi_r(\vec{x}_i, t)} = \frac{\delta H}{\delta \phi_r(x)}. \tag{2.1.25}$$

In terms of the discrete coordinates and conjugate momenta we can now apply the quantization rules of Quantum Mechanics to obtain a quantum field theory. That is, we start with a Lagranian density in terms of products of quantum field operators (in what follows we will use capital letters to denote quantum field theoretic quantities as a reminder that they are quantum mechanical operators)

$$\mathcal{L} = \mathcal{L}(\Phi_r, \partial_\mu \Phi_r). \tag{2.1.26}$$

(Even this first step is non-trivial, since products of fields are not always well defined due to their distributional nature. We will refine this step later, but for now we continue.) Since now Φ is a quantum operator we face our first problem in simply carrying over classical operations to the quantum case. Specifically it is no longer necessary that the variation in the fields, $\delta\Phi$, commute with the field operators and derivatives of the field operators. (If one assumes that the variations of both the fields and their conjugate momenta are independent then their CCR, to be given, imply that they commute.) Hence a priori it is not certain that the variation of the Lagrangian can be factorized into the Euler-Lagrange equations times $\delta\Phi$ and an action principle obtained. For example, the variation of Φ^2 is $\delta\Phi\Phi + \Phi\delta\Phi$, for arbitrary variation this is not necessarily $2\Phi\delta\Phi$. Since the action principle was used to derive the Euler-Lagrange field equations which describe the dynamical space-time evolution of the fields we must hypothesize these instead. That is we will consider field theories for which the Euler-Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial \Phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_r} = 0 \tag{2.1.27}$$

are by fiat the field equations. Here the derivatives of $\mathcal{L}$ with respect to the quantum fields (and for quantum field derivatives in general) are defined just as a classical derivative where the fields in a product are kept in their original order as the chain rule is applied. For instance, $\frac{\partial}{\partial\Phi}\Phi^4 = 4\Phi^3$. On the other hand if we differentiate a product of fields like $\Phi\dot{\Phi}\Phi$ with respect to Φ the operators must be kept in their original order,

$$\begin{aligned}\frac{\partial}{\partial\Phi}\Phi\dot{\Phi}\Phi &= \dot{\Phi}\Phi + \Phi\dot{\Phi} = 2\Phi\dot{\Phi} + [\dot{\Phi}, \Phi] \\ &\neq 2\Phi\dot{\Phi}.\end{aligned}$$

As we see the original term $\Phi\dot{\Phi}\Phi = \Phi^2\dot{\Phi} + \Phi[\dot{\Phi}, \Phi]$, hence ignoring the commutator in the derivative is like throwing away the linear Φ term in the original product. Hence we will view the Lagrangian as a short hand way of summarizing the dynamics of the fields, which is defined to be the Euler-Lagrange equations formally derived from the Lagrangian. The approach, as we will see when we discuss specific models, will be to define products of quantum fields, called normal products, with the property that operator ordering within the normal product is irrelevant and that the field equations are the normal product of the fields in the Euler-Lagrange equations. In general we will ignore these ordering questions at first and use the definition of operators and field equations suggested by the classical theory. So when a specific composite operator, like H or $\mathcal{P}^\mu$, is defined we will keep the order fixed according to that definition and proceed to study the consequences. When necessary we will return to the classical definition and re-define the ordering of the composite operators in a quantum field theoretically consistent manner. This will be done in the framework of specific models. In fact some fields obey equations of motion which are not derivable from local Lagrangians, their field equations are said to have anomalous terms in them. The form of anomalies in QFT is an extremely important subject since it deals with purely quantum mechanical corrections to field equations. The discussion of such models will be taken up in the study of renormalization which is beyond the scope of these notes.

The quantization rules for the operators Φ_r are stated in terms of equal time commutation relations for the fields and the conjugate "momentum" fields, defined in an analogous procedure as followed in the classical case. We divide three-space into cells $\delta\vec{x}_i, i = 1, 2, 3, \cdots$ and replace the quantum fields with their averages in the cell

$$\bar{\Phi}_r(\vec{x}_i,t)=\frac{1}{\delta\vec{x}_i}\int_{\delta\vec{x}_i}d^3x\Phi_r(\vec{x},t)\approx\Phi_r(\vec{x}_i,t). \tag{2.1.28}$$

These coordinate operators are denoted

$$Q_{ri}(t)\equiv\bar{\Phi}_r(\vec{x}_i,t)\approx\Phi_r(\vec{x}_i,t) \tag{2.1.29}$$

and the velocity operators

$$\dot{Q}_{ri}=\dot{\Phi}_r(\vec{x}_i,t). \tag{2.1.30}$$

Then again

$$\mathcal{L}_i=\mathcal{L}_i(Q_{ri},\dot{Q}_{ri},Q'_{ri}) \tag{2.1.31}$$

where we defined the spatial derivatives by their nearest neighbor differences, hence, the Q'_{ri} in $\mathcal{L}_i$. The Lagrangian is

$$L(t)=\int d^3x\mathcal{L}=\sum_i\delta\vec{x}_i\mathcal{L}(Q_{ri},\dot{Q}_{ri},Q'_{ri}). \tag{2.1.32}$$

The conjugate momentum operators are defined similarly as

$$P_{ri}(t)\equiv\frac{\partial L(t)}{\partial\dot{Q}_{ri}(t)}=\sum_j\delta\vec{x}_j\frac{\partial\mathcal{L}_j(t)}{\partial\dot{Q}_{ri}(t)}=\delta\vec{x}_i\frac{\partial\mathcal{L}_i}{\partial\dot{Q}_{ri}(t)}\equiv\Pi_r(\vec{x}_i,t)\delta\vec{x}_i \tag{2.1.33}$$

and the Hamiltonian

$$\begin{aligned}H(Q_{ri},P_{ri})&\equiv\sum_i P_{ri}\dot{Q}_{ri}-L\\&=\sum_i\delta\vec{x}_i\left[\frac{\partial\mathcal{L}_i}{\partial\dot{Q}_{ri}}\dot{Q}_{ri}-\mathcal{L}_i\right]\\&=\sum_i\delta\vec{x}_i\left[\Pi_r(\vec{x}_i,t)\dot{\Phi}_r(\vec{x}_i,t)-\mathcal{L}_i\right]\end{aligned}. \tag{2.1.34}$$

(As in quantum mechanics questions of operator ordering may arise here. For example, if one has classically pq (=qp) in an expression, should this be replaced by PQ, QP or $\frac{1}{2}$(PQ + QP) in the quantum mechanical case? We will discuss operator ordering in more detail in the context of specific models later.) Going over to the continuum limit with $\Pi_r(\vec{x}_i,t)\to\Pi_r(\vec{x},t)$ we find

$$\Pi_r(x)=\frac{\partial\mathcal{L}}{\partial\dot{\Phi}_r(x)} \tag{2.1.35}$$

and

$$H = \int d^3x\mathcal{H} \tag{2.1.36}$$

with the Hamiltonian density

$$\mathcal{H} = \Pi_r(x)\dot{\Phi}_r(x) - \mathcal{L}(\Phi_r, \partial_\mu\Phi_r). \tag{2.1.37}$$

Since $Q_{ri}(t)$ and $P_{ri}(t)$ depend on time they are Heisenberg operators and we demand that they obey the usual quantum mechanical equal time commutation relations

$$\begin{aligned}
[Q_{ri}(t), P_{sj}(t)] &= i\hbar\delta_{rs}\delta_{ij} \\
[Q_{ri}(t), Q_{sj}(t)] = 0 &= [P_{ri}(t), P_{sj}(t)]\,.
\end{aligned} \tag{2.1.38}$$

Furthermore, they obey the Heisenberg equations of motion (these are the quantum analogues of the classical Poisson bracket formulation of the Hamilton equations of motion), as can be explicitly verified in each case,

$$\begin{aligned}
[H, Q_{ri}(t)] &= -i\hbar\frac{\partial Q_{ri}}{\partial t} \\
[H, P_{ri}(t)] &= -i\hbar\frac{\partial P_{ri}}{\partial t}
\end{aligned} \tag{2.1.39}$$

that is,

$$\begin{aligned}
[Q_{ri}(t), \Pi_s(\vec{x}_j, t)] &= i\hbar\delta_{rs}\frac{\delta_{ij}}{\delta\vec{x}_j} \\
[Q_{ri}, Q_{sj}(t)] = 0 &= [\Pi_r(\vec{x}_i, t), \Pi_s(\vec{x}_j, t)]
\end{aligned} \tag{2.1.40}$$

and

$$\begin{aligned}
[H, Q_{ri}(t)] &= -i\hbar\frac{\partial Q_{ri}(t)}{\partial t} \\
[H, \Pi_r(\vec{x}_i, t)] &= -i\hbar\frac{\partial \Pi_r(\vec{x}_i, t)}{\partial t}.
\end{aligned} \tag{2.1.41}$$

Going over to the continuum limit we find that

$$\frac{\delta_{ij}}{\delta\vec{x}_j} = \delta^3(\vec{x} - \vec{y}) \tag{2.1.42}$$

where $\vec{x} \in \delta\vec{x}_i$ and $\vec{y} \in \delta\vec{x}_j$ since

$$\begin{aligned}\int d^3y\delta^3(\vec{x}-\vec{y})f(\vec{y}) &= f(\vec{x}) \\ &= \sum_j \delta\vec{x}_j \frac{\delta_{ij}}{\delta\vec{x}_j} f(\vec{x}_j)\,, \\ &= f(\vec{x}_i) = f(\vec{x})\end{aligned} \tag{2.1.43}$$

so that the quantum fields obey the ETCR

$$\begin{aligned}[\Phi_r(\vec{x},t),\Pi_s(\vec{y},t)] &= i\hbar\delta_{rs}\delta^3(\vec{x}-\vec{y}) \\ [\Phi_r(\vec{x},t),\Phi_s(\vec{y},t)] &= 0 = [\Pi_r(\vec{x},t),\Pi_s(\vec{y},t)]\end{aligned} \tag{2.1.44}$$

and the Heisenberg equations of motion

$$\begin{aligned}[H,\Phi_r(x)] &= -i\hbar\frac{\partial\Phi_r(x)}{\partial t} \\ [H,\Pi_r(x)] &= -i\hbar\frac{\partial\Pi_r(x)}{\partial t}.\end{aligned} \tag{2.1.45}$$

Note that the discretized version of the QFT yields the mechanical interpretation of QFT as an infinite collection of quantum mechanical generalized coordinates.

Before proceeding further let's consider an example with which we are already familiar, the noninteracting, Hermitian, scalar (spin zero) field with mass m. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - m^2\Phi^2. \tag{2.1.46}$$

The Euler-Lagrange equations describing the time evolution of the field are

$$\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi} = 0 = -(\partial^2+m^2)\Phi. \tag{2.1.47}$$

The quantization rules are based on the Hamiltonian approach so we introduce the momentum field canonically conjugate to Φ by

$$\Pi(x) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\Phi}(x)}. \tag{2.1.48}$$

Now

$$\mathcal{L} = \frac{1}{2}\dot{\Phi}\dot{\Phi} - \frac{1}{2}\vec{\bigtriangledown}\Phi\cdot\vec{\bigtriangledown}\Phi - \frac{1}{2}m^2\Phi^2 \tag{2.1.49}$$

hence

$$\Pi(x) = \dot{\Phi}(x). \tag{2.1.50}$$

The Hamiltonian density is

$$\mathcal{H} \equiv \Pi(x)\dot{\Phi}(x) - \mathcal{L}(x) = \dot{\Phi}\dot{\Phi} - \frac{1}{2}\dot{\Phi}\dot{\Phi} + \frac{1}{2}\vec{\bigtriangledown}\Phi \cdot \vec{\bigtriangledown}\Phi + \frac{1}{2}m^2\Phi^2$$

$$= \frac{1}{2}\dot{\Phi}\dot{\Phi} + \frac{1}{2}\vec{\bigtriangledown}\Phi \cdot \vec{\bigtriangledown}\Phi + \frac{1}{2}m^2\Phi^2. \tag{2.1.51}$$

The canonical quantization rules are

$$[\Pi(\vec{x},t), \Phi(\vec{y},t)] = -i\hbar\delta^3(\vec{x}-\vec{y}) \tag{2.1.52}$$

which yield the ETCR

$$\left[\dot{\Phi}(\vec{x},t), \Phi(\vec{y},t)\right] = -i\hbar\delta^3(\vec{x}-\vec{y}) \tag{2.1.53}$$

or, with the help of an equal time delta function, we write this as

$$\delta(x^0-y^0)\left[\dot{\Phi}(x), \Phi(y)\right] = -i\hbar\delta^4(x-y) \tag{2.1.54}$$

and

$$\delta(x^0-y^0)\left[\Phi(x), \Phi(y)\right] = 0 = \delta(x^0-y^0)\left[\dot{\Phi}(x), \dot{\Phi}(y)\right]. \tag{2.1.55}$$

The Heisenberg equations of motion are

$$[H, \Phi(x)] = -i\hbar\dot{\Phi}; \quad \left[H, \dot{\Phi}(x)\right] = -i\hbar\ddot{\Phi}(x). \tag{2.1.56}$$

Using the canonical commutation relations we can calculate

$$\delta(x^0-y^0)\left[\mathcal{H}(y), \dot{\Phi}(x)\right] = \delta(x^0-y^0)\frac{1}{2}\vec{\bigtriangledown}_y\Phi(y).\left[\vec{\bigtriangledown}_y\Phi(y), \dot{\Phi}(x)\right]$$

$$+\delta(x^0-y^0)\frac{1}{2}\left[\vec{\bigtriangledown}_y\Phi(y), \dot{\Phi}(x)\right].\vec{\bigtriangledown}_y\Phi(y)$$

$$+\delta(x^0-y^0)\left\{\frac{1}{2}m^2\Phi(y)\left[\Phi(y), \dot{\Phi}(x)\right] + \frac{1}{2}m^2\left[\Phi(y), \dot{\Phi}(x)\right]\Phi(y)\right\}$$

$$= \delta(x^0-y^0)(+i\hbar)\left\{\vec{\bigtriangledown}_y\Phi(y).\vec{\bigtriangledown}_y\delta^3(\vec{x}-\vec{y}) + m^2\Phi(y)\delta(\vec{x}-\vec{y})\right\}. \tag{2.1.57}$$

Now

$$H = \int d^3y\mathcal{H}(y) \tag{2.1.58}$$

so that

$$\delta(x^0 - y^0)\left[H, \dot{\Phi}(x)\right] = i\hbar\delta(x^0 - y^0)\left\{- \bigtriangledown^2 \Phi(x) + m^2\Phi(x)\right\}. \tag{2.1.59}$$

Furthermore, there is no explicit time dependence in our theory so H is a constant $[H, H] = \dot{H} = 0$. So we can integrate over y_0 to obtain for Heisenberg's equation of motion

$$\begin{aligned}\left[H, \dot{\Phi}(x)\right] &= i\hbar\left[- \bigtriangledown^2 \Phi(x) + m^2\Phi(x)\right] \\ &= -i\hbar\ddot{\Phi}\end{aligned}. \tag{2.1.60}$$

This implies

$$\ddot{\Phi} - \bigtriangledown^2\Phi + m^2\Phi(x) = 0 \tag{2.1.61}$$

or

$$(\partial^2 + m^2)\Phi = 0 \tag{2.1.62}$$

and we obtain the Euler-Lagrange equation of motion as we should.

Before Fourier transforming to momentum space to recapture the particle interpretation as discussed in the introduction, let us consider symmetries in quantum field theory. In particular we would like to relate the time translation operator $\mathcal{P}^0$ discussed in the quantum mechanics review to H above. Further since Φ and Π are dynamical degrees of freedom we would like a method for constructing all symmetry generators in terms of them. Specifically we would like to construct $\mathcal{P}^\mu$ and $\mathcal{M}^{\mu\nu}$ the generators of the Poincare' transformations since a subset of these operators will form a CSCO whose eigenstates will be the particle states of the theory. The Lagrangian formulation of field theory will be particularly useful for this procedure. However, first recall, from the Hamiltonian point of view symmetry transformations are related to operators that commute with the Hamiltonian, that is, are constants in time since the Heisenberg equation of motion for an operator Q(t) is

$$-i\hbar\frac{\partial Q(t)}{\partial t} = [H, Q(t)]\,. \tag{2.1.63}$$

If $\dot{Q} = 0$ this implies $[H, Q] = 0$ and Q describes an invariance of H. In general symmetries are divided into two types space-time symmetries and internal symmetries. Space-time symmetries are transformations of the coordinate system while internal symmetries do not involve the coordinate system only the fields. In both cases the invariance of the system under such symmetry transformations will lead to conservation laws.

Recall for symmetry transformations of our system which are represented by unitary or antiunitary operators we have a one-to-one correspondence among the states of our Hilbert space given by

$$|\psi' >= U|\psi > \tag{2.1.64}$$

that preserves transition probabilities

$$| < \psi'|\phi' > |^2 = | < \psi|\phi > |^2. \tag{2.1.65}$$

Furthermore, operator matrix elements transform as finite dimensional matrix representations of the symmetry group

$$< \psi'|A_r(x')|\phi' >= S_{rs} < \psi|A_s(x)|\phi > \tag{2.1.66}$$

that is,

$$U^\dagger A_r(x')U = S_{rs}A_s(x). \tag{2.1.67}$$

In particular if these transformations belong to some continuous group $\mathcal{G}$ with elements $g(\alpha)$ depending on the group parameters $\alpha_i,\ i = 1, 2, \cdots, A$, and $A = dim\mathcal{G}$, then for $g_1 \cdot g_2 = g$ we have that

$$U(g_1)U(g_2) = U(g) \tag{2.1.68}$$

where in general we can take the phase equal to one which we will assume here. Consequently,

$$U^\dagger(g_2)U^\dagger(g_1)A_r(x')U(g_1)U(g_2) = U^\dagger(g)A_r(x')U(g) \tag{2.1.69}$$

which implies

$$S_{rs}(g_1)S_{st}(g_2)A_t(x) = S_{rt}(g)A_t(x) \tag{2.1.70}$$

that is

$$S(g_1)S(g_2) = S(g). \tag{2.1.71}$$

S(g) is a finite dimensional matrix representation of the group multiplication law. For transformations continuously connected to the identity we have that

$$U(g(\alpha)) = e^{i\alpha_i Q^i}. \tag{2.1.72}$$

Since

$$U^{-1} = U^\dagger \quad , \tag{2.1.73}$$

implies

$$e^{-i\alpha_i Q^i} = e^{-i\alpha^i Q_i^\dagger} \quad , \tag{2.1.74}$$

we have

$$Q_i = Q_i^\dagger \quad , \tag{2.1.75}$$

Q_i is Hermitian and is known as the charge or generator of the transformations.

If U corresponds to an invariance of the system (the eigenstates are the same) then

$$H = U^\dagger H U \ , \tag{2.1.76}$$

the Hamiltonian is invariant. This implies that the transition probability of state $|\phi(t_0) >$ to evolve into state $|\psi(t) >$ at time t is unchanged for $|\phi'(t_0) >$ to evolve into $|\psi'(t) >$ (in the Schrödinger picture)

$$| < \psi(t)|U(t,t_0)|\phi(t_0) > |^2 = | < \psi'(t)|U(t,t_0)|\phi'(t_0) > |^2 \tag{2.1.77}$$

implies

$$U^\dagger(g)U(t,t_0)U(g) = U(t,t_0) \tag{2.1.78}$$

but $U(t,t_0) = e^{-iH(t-t_0)}$ so $[U(t,t_0), U(g)] = 0$. Conversely, if the equation of motion is invariant, $[U(t,t_0), U(g)] = 0$, then the Hamiltonian is invariant since

$$U(t_0 + \delta t, t_0) \approx 1 - iH\delta t \tag{2.1.79}$$

so that

$$[H, U(g)] = 0. \tag{2.1.80}$$

So the invariance of the law of motion implies a symmetry of the Hamiltonian and a symmetry of the Hamiltonian implies an invariance of the law of motion. For $U(g(\alpha)) = e^{i\alpha_i Q^i}$ and α_i small we find that

$$U^\dagger H U \equiv H + \delta H \tag{2.1.81}$$

with δH the variation of H. But

$$U^\dagger H U = H - i\alpha_i \left[Q^i, H\right] \tag{2.1.82}$$

thus

$$\delta H = -i\alpha_i \left[Q_i, H\right] \tag{2.1.83}$$

if H is invariant then $U^\dagger H U = H$, that is $\delta H = 0$, so $[Q^i, H] = 0$. By the Heisenberg equation of motion

$$i\hbar \dot{Q}^i = \left[Q^i, H\right] \tag{2.1.84}$$

so $\dot{Q}^i = 0$ is a constant of motion, Q^i is a conserved quantity.

The Lagrangian formulation of QFT allows for a straightfoward construction of the charges Q associated with symmetries of $\mathcal{L}$ (and hence H). This procedure is incorporated in Noether's Theorem. Rather than review Noether's Theorem in classical field theory and then repeat the process with appropriate changes in the quantum field theoretic case we will proceed directly to the quantum case. As always we are using the classical theory as a guide and we should expect some changes when the fields become operators. Especially we will need to clarify questions of operator ordering. Here our "give and take" approach will be used. We will construct currents and charges that do the job we want based on the classical manipulations. When we use the definitions in our specfic spin 0, $\frac{1}{2}$, 1 free field theories we will find that we must come back with a finer definition for some quantities so that the operator nature of the fields is taken into account consistently. As usual we start with a Lagrangian

$$\mathcal{L} = \mathcal{L}(\Phi_r, \partial_\mu \Phi_r) \tag{2.1.85}$$

and ask for $\mathcal{L}$ to be invariant under transformations belonging to the group $\mathcal{G}$

$$\mathcal{L}(x) = U^\dagger(g)\mathcal{L}(x')U(g) \equiv \mathcal{L} + \delta\mathcal{L}. \tag{2.1.86}$$

The middle term is evaluated at x' if $\mathcal{G}$ is a space- time symmetry group. Note this implies that the equations of motion in the transformed system are the same as in the original system since $\mathcal{L}$ is the same. Also

$$S' = \int d^4x' U^\dagger(g)\mathcal{L}(x')U(g)$$

$$= \int d^4x \left|\frac{\partial x'}{\partial x}\right| U^\dagger(g)\mathcal{L}(x')U(g) \tag{2.1.87}$$

we consider the Poincare' transformations only, so $|\frac{\partial x'}{\partial x}| = 1$ and if

$$U^\dagger \mathcal{L}(x')U = \mathcal{L}(x) \tag{2.1.88}$$

we have

$$S' = \int d^4x\mathcal{L}(x) = S \tag{2.1.89}$$

that the action is invariant.

Now under these transformations the fields vary

$$U^\dagger(g)\Phi_r(x')U(g) = S_{rs}(g)\Phi_s(x) \tag{2.1.90}$$

or for infinitesimal variations

$$U(g) = e^{i\alpha_i Q^i} \approx 1 + i\alpha_i Q^i \tag{2.1.91}$$

while

$$S_{rs}(g) = \left(e^{-\alpha_i D^i}\right)_{rs} \approx 1 - \alpha_i D^i_{rs} \tag{2.1.92}$$

where $D^i_{rs}$is a matrix. So inverting,

$$U(g)\Phi_r(x)U^\dagger(g) = S^{-1}_{rs}(g)\Phi_s(x') \tag{2.1.93}$$

implies that

$$\Phi_r(x) + i\alpha_i\left[Q^i, \Phi_r(x)\right] = \Phi_r(x') + \alpha_i D^i_{rs}\Phi_s(x). \tag{2.1.94}$$

Letting $x'^\mu = x^\mu + \alpha_i\delta^i x^\mu$, we obtain the intrinsic variation of Φ_r

$$-i\alpha_i\bar{\delta}^i\Phi_r(x) \equiv -i\left(U(g)\Phi_r(x)U^\dagger(g) - \Phi_r(x)\right)$$

$$= \alpha_i\left[Q^i, \Phi_r(x)\right]$$

$$= -i\alpha_i\left[\delta^i x^\mu\partial_\mu\Phi_r(x) + D^i_{rs}\Phi_s(x)\right]. \tag{2.1.95}$$

So we are now in a position to calculate the intrinsic variation of the Lagrangian

$$U^\dagger(g)\mathcal{L}(x)U(g) = \mathcal{L}(x) - i\alpha_i\left[Q^i, \mathcal{L}(x)\right]$$

$$\equiv \mathcal{L}(x) - \alpha_i \bar{\delta}^i \mathcal{L}(x). \tag{2.1.96}$$

Since $\mathcal{L}$ is a function of Φ_r and $\partial_\mu \Phi_r$, its intrinsic variation is just given by the chain rule operation

$$\bar{\delta}^i \mathcal{L} = \left(\bar{\delta}^i \Phi_r \frac{\partial}{\partial \Phi_r} \right) \mathcal{L} + \left(\bar{\delta}^i \partial_\mu \Phi_r \frac{\partial}{\partial \partial_\mu \Phi_r} \right) \mathcal{L} \tag{2.1.97}$$

where it is understood that the variations times the derivatives act as units as they are brought inside $\mathcal{L}$ to the field upon which they act. Here, as in what immediately follows, we will proceed formally as though $\bar{\delta}^i \Phi_r$ commutes with Φ_r and $\partial_\mu \Phi_r$. We may then use the manipulations of classical field theory. Eventually we will define the composite operators making up $\mathcal{L}$ by means of normal products of fields so that the fields within a product commute. Hence we can factor out the field variations in $\bar{\delta}^i \mathcal{L}$ to yield

$$\bar{\delta}^i \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \Phi_r} \bar{\delta}^i \Phi_r + \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_r} \bar{\delta}^i \partial_\mu \Phi_r. \tag{2.1.98}$$

But we have that

$$\begin{aligned} \bar{\delta}^i \partial_\mu^x \Phi_r(x) &= U(g) \partial_\mu^x \Phi_r(x) U^\dagger - \partial_\mu^x \Phi_r(x) \\ &= \partial_\mu^x [U \Phi_r(x) U^\dagger - \Phi_r(x)] = \partial_\mu \bar{\delta}^i \Phi_r \end{aligned} \tag{2.1.99}$$

so that differentiating by parts implies

$$\begin{aligned} \bar{\delta}^i \mathcal{L}(x) = &\left(\frac{\partial \mathcal{L}}{\partial \Phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_r} \right) \bar{\delta}^i \Phi_r(x) \\ &+ \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_r} \bar{\delta}^i \Phi_r \right]. \end{aligned} \tag{2.1.100}$$

But the Euler-Lagrange equations of motion require that

$$\frac{\partial \mathcal{L}}{\partial \Phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_r} = 0, \tag{2.1.101}$$

hence

$$\bar{\delta}^i \mathcal{L}(x) = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_r} \bar{\delta}^i \Phi_r \right]. \tag{2.1.102}$$

It follows that

$$U^\dagger(g)\mathcal{L}(x)U(g) = \mathcal{L}(x) - \alpha_i \bar{\delta}^i \mathcal{L}(x) \tag{2.1.103}$$

or at x'

$$\begin{aligned} U^\dagger(g)\mathcal{L}(x')U(g) &= \mathcal{L}(x') - \alpha_i \bar{\delta}^i \mathcal{L}(x) \\ &= \mathcal{L}(x) + \alpha_i \left[\delta^i x^\mu \partial_\mu \mathcal{L} - \bar{\delta}^i \mathcal{L}\right] \end{aligned} \tag{2.1.104}$$

and consequently,

$$\begin{aligned} \delta\mathcal{L}(x) &= U^\dagger(g)\mathcal{L}(x')U(g) - \mathcal{L}(x) \\ &= -\alpha_i \left[\bar{\delta}^i \mathcal{L} - \delta^i x^\mu \partial_\mu \mathcal{L}\right] \end{aligned} . \tag{2.1.105}$$

By letting $-\alpha_i \bar{\delta}^i \equiv \bar{\delta}$ and $+\alpha_i \delta^i x^\mu \equiv \delta x^\mu$ we have

$$\delta\mathcal{L}(x) = \bar{\delta}\mathcal{L} + \delta x^\mu \partial_\mu \mathcal{L}(x). \tag{2.1.106}$$

However, equation (2.1.102) gave

$$\bar{\delta}\mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_r} \bar{\delta}\Phi_r \right) \tag{2.1.107}$$

so that equation (2.1.106) becomes

$$\begin{aligned} \delta\mathcal{L} &= \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_r} \bar{\delta}\Phi_r \right] + \delta x^\mu \partial_\mu \mathcal{L}(x) \\ &= \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_r} \bar{\delta}\Phi_r + \delta x^\mu \mathcal{L} \right] - \left(\partial_\mu \delta x^\mu\right) \mathcal{L}(x). \end{aligned} \tag{2.1.108}$$

Recall that for the Poincare' group

$$\begin{aligned} x'^\mu &= x^\mu + \omega^{\mu\nu} x_\nu + \epsilon^\mu \\ \omega^{\mu\nu} &= -\omega^{\nu\mu} \end{aligned} \tag{2.1.109}$$

which implies that

$$\partial_\mu \delta x^\mu = \omega^{\mu\nu} g_{\mu\nu} = 0. \tag{2.1.110}$$

Thus, we obtain Noether's Theorem:

$$\delta\mathcal{L} = \partial_\mu J^\mu$$

$$J^\mu \equiv \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_r} \bar{\delta} \Phi_r + \delta x^\mu L \right] \tag{2.1.111}$$

where J^μ is the current. If the Lagrangian is invariant

$$\delta \mathcal{L} = 0 = U^\dagger \mathcal{L}(x') U - \mathcal{L}(x) \tag{2.1.112}$$

the current is conserved

$$\partial_\mu J^\mu = 0. \tag{2.1.113}$$

The charge Q^i is a time independent operator given by

$$Q^i \equiv \int d^3x J^{i0}(x) \tag{2.1.114}$$

which generates symmetry transformations. PROOF:

$$\dot{Q}^i = \int d^3x \partial_0 J^{i0} \tag{2.1.115}$$

but

$$\partial_\mu J^\mu = 0 = \partial_0 J^0 - \vec{\bigtriangledown} \cdot \vec{J} \tag{2.1.116}$$

which implies

$$\dot{Q}^i = -\int d^3x \vec{\bigtriangledown} \cdot \vec{J} = -\int_{S_\infty} d\vec{S} \cdot \vec{J} = 0 \tag{2.1.117}$$

since the fields tend to zero as the $|\vec{x}| \to \infty$. Thus, the charge, Q^i , is a conserved quantity. The charge density is

$$J^0 = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_r} \bar{\delta} \Phi_r + \delta x^0 \mathcal{L}, \tag{2.1.118}$$

but

$$\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_r} = \Pi_r(x) \tag{2.1.119}$$

so that

$$J^0 = \Pi_r \bar{\delta} \Phi_r + \delta x^0 \mathcal{L} \tag{2.1.120}$$

and

$$Q = \int d^3x \Pi_r \bar{\delta} \Phi_r + \int d^3x \delta x^0 \mathcal{L}. \tag{2.1.121}$$

Recall that

$$\mathcal{L} = -\mathcal{H} + \Pi_r \dot{\Phi}_r \tag{2.1.122}$$

and

$$\delta x^\mu = \epsilon_\nu g^{\nu\mu} + \omega_{\nu\lambda} x^\lambda g^{\nu\mu} \tag{2.1.123}$$

for Poincare' transformations. Presently, it is simpler to consider internal symmetries and space-time symmetries separately.

1)Internal Symmetries $\delta x^\mu = 0$

$$\bar{\delta}\Phi_r(x) = -\alpha_i \bar{\delta}^i \Phi_r(x) = -\alpha_i D^i_{rs}\Phi_s(x) \tag{2.1.124}$$

$$\delta\mathcal{L} = \bar{\delta}\mathcal{L} = \partial_\mu J^\mu \tag{2.1.125}$$

with

$$J^\mu = \left[\frac{\partial \mathcal{L}}{\partial\partial_\mu \Phi_r}\left(-\alpha_i D^i_{rs}\Phi_s\right)\right] \equiv -\alpha_i J^\mu_i \tag{2.1.126}$$

so that we have

$$J^\mu_i = \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_r} D^i_{rs}\Phi_s. \tag{2.1.127}$$

If $\delta\mathcal{L} = 0$ this implies $\partial_\mu J^\mu_i = 0$ and

$$Q_i = \int d^3x J^0_i(x) = \int d^3x \Pi_r D^i_{rs}\Phi_s. \tag{2.1.128}$$

Already $\dot{Q}_i = 0$, so

$$[Q_i, \Phi_r(x)] = \int d^3y \left[\Pi_t(y) D^i_{ts}\Phi_s(y), \Phi_r(x)\right]. \tag{2.1.129}$$

Since Q_i is time independent its integrand can be taken at $y^0 = x^0$ so that the equal time commutator yields

$$[Q_i, \Phi_r(x)] = \int d^3y - i\delta_{tr}\delta^3(\vec{y}-\vec{x}) D^i_{ts}\Phi_s(y) = -iD^i_{rs}\Phi_s(x), \tag{2.1.130}$$

which agrees with our previously derived result (2.1.95) and (2.1.124). So indeed this is Q_i.

2)Translations $\delta x^\mu = g^{\mu\nu}\epsilon_\nu$

$$\bar{\delta}\Phi_r(x) = -\alpha_i \bar{\delta}^i \Phi_r = -\delta x^\mu \partial_\mu \Phi_r = -\epsilon_\nu g^{\mu\nu} \partial_\mu \Phi_\nu, \tag{2.1.131}$$

since $D_{rs} = 0$ for translations i.e. $S_{rs} = \delta_{rs}$ only. We have that

$$\delta\mathcal{L} = \bar{\delta}\mathcal{L} + \delta x^\mu \partial_\mu \mathcal{L} = \partial_\mu J^\mu \tag{2.1.132}$$

with

$$\begin{aligned} J^\mu &= \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_r} \left(-\epsilon_\nu g^{\mu\nu} \partial_\mu \Phi_r \right) + g^{\mu\nu} \epsilon_\nu \mathcal{L} \right] \\ &= -\epsilon_\nu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_r} \partial^\nu \Phi_r - g^{\mu\nu} \mathcal{L} \right] \\ &= -\epsilon_\nu T^{\mu\nu}(x) \end{aligned} \tag{2.1.133}$$

where the current $T^{\mu\nu}$ is given by

$$T^{\mu\nu}(x) \equiv \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_r} \partial^\nu \Phi_r - g^{\mu\nu} \mathcal{L} \tag{2.1.134}$$

and is called the energy-momentum tensor. If $\delta\mathcal{L} = 0$, i.e. $\mathcal{L}$ is translationally invariant, then we have that

$$\partial_\mu T^{\mu\nu} = 0. \tag{2.1.135}$$

Furthermore, the charge

$$Q^\nu \equiv \mathcal{P}^\nu \equiv \int d^3x T^{0\nu} \tag{2.1.136}$$

is called the energy-momentum operator or translation operator. Now,

$$\begin{aligned} [\mathcal{P}^\nu, \Phi_r(x)] &= \int d^3y \left[T^{0\nu}(y), \Phi_r(x) \right] \\ &= \int d^3y \left[\left(\Pi_r \partial^\nu \Phi_r - g^{0\nu} \mathcal{L} \right)(y), \Phi_r(x) \right] \end{aligned} \tag{2.1.137}$$

So for $\nu = 0$ we have

$$\begin{aligned} \mathcal{P}^0 = \int d^3x T^{00} &= \int d^3x \left(\Pi_r \dot{\Phi}_r - \mathcal{L} \right) \\ &= \int d^3x \mathcal{H} \\ &= H \end{aligned} \tag{2.1.138}$$

and $[\mathcal{P}^0, \Phi_r(x)] = [H, \Phi_r(x)] = -i\dot{\Phi}_r(x)$, which agrees with (2.1.95). For $\mu = 1, 2, 3$ we have

$$\vec{\mathcal{P}} = \int d^3x T^{0i} = -\int d^3x \Pi_r \vec{\bigtriangledown}\Phi_r \tag{2.1.139}$$

so that

$$\begin{aligned}\left[\vec{\mathcal{P}}, \Phi_r(x)\right] &= -\int d^3y \left[\Pi_s(y), \Phi_r(x)\right] \vec{\bigtriangledown}\Phi_s(y) \\ &= i\vec{\bigtriangledown}\Phi_r(x)\end{aligned}. \tag{2.1.140}$$

Hence

$$[\mathcal{P}^\mu, \Phi_r(x)] = -i\partial^\mu \Phi_r(x) \tag{2.1.141}$$

which is in agreement with equation (2.1.95).

3) Lorentz Transformations $\delta x^\mu = g^{\mu\nu}\omega_{\nu\lambda}x^\lambda = \omega^{\mu\nu}x_\nu$

$$\begin{aligned}\bar{\delta}\Phi_r &= -\delta x^\mu \partial_\mu \Phi_r - \frac{\omega_{\mu\nu}}{2} D^{\mu\nu}_{rs}\Phi_s \\ &= -\omega^{\mu\nu}x_\nu \partial_\mu \Phi_r - \frac{\omega_{\mu\nu}}{2} D^{\mu\nu}_{rs}\Phi_s \\ &= \frac{\omega_{\mu\nu}}{2}\left[\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\Phi_r - D^{\mu\nu}_{rs}\Phi_s\right]\end{aligned} \tag{2.1.142}$$

where the $D^{\mu\nu}_{rs}$ are the tensor or spinor representations of the Lorentz group studied earlier. So

$$\delta\mathcal{L} = \partial_\mu J^\mu \tag{2.1.143}$$

with

$$\begin{aligned}J^\mu &= \left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_r}\frac{\omega_{\nu\rho}}{2}\left[\left(x^\nu\partial^\rho - x^\rho\partial^\nu\right)\Phi_r - D^{\nu\rho}_{rs}\Phi_s\right] + g^{\mu\nu}\omega_{\nu\rho}x^\rho\mathcal{L}\right] \\ &\equiv \frac{\omega_{\nu\rho}}{2}M^{\mu\nu\rho}(x),\end{aligned} \tag{2.1.144}$$

where $M^{\mu\nu\rho}$ is called the angular momentum tensor. We can write it as

$$\frac{\omega_{\nu\rho}}{2}M^{\mu\nu\rho} = \frac{\omega_{\nu\rho}}{2}x^\nu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_r}\partial^\rho\Phi_r - g^{\mu\rho}\mathcal{L}\right)$$

$$-\frac{\omega_{\nu\rho}}{2}x^\rho\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_r}\partial^\nu\Phi_r - g^{\mu\nu}\mathcal{L}\right) - \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_r}D^{\nu\mu}_{rs}\Phi_s\frac{\omega_{\nu\rho}}{2}$$

$$+\frac{\omega_{\nu\rho}}{2}\left(x^\nu g^{\mu\rho}\mathcal{L} - x^\rho g^{\mu\nu}\mathcal{L} + 2x^\rho g^{\mu\nu}\mathcal{L}\right)$$

$$= \frac{\omega_{\nu\rho}}{2}\left(x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} - \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_r}D^{\nu\rho}_{rs}\Phi_s\right) + \frac{\omega_{\nu\rho}}{2}\left(x^\nu g^{\mu\rho}\mathcal{L} + x^\rho g^{\mu\nu}\mathcal{L}\right). \quad (2.1.145)$$

Since the second term in brackets on the right hand side is $\nu - \rho$ symmetric it vanishes when multiplied by $\omega_{\nu\rho}$. The first term in brackets is $\nu-\rho$ anti-symmetric hence we have that

$$\begin{aligned} M^{\mu\nu\rho} &= x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} - \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_r}D^{\nu\rho}_{rs}\Phi_s \\ &= -M^{\mu\rho\nu}. \end{aligned} \quad (2.1.146)$$

Note that we have simply

$$\partial_\mu M^{\mu\nu\rho} = T^{\nu\rho} - T^{\rho\nu} - \partial_\mu\left(\Pi^\mu_r D^{\nu\rho}_{rs}\Phi_s\right) \quad (2.1.147)$$

where

$$\Pi^\mu_r \equiv \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_r}. \quad (2.1.148)$$

¿From (2.1.143) we know this must be zero, but let's check it

$$T^{\nu\rho} = \Pi^\nu_r\partial^\rho\Phi_r - g^{\nu\rho}\mathcal{L} \quad (2.1.149)$$

hence

$$T^{\nu\rho} - T^{\rho\nu} = \Pi^\nu_r\partial^\rho\Phi_r - \Pi^\rho_r\partial^\nu\Phi_r, \quad (2.1.150)$$

while

$$\begin{aligned} &-\partial_\mu\left(\Phi^\mu_r D^{\nu\rho}_{rs}\Phi_s\right) = \\ &-\frac{\partial\mathcal{L}}{\partial\Phi_r}D^{\nu\rho}_{rs}\Phi_s - \Pi^\mu_r\partial_\mu D^{\nu\rho}_{rs}\Phi_s. \end{aligned} \quad (2.1.151)$$

Now the variation yields

$$\begin{aligned} \delta\mathcal{L} &= \bar{\delta}\mathcal{L} + \delta x^\mu\partial_\mu\mathcal{L} \\ &= \frac{\partial\mathcal{L}}{\partial\Phi_r}\bar{\delta}\Phi_r + \Pi^\mu_r\partial_\mu\bar{\delta}\Phi_r + \delta x^\mu\partial_\mu\mathcal{L} \end{aligned}. \quad (2.1.152)$$

Performing the necessary algebra this becomes

$$\delta^{\nu\rho}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\Phi_r}\left[\left(x^\nu\partial^\rho - x^\rho\partial^\nu\right)\Phi_r - D^{\nu\rho}_{rs}\Phi_s\right]$$

$$
\begin{aligned}
&+\Pi_r^\mu \partial_\mu \left[\left(x^\nu \partial^\rho - x^\rho \partial^\nu \right) \Phi_r - D_{rs}^{\nu\rho} \Phi_s \right] - \left(x^\nu \partial^\rho - x^\rho \partial^\nu \right) \mathcal{L} \\
&= -\frac{\partial \mathcal{L}}{\partial \Phi_r} D_{rs}^{\nu\rho} \Phi_s + x^\nu \frac{\partial \mathcal{L}}{\partial \Phi_r} \partial^\rho \Phi_r - x^\rho \frac{\partial \mathcal{L}}{\partial \Phi_r} \partial^\nu \Phi_r \\
&\quad -\Pi_r^\mu \partial_\mu D_{rs}^{\nu\rho} \Phi_s + \Phi_r^\nu \partial^\rho \Phi_r - \Pi_r^\rho \partial^\nu \Phi_r \\
&\quad +x^\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_r} \partial^\rho \partial_\mu \Phi_r - x^\rho \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_r} \partial^\nu \partial_\mu \Phi_r \\
&\quad -x^\nu \partial^\rho \mathcal{L} + x^\rho \partial^\nu \mathcal{L} \\
&= -\frac{\partial \mathcal{L}}{\partial \Phi_r} D_{rs}^{\nu\rho} \Phi_s - \Phi_r^\mu \partial_\mu D_{rs}^{\nu\rho} \Phi_s + \Phi_r^\nu \partial^\rho \Phi_r - \Pi_r^\rho \partial^\nu \Phi_r \\
&\quad +x^\nu \partial^\rho \mathcal{L} - x^\rho \partial^\nu \mathcal{L} - x^\nu \partial^\rho \mathcal{L} + x^\rho \partial^\nu \mathcal{L}.
\end{aligned}
\tag{2.1.153}
$$

Hence we have checked explicitly that

$$
\delta^{\nu\rho} \mathcal{L} = T^{\nu\rho} - T^{\rho\nu} - \partial_\mu \left(\Pi_r^\mu D_{rs}^{\nu\rho} \Phi_s \right), \tag{2.1.154}
$$

so

$$
\partial_\mu M^{\mu\nu\rho} = \delta^{\nu\rho} \mathcal{L} = 0 \tag{2.1.155}
$$

as proved before in general. So if $\mathcal{L}$ is Lorentz invariant we have

$$
T^{\nu\rho} - T^{\rho\nu} = \partial_\mu \left(\Pi_r^\mu D_{rs}^{\nu\rho} \Phi_s \right). \tag{2.1.156}
$$

We can always construct a symmetric energy momentum tensor that leads to the same $\mathcal{P}^\mu$ and is conserved as shown by F. J. Belinfante (Purdue professor). Let

$$
\begin{aligned}
H^{\rho\mu\nu} &\equiv \Pi_r^\rho D_{rs}^{\mu\nu} \Phi_s \\
&= -H^{\rho\nu\mu},
\end{aligned}
\tag{2.1.157}
$$

then

$$
T^{\mu\nu} - T^{\nu\mu} = \partial_\rho H^{\rho\mu\nu}. \tag{2.1.158}
$$

Then we can always define a symmetric energy-momentum tensor (the Belinfante tensor, the construction process being called the Belinfante improvement proceedure) by

$$
\Theta^{\mu\nu} \equiv T^{\mu\nu} - \partial_\rho G^{\rho\mu\nu} \tag{2.1.159}
$$

where

$$G^{\rho\mu\nu} \equiv \frac{1}{2}\left[H^{\rho\mu\nu} + H^{\mu\nu\rho} + H^{\nu\mu\rho}\right]. \tag{2.1.160}$$

1) First we have

$$\partial_\mu \Theta^{\mu\nu} = \partial_\mu T^{\mu\nu}. \tag{2.1.161}$$

PROOF:

$$\partial_\mu \Theta^{\mu\nu} - \partial_\mu T^{\mu\nu} = -\partial_\mu \partial_\rho G^{\rho\mu\nu} \tag{2.1.162}$$

but

$$\begin{aligned} G^{\mu\rho\nu} &= \frac{1}{2}\left[H^{\mu\rho\nu} + H^{\rho\nu\mu} + H^{\nu\rho\mu}\right] \\ &= \frac{1}{2}\left[-H^{\mu\nu\rho} - H^{\rho\mu\nu} - H^{\nu\mu\rho}\right] \end{aligned}. \tag{2.1.163}$$

So

$$G^{\rho\mu\nu} = \frac{1}{2}\left[H^{\rho\mu\nu} + H^{\mu\nu\rho} + H^{\nu\mu\rho}\right] \tag{2.1.164}$$

and

$$G^{\rho\mu\nu} = -G^{\mu\rho\nu}. \tag{2.1.165}$$

Thus,

$$\partial_\mu \partial_\rho G^{\rho\mu\nu} \equiv 0. \tag{2.1.166}$$

2) Secondly $\Theta^{\mu\nu}$ is to be symmetric

$$\Theta^{\mu\nu} = \Theta^{\nu\mu}, \tag{2.1.167}$$

so

$$\begin{aligned} &T^{\mu\nu} - T^{\nu\mu} \\ &= -\frac{1}{2}\partial_\rho\left[H^{\rho\nu\mu} + H^{\nu\mu\rho} + H^{\mu\nu\rho} - H^{\rho\mu\nu} - H^{\mu\nu\rho} - H^{\nu\mu\rho}\right] \\ &= +\partial_\rho H^{\rho\mu\nu} \end{aligned} \tag{2.1.168}$$

as required. Hence we also have

$$2') \qquad \partial_\nu \Theta^{\mu\nu} = 0. \tag{2.1.169}$$

3) Thirdly the charge is unchanged

$$\begin{aligned} \mathcal{P}^\nu = \int d^3x \Theta^{0\nu} &= \int d^3x T^{0\nu} - \int d^3x \partial_\rho G^{\rho 0\nu} \\ &= \int d^3x T^{0\nu} - \int d^3x \partial_0 G^{00\nu} \end{aligned} \tag{2.1.170}$$

but

$$G^{00\nu} = 0 \tag{2.1.171}$$

hence

$$\mathcal{P}^\nu = \int d^3x \Theta^{0\nu} = \int d^3x T^{0\nu}. \tag{2.1.172}$$

Similarly we can Belinfante improve the angular momentum tensor, denote the improved tensor by $M_B^{\mu\nu\rho}$, so that

$$M_B^{\mu\nu\rho} \equiv x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu}. \tag{2.1.173}$$

First note that

$$\partial_\mu M_B^{\mu\nu\rho} = \Theta^{\nu\rho} - \Theta^{\rho\nu} = 0. \tag{2.1.174}$$

So we have

$$\begin{aligned} M_B^{\mu\nu\rho} &= M^{\mu\nu\rho} + H^{\mu\nu\rho} - x^\nu \partial_\lambda G^{\lambda\mu\rho} + x^\rho \partial_\lambda G^{\lambda\mu\nu} \\ &= M^{\mu\nu\rho} + H^{\mu\nu\rho} + \partial_\lambda \left[x^\rho G^{\lambda\mu\nu} - x^\nu G^{\lambda\mu\rho} \right] + H^{\mu\nu\rho} \\ &+ \frac{1}{2} \left[H^{\nu\mu\rho} + H^{\mu\rho\nu} + H^{\rho\mu\nu} - H^{\rho\mu\nu} - H^{\mu\nu\rho} - H^{\nu\mu\rho} \right]. \end{aligned} \tag{2.1.175}$$

This is just a Belinfante improvement to $M^{\mu\nu\rho}$ given by

$$M_B^{\mu\nu\rho} = M^{\mu\nu\rho} + \partial_\lambda \left[x^\rho G^{\lambda\mu\nu} - x^\nu G^{\lambda\mu\rho} \right]. \tag{2.1.176}$$

Hence the charge is also unchanged

$$\begin{aligned} \int d^3x M_B^{0\nu\rho} &= \int d^3x M^{0\nu\rho} + \int d^3x \partial_0 \left[x^\rho G^{00\nu} - x^\nu G^{00\rho} \right] \\ &= \int d^3x M^{0\nu\rho} \\ &= \mathcal{M}^{\nu\rho} \end{aligned} \tag{2.1.177}$$

yielding the generator of the Lorentz transformations. Also

$$\begin{aligned} [\mathcal{M}^{\nu\rho}, \Phi_r(x)] &= -i\bar{\delta}^{\nu\rho} \Phi_r(x) \\ &= -i \left[(x^\nu \partial^\rho - x^\rho \partial^\nu) \Phi_r(x) - D_{rs}^{\nu\rho} \Phi_s(x) \right], \end{aligned} \tag{2.1.178}$$

which can be checked explicitly as in the $\mathcal{P}^\mu$ case. In general, any current can be Belinfante improved

$$J_B^\mu \equiv J^\mu + \partial_\rho G^{\rho\mu} \tag{2.1.179}$$

if there exist tensors $G^{\rho\mu}$ so that

$$G^{\rho\mu} = -G^{\mu\rho}, \tag{2.1.180}$$

then

$$\partial_\rho \partial_\mu G^{\rho\mu} \equiv 0 \tag{2.1.181}$$

and

$$Q = \int d^3x J_B^0 = \int d^3x J^0. \tag{2.1.182}$$

We now turn our attention to the specific Lagrangians describing spin 0, $\frac{1}{2}$, and 1 particles that are non-interacting.