formula follows from the finite dimensional Rotation matrix for orbital angular momentum $\left(D^{(\ell)}_{m0}(R(\vec{\theta})) = \left(\frac{4\pi}{2\ell+1}\right)^{1/2} Y_\ell^{m*}(\theta,\varphi) \right)$

which implies the addition theorem

$$\frac{2\ell+1}{4\pi} P_\ell(\cos\alpha) = \sum_{m=-\ell}^{+\ell} (-1)^m Y_\ell^{m}(\theta,\varphi) Y_\ell^{-m}(\theta_{\vec{p}},\varphi_{\vec{p}})$$

where α is the angle between the vectors with spherical coordinates (θ,φ) and $(\theta_{\vec{p}},\varphi_{\vec{p}})$

(see Gottfried: chapter 34, 35
Cohen-Tannoudji, Diu, Laloë; Chapter VI, A_{VI},
Chapter VIII, $A_{VIII}2$.)

5.3.7. Addition of angular momentum and Clebsch-Gordon Coefficients

We now consider a system made up of the union of 2 substystems such that the two partial angular momenta $\vec{J}_1$ and $\vec{J}_2$ commute. For instance we can have 2 particles each with individual angular momentum $\vec{J}_i$ or we can consider the case of orbital $\vec{L}$ and spin $\vec{S}$

angular momenta. Then we have $\vec{J}_1$ and $\vec{J}_2$ such that

$$[J_1^i, J_1^j] = i\hbar\epsilon_{ijk}J_1^k$$

$$[J_2^i, J_2^j] = i\hbar\epsilon_{ijk}J_2^k$$

but $$[J_1^i, J_2^j] = 0.$$

The state space of subsystem 1, $\mathcal{H}_1$, is spanned by the set of standard basis vectors $\{|k_1, j_1, m_1\rangle\}$ which are the mutual eigenvectors of the system 1 CSCO $\{A_1, \vec{J}_1^2, J_{1z}\}$

$$A_1|k_1, j_1, m_1\rangle = a_{k_1}|k_1, j_1, m_1\rangle$$

$$\vec{J}_1^2|k_1, j_1, m_1\rangle = j_1(j_1+1)\hbar^2|k_1, j_1, m_1\rangle$$

$$J_{1z}|k_1, j_1, m_1\rangle = m_1\hbar|k_1, j_1, m_1\rangle$$

and within each subspace of fixed (k_1, j_1) $\mathcal{H}_1(k_1, j_1)$ we have

$$J_{1\pm}|k_1, j_1, m_1\rangle = \hbar\sqrt{j_1(j_1+1) - m_1(m_1\pm1)}\;\times$$

$$\times|k_1, j_1, m_1\pm1\rangle.$$

Similarly for the state space $\mathcal{H}_2$ of subsystem 2, its standard basis vectors are $\{|k_2, j_2, m_2\rangle\}$ the mutual eigenvectors of the subsystem 2 CSCO $\{A_2, \vec{J}_2^{\,2}, J_{2z}\}$

$$A_2|k_2, j_2, m_2\rangle = a_{k_2}|k_2, j_2, m_2\rangle$$

$$\vec{J}_2^{\,2}|k_2, j_2, m_2\rangle = j_2(j_2+1)\hbar^2|k_2, j_2, m_2\rangle$$

$$J_{2z}|k_2, j_2, m_2\rangle = m_2\hbar\,|k_2, j_2, m_2\rangle$$

and

$$J_{2\pm}|k_2, j_2, m_2\rangle = \hbar\sqrt{j_2(j_2+1) - m_2(m_2\pm 1)}\;\times$$

$$\times\,|k_2, j_2, m_2 \pm 1\rangle.$$

Hence the state space of the entire system is the direct product of $\mathcal{H}_1$ and $\mathcal{H}_2$

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \quad \text{with basis}$$

vectors of the form

$$|k_1, k_2; j_1, j_2; m_1, m_2\rangle = |k_1, j_1, m_1\rangle \otimes |k_2, j_2, m_2\rangle.$$

Since $\mathcal{H}_i = \oplus \mathcal{H}_i(k_i, j_i)$ we have that $\mathcal{H}$ is a direct sum of the subspaces $\mathcal{H}(k_1, k_2; j_1, j_2)$ obtained by taking the direct product of $\mathcal{H}_1(k_1, j_1)$ and $\mathcal{H}_2(k_2, j_2)$ for fixed (k_1, j_1) and (k_2, j_2)

$$\mathcal{H} = \oplus\, \mathcal{H}(k_1, k_2; j_1, j_2)$$

with

$$\mathcal{H}(k_1, k_2; j_1, j_2) = \mathcal{H}_1(k_1, j_1) \otimes \mathcal{H}_2(k_2, j_2)$$

The dimension of $\mathcal{H}_i(k_i, j_i)$ is $(2j_i+1)$; hence the dimension of $\mathcal{H}(k_1, k_2; j_1, j_2)$ is the product

$$(2j_1+1)(2j_2+1).$$

Further recall that $\vec{J}_i \mathcal{H}_i(k_i, j_i) = \mathcal{H}_i(k_i, j_i)$, hence $\vec{J}_1$ and $\vec{J}_2$ leave $\mathcal{H}(k_1, k_2; j_1, j_2)$ invariant

$$\vec{J}_i \mathcal{H}(k_1, k_2; j_1, j_2) = \mathcal{H}(k_1, k_2; j_1, j_2)$$

for $i = 1, 2$.

The total angular momentum of the composite system is defined by

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$ (recall $\vec{J}_i$ acts only on $\mathcal{H}_i$). Since each $\vec{J}_i$ obey the SU(2) algebra, so does $\vec{J}$

i.e. $$[J_1^i + J_2^i, J_1^j + J_2^j] = [J_1^i, J_1^j]$$
$$+ [J_2^i, J_2^j] + [J_1^i, J_2^j] + [J_2^i, J_1^j]$$
$$= i\hbar\epsilon_{ijk}(J_1^k + J_2^k) .$$

Since $[\vec{J}_i, \vec{J}_{1,2}^2] = 0 \Rightarrow [\vec{J}, \vec{J}_{1,2}^2] = 0$ and $[J_z, \vec{J}_{1,2}^2] = [J_{1z} + J_{2z}, \vec{J}_{1,2}^2] = 0$.

As well $[J_z, J_{1z}] = 0 = [J_z, J_{2z}]$.

Since $$\vec{J}^2 = \vec{J}_1^2 + \vec{J}_2^2 + 2\vec{J}_1 \cdot \vec{J}_2$$
$$= \vec{J}_1^2 + \vec{J}_2^2 + 2J_{1z}J_{2z}$$
$$+ 2(J_{1x}J_{2x} + J_{1y}J_{2y})$$

Recalling that $J_{i\pm} \equiv J_{ix} \pm i J_{iy}$
we have

$$J_{ix} = \frac{1}{2}(J_{i-} + J_{i+})$$

$$J_{iy} = \frac{i}{2}(J_{i-} - J_{i+})$$

So

$$J_{1x}J_{2x} + J_{1y}J_{2y} = \frac{1}{4}(J_{1-} + J_{1+})(J_{2-} + J_{2+}) - \frac{1}{4}(J_{1-} - J_{1+})(J_{2-} - J_{2+})$$

$$= \frac{1}{2}(J_{1+}J_{2-} + J_{1-}J_{2+})$$

So $\vec{J}^2 = \vec{J}_1^2 + \vec{J}_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$.

Hence $[\vec{J}^2, J_{1z}] = [J_{1+}J_{2-} + J_{1-}J_{2+}, J_{1z}]$

$$= [J_{1+}, J_{1z}]J_{2-} + [J_{1-}, J_{1z}]J_{2+}$$

$$= -\hbar J_{1+}J_{2-} + \hbar J_{1-}J_{2+}$$

$$= -\hbar(J_{1+}J_{2-} - J_{1-}J_{2+}) \neq 0$$

Similarly $[\vec{J}^2, J_{2z}] = \hbar(J_{1+}J_{2-} - J_{1-}J_{2+}) \neq 0$

To summarize: $[\vec{J}^2, \vec{J}_1^2] = 0 = [\vec{J}^2, \vec{J}_2^2]$

$$[J_z, \vec{J}_1^2] = 0 = [J_z, \vec{J}_2^2]$$

even $[J_z, J_{1z}] = 0 = [J_z, J_{2z}]$

but $[\vec{J}^2, J_{1z}] \neq 0$

$$[\vec{J}^2, J_{2z}] \neq 0.$$

Thus we can consider the CSCO $\{A_1, A_2, \vec{J}_1^2, \vec{J}_2^2, \vec{J}^2, J_z\}$. The set of mutual eigenvectors will be a basis; denote them

$$|k_1, k_2; j_1, j_2; J, M\rangle.$$

This set will be useful for studying properties of the total angular momentum. The basis vectors for the CSCO $\{A_1 A_2; \vec{J}_1^2, \vec{J}_2^2; J_{1z}, J_{2z}\}$, $|k_1, k_2; j_1, j_2; m_1, m_2\rangle$, are useful for studying the properties of the individual angular momenta $\vec{J}_1$ and $\vec{J}_2$.

Since $[\vec{J}^2, J_{1z}] \neq 0$, these two sets of basis vectors are not the same. However since $\vec{J}_i \mathcal{H}(k_1,k_2;j_1,j_2) = \mathcal{H}(k_1,k_2;j_1,j_2)$ we have that $\vec{J} = \vec{J}_1 + \vec{J}_2$ leaves $\mathcal{H}(k_1,k_2;j_1,j_2)$ invariant. Hence it is a direct sum of orthogonal subspaces, labelled by $\mathcal{H}(k_1,k_2;p_J,J)$ each of which is invariant under $\vec{J}$, and therefore $\vec{J}^2, J_z, J_+, J_-$;

each J picks out a subspace of $\mathcal{H}(k_1,k_2;j_1,j_2)$

$$\mathcal{H}(k_1,k_2;j_1,j_2) = \bigoplus_J \mathcal{H}(k_1,k_2;p_J,J)^{j_1,j_2;}$$

where p_J counts the number of spaces associated with J for j_1 and j_2. It is only the $(2j_1+1)(2j_2+1)$ basis vectors that need to be transformed.

Thus we must determine 1) given j_1 and j_2 what are the allowed values of J and how many subspaces $\mathcal{H}(k_1,k_2;p_J,J)$ occur for them, i.e. what is p_J? 2) How do we expand the eigenvectors of $\vec{J}^2, J_z$, $|k_1,k_2;j_1,j_2;J,M\rangle$, belonging to $\mathcal{H}(k_1,k_2;j_1,j_2)$ in terms of the $\{|k_1,k_2;j_1,j_2;m_1,m_2\rangle\}$ basis?

We begin by determining the

eigenvalues of J_z and their degree of degeneracy. Let $j_1 \geq j_2$ and consider the $\mathcal{H}(k_1,k_2;j_1,j_2)$ subspace, it is

$(2j_1+1)(2j_2+1)$ dimensional. The eigenvectors

$|k_1,k_2;j_1,j_2;m_1,m_2\rangle$ are already eigenstates

of J_z

$$J_z|k_1,k_2;j_1,j_2;m_1,m_2\rangle = (J_{1z}+J_{2z})|k_1,k_2;j_1,j_2;m_1,m_2\rangle$$
$$= (m_1+m_2)\hbar\,|k_1,k_2;j_1,j_2;m_1,m_2\rangle$$

with J_z eigenvalue $M = m_1+m_2$. Since $m_1 = -j_1,\dots,+j_1$ and $m_2 = -j_2,\dots,+j_2$; we have

that M can take the values

$$M = -(j_1+j_2),\ -(j_1+j_2)+1,\ \dots,\ (j_1+j_2)-1,\ (j_1+j_2)$$

Since $M = m_1+m_2$ it is possible for each of these values to occur more than once. The number of times the value M can occur for a particular (j_1,j_2) is the degree of degeneracy of M, denote this number by

$$g_{j_1j_2}(M).$$

On the other hand, each value of J can occur ($P_J =$) $P_{j_1j_2}(J)$ times. For each J there are the $(2J+1)$ values of $M = -J, -J+1, \dots, +J$ and the associated eigenvectors. If $g_{j_1j_2}(M)$ is the degeneracy of M, then

$$g_{j_1j_2}(M) = \sum_{J \geq |M|}' P_{j_1j_2}(J)$$

So

$$g_{j_1j_2}(J) = P_{j_1j_2}(J) + P_{j_1j_2}(J+1) + \cdots$$

and

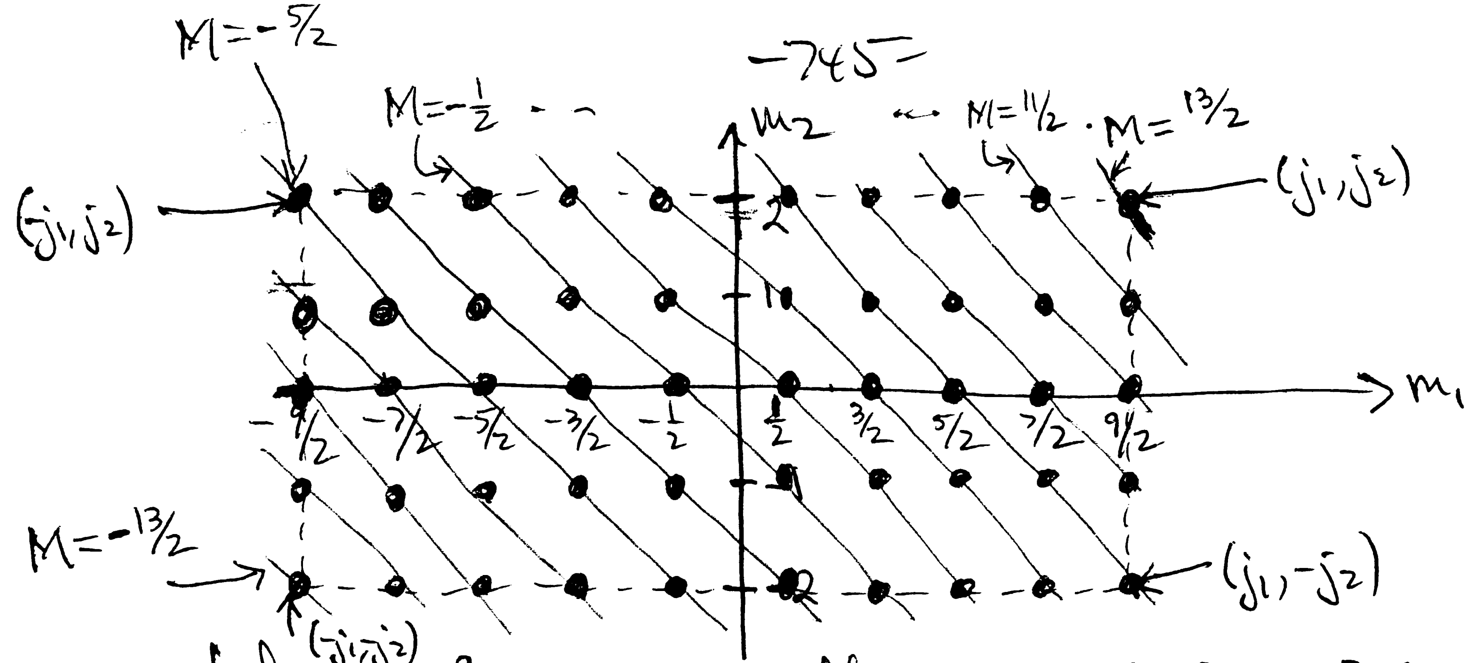
$$g_{j_1j_2}(J+1) = P_{j_1j_2}(J+1) + P_{j_1j_2}(J+2) + \cdots$$

$\Rightarrow$

$$\boxed{P_{j_1j_2}(J) = g_{j_1j_2}(J) - g_{j_1j_2}(J+1)}$$

Hence to determine $P_{j_1j_2}(J)$, we need only $g_{j_1j_2}(M)$ for each M. $g_{j_1j_2}(M)$ is just the number of ways (m_1, m_2) can add up to $M = m_1 + m_2$ given that $m_i = -j_i, \dots, +j_i$.

This is most easily determined by plotting m_1 and m_2 for their allowed values.



let $j_1 = \frac{9}{2}$; $j_2 = 2$ then $m_1 = -\frac{9}{2}, -\frac{7}{2}, \dots, \frac{7}{2}, \frac{9}{2}$

$m_2 = -2, -1, 0, +1, +2$

are the allowed values of m_1 and m_2. Each allowed point is given by its coordinates (m_1, m_2). The curve $M = m_1 + m_2$ (i.e. $m_2 = -m_1 + M$) is a negatively sloped diagonal. The number of allowed (m_1, m_2) points intersected by the M-labelled diagonal is the particular $g_{j_1 j_2}(M)$. Thus we find

$$g_{j_1 j_2}(M) = \begin{cases} 0 & , \text{if } |M| > j_1 + j_2 \\ j_1 + j_2 + 1 - |M| & , \text{if } j_1 + j_2 \geq |M| \geq |j_1 - j_2| \\ 2j_2 + 1 & , \text{if } |j_1 - j_2| \geq |M| \geq 0 \end{cases}$$

Thus we find non-zero $P_{j_1 j_2}$ that are

$$\boxed{\begin{array}{l} P_{j_1 j_2}(J) = 1 \quad \text{for} \\ J = j_1+j_2,\, j_1+j_2-1, \ldots, |j_1-j_2|. \end{array}}$$

(For $J > j_1+j_2$; $P_{j_1j_2}(J) = 0$ since $g_{j_1j_2}(M) = 0$ for $|M| > j_1+j_2$. Also when $g_{j_1j_2}(M) \atop {g_{j_1j_2}(M+1)}$ is constant we find $P_{j_1j_2}(J) = 0$ for $\cancel{J} < j_1 - j_2$.

Thus $j_1+j_2 \geq J \geq |j_1-j_2|$.) Hence we have the

<u>Fundamental Addition Theorem For Angular Momentum</u>: $\left(\mathcal{H}(k_1,k_2;j_1,j_2) = \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} \mathcal{H}(k_1,k_2;\cancel{j_1 j_2}\, J)\right)$

In each subspace $\mathcal{H}(k_1,k_2;j_1,j_2)$ (of dimension $(2j_1+1)(2j_2+1)$) spanned by the vectors $\{|k_1,k_2;j_1,j_2;m_1,m_2\rangle$ with $m_i = -j_i, \ldots, +j_i\}$

1) The eigenvalues of $\vec{J}^2$ are such that

$$J = (j_1+j_2), (j_1+j_2)-1, \ldots, |j_1-j_2|.$$

2) With each value of J there is only one

invariant subspace

$$\mathcal{H}(k_1,k_2;j_1,j_2;p_J=1,J) \equiv \mathcal{H}(k_1,k_2;j_1,j_2;J).$$

OK

This subspace has $(2J+1)$ dimensions with basis vectors labelled by $M=-J,\ldots,+J$; $\{|k_1,k_2;j_1,j_2;J,M\rangle\}$. So given J, the subspace $\mathcal{H}(k_1,k_2;\cancel{j_1j_2};J)$ OK is completely specified; giving M, specifies one vector in it uniquely and therefore in $\mathcal{H}(k_1,k_2;j_1,j_2)$ uniquely. Thus $\vec{J}^2$ and J_z are a CSCO in $\mathcal{H}(k_1,k_2;j_1,j_2)$ i.e. given (J,M) only one vector in $\mathcal{H}(k_1,k_2;j_1,j_2)$ is uniquely specified.

Note: we can check directly that the number of vectors given by (J,M) in $\mathcal{H}(k_1,k_2;j_1,j_2)$ indeed equals the dimension $(2j_1+1)(2j_2+1)$ of this subspace. The number of vectors given J are $2J+1$, the allowed values of M. Thus summing over J, we have the

$D_{j_1 j_2} \equiv$ number of vectors $= \sum_{J=j_1-j_2}^{j_1+j_2} (2J+1)$ for $j_1 \geq j_2$

Letting $J = j_1 - j_2 + n$; this becomes

$$D_{j_1 j_2} = \sum_{n=0}^{2j_2} [2(j_1 - j_2 + n) + 1]$$

$$= [2(j_1 - j_2) + 1](2j_2 + 1) + \sum_{n=0}^{2j_2} 2n$$

$$= [2(j_1 - j_2) + 1](2j_2 + 1) + 2\,\frac{2j_2(2j_2+1)}{2}$$

$$= (2j_1 + 1)(2j_2 + 1).$$

For example: suppose we have 2 spin $\frac{1}{2}$ particles $j_1 = \frac{1}{2}$; $j_2 = \frac{1}{2}$; the $\mathcal{H}(k_1, k_2; j_1 = \frac{1}{2}, j_2 = \frac{1}{2})$ space is spanned by the standard basis vectors (suppress k_1, k_2)

$$|j_1 = \tfrac{1}{2}, j_2 = \tfrac{1}{2}; m_{s_1}, m_{s_2}\rangle$$

$$= |j_1 = \tfrac{1}{2}, m_{s_1}\rangle \otimes |j_2 = \tfrac{1}{2}, m_{s_2}\rangle$$

with $m_{s_{1z}} = \pm\frac{1}{2}$. The $\mathcal{H}(\frac{1}{2},\frac{1}{2})$ space is $(2j_1+1)(2j_2+1) = 2\cdot 2 = 4$ dimensional.

The 4 basis vectors are

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle \; .$$

Now we see that J has allowed values $J = j_1+j_2, j_1+j_2-1, \ldots, |j_1-j_2|$

$$= 1, 0 \quad \text{only}.$$

Further for $J=1$, M can have $M=1,0,-1$ values and for $J=0$, M can only be $M=0$.

Then we have 4 vectors $|j_1,j_2;J,M\rangle$, the new basis vectors. Since

$J_z = J_{1z} + J_{2z}$, we have

$$J_z |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle = (\tfrac{1}{2}+\tfrac{1}{2})\hbar |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$J_z |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle = (-\tfrac{1}{2}-\tfrac{1}{2})\hbar |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$J_z |\tfrac{1}{2},\pm\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\mp\tfrac{1}{2}\rangle = 0 .$$

Further since

$$\vec{J}^2 = \vec{J}_1^2 + \vec{J}_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$$

we have that

$$\vec{J}^2 |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle = (\overbrace{\tfrac{3}{4}+\tfrac{3}{4}+2\tfrac{1}{4}}^{=2})\hbar^2 |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$= 1(1+1)\hbar^2 |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle$$

also

$$\vec{J}^2 |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle = 1(1+1)\hbar^2 |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

since $J_{i+} |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle = 0$

$$J_{i-} |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle = 0$$

Thus

$$|j_1=\tfrac{1}{2}, j_2=\tfrac{1}{2}; J=1, M=1\rangle$$

$$= |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$|j_1=\tfrac{1}{2}, j_2=\tfrac{1}{2}; J=1, M=-1\rangle$$
$$= |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle .$$

Now $|\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, \mp\tfrac{1}{2}\rangle$ are the $M=0$ eigenvectors of J_z; but they are not eigenvectors of $\vec{J}^2$

$$\vec{J}^2 |\tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \left(\tfrac{3}{4} + \tfrac{3}{4} - \tfrac{2}{4}\right)\hbar^2 |\tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$
$$+ J_{1-}|\tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes J_{2+}|\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$= \hbar^2 |\tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$
$$+ \hbar^2 \underbrace{\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)}}_{=1} |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes$$
$$\underbrace{\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) + \tfrac{1}{2}(\tfrac{1}{2}+1)}}_{=1} |\tfrac{1}{2}, +\tfrac{1}{2}\rangle$$
$$= \hbar^2 \left(|\tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle + |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, +\tfrac{1}{2}\rangle \right)$$

Likewise

$$\vec{J}^2 |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, +\tfrac{1}{2}\rangle =$$
$$= \hbar^2 \left(|\tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, +\tfrac{1}{2}\rangle + |\tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \right)$$

If we take the sum and difference of these $|j_1,j_2;m_1,m_2\rangle$ basis vectors, we have that

$$\vec{J}^2\left[|\tfrac{1}{2},\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle+|\tfrac{1}{2},-\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},+\tfrac{1}{2}\rangle\right]$$

$$=2\hbar^2\left[|\tfrac{1}{2},\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle+|\tfrac{1}{2},-\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},+\tfrac{1}{2}\rangle\right]$$

and

$$\vec{J}^2\left[|\tfrac{1}{2},\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle-|\tfrac{1}{2},-\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},+\tfrac{1}{2}\rangle\right]$$
$$=0$$

Hence the missing $J=1, M=0$ eigenvector is given by

$$|j_1=\tfrac{1}{2},j_2=\tfrac{1}{2};J=1,M=0\rangle$$
$$=\frac{1}{\sqrt{2}}\left[|\tfrac{1}{2},\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle+|\tfrac{1}{2},-\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle\right]$$

while the $J=0, M=0$ singlet state is

$|j_1=\frac{1}{2}, j_2=\frac{1}{2}; J=0, M=0\rangle$

$$= \frac{1}{\sqrt{2}}\left[|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle - |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},+\tfrac{1}{2}\rangle\right]$$

The 3; $J=1$ vectors are called the <u>triplet</u>, spin 1 states. Note they are even under the interchange of $1\leftrightarrow 2$ while the singlet $J=0$ state is odd under interchange $1\leftrightarrow 2$ (i.e. goes into minus itself.)

The $\frac{1}{\sqrt{2}}$ factors normalized the $|j_1 j_2; J, M\rangle$ states to one and we choose the phases so that all overall coefficients are real & positive i.e. $\frac{+1}{\sqrt{2}}$, for the order given.

for the $|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle$ vector

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Thus we see that within each $\mathcal{H}(k_1 k_2; j_1, j_2)$ $(2j_1+1)(2j_2+1)$ dimensional space, the $|k_1, k_2; j_1, j_2; J, M\rangle$ vectors are simple changes of basis vectors from the $|k_1, k_2; j_1, j_2; m_1, m_2)$ eigenbasis.

We would now like to generalize the above construction to arbitrary j_1 and j_2. Given the $\{A_1, A_2; \vec{J}_1^2, \vec{J}_2^2; J_{1z}, J_{2z}\}$ basis vectors $\{|k_1, j_1, m_1\rangle \otimes |k_2, j_2, m_2\rangle\}$ in $\mathcal{H}(k_1, k_2; j_1, j_2)$ we desire to construct the

$$\{A_1, A_2; \vec{J}_1^2, \vec{J}_2^2; \vec{J}^2, J_z\} \text{ basis vectors}$$

$$\{|k_1, k_2; j_1, j_2; J, M\rangle\} \text{ in } \mathcal{H}(k_1, k_2; j_1, j_2; J)$$

for each J.

This will involve J values

$$J = j_1 + j_2,\ j_1 + j_2 - 1,\ \ldots,\ |j_1 - j_2|$$

and their corresponding M values $M = -J, \ldots, +J$.

That is the space $\mathcal{H}(k_1, k_2; j_1, j_2)$ has the decomposition into fixed J subspaces

$$\mathcal{H}(k_1, k_2; j_1, j_2) = \mathcal{H}(k_1, k_2; j_1, j_2; J = j_1 + j_2) \oplus \mathcal{H}(k_1, k_2; j_1, j_2; J = j_1 + j_2 - 1) \oplus \cdots \oplus \mathcal{H}(k_1, k_2; j_1, j_2; J = |j_1 - j_2|)$$

We will determine the expansion of the (J,M) eigenvectors of each subspace. That is each vector in $\mathcal{H}(k_1,k_2;j_1,j_2)_J$ has the expansion; for instance

$$|k_1,k_2;j_1,j_2;J,M\rangle = \sum_{m_1=-j_1}^{+j_1} \sum_{m_2=-j_2}^{+j_2} |k_1,k_2;j_1,j_2;m_1,m_2\rangle$$

$$\times\langle j_1,j_2;m_1,m_2|J,M\rangle.$$

The coefficients $\langle j_1,j_2;m_1,m_2|J,M\rangle$

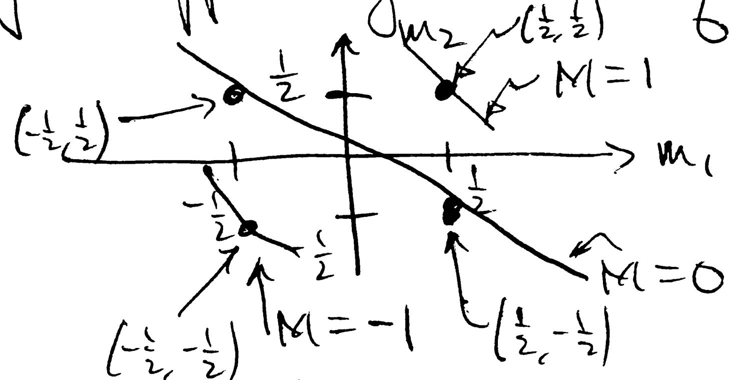
are called <u>Clebsch-Gordon coefficients</u>. As we have seen they are independent of (k_1,k_2). Indeed, the angular momentum operators $\vec{J}_1$ and $\vec{J}_2$ have matrices in their standard basis that are independent of k_1,k_2, consequently the $\vec{J}^2$ and J_z matrices are independent of $k_1 k_2$. Hence the components of their common eigenvectors are independent of k_1,k_2.

We can use the $j_1=\frac{1}{2}$, $j_2=\frac{1}{2}$ example to see how a general method for find $\langle j_1,j_2;m_1,m_2|JM\rangle$ can be determined.

The space $\mathcal{H}(k_1,k_2; j_1=\frac{1}{2}, j_2=\frac{1}{2}) = \mathcal{H}(k_1,k_2, j_1+j_2=1)$

$\oplus\, \mathcal{H}(k_1,k_2, j_1-j_2=0)$

The vectors (surpressing k_1,k_2 and j_1,j_2) $|J,M\rangle$ can be found by considering first the largest value of $J = j_1+j_2 = 1$; this is just the $m_1=j_1$, $m_2=j_2$ upper right corner of the m_1-m_2 plot



This is the only vector with $M=j_1+j_2$; i.e. $g_{j_1 j_2}(M=j_1+j_2) = 1$. Thus it is also an eigenvector of $\vec{J}^2$ with $J=j_1+j_2$ as we said above. So

$$|J=j_1+j_2, M=j_1+j_2\rangle = |j_1 j_2; j_1=m_1, m_2=j_2\rangle$$

in our case $|J=1, M=1\rangle = |\tfrac{1}{2},\tfrac{1}{2}; \tfrac{1}{2},\tfrac{1}{2}\rangle$

$$= |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle$$

as we know.

We can next find all of the other vectors in $\mathcal{H}(J=1)$ by applying J_- to this vector

$$J_- |J=1, M=1\rangle = \hbar\sqrt{J(J+1) - 1(1-1)}\; |J=1, M=0\rangle$$

$$= \hbar\sqrt{2}\; |J=1, M=0\rangle$$

$\Rightarrow$ So

$$|J=1, M=0\rangle = \frac{1}{\hbar\sqrt{2}} J_- |J=1, M=1\rangle$$

Now $\vec{J} = \vec{J}_1 + \vec{J}_2$ and

$$J_\pm = J_x \pm i J_y = J_{1x} \pm i J_{1y} + J_{2x} \pm i J_{2y}$$

$$= J_{1\pm} + J_{2\pm}$$

hence

$$J_- |J=1, M=1\rangle = (J_{1-} + J_{2-}) |\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$= \hbar\left[|\tfrac{1}{2}, +\tfrac{1}{2}; -\tfrac{1}{2}, \tfrac{1}{2}\rangle + |\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\rangle \right]$$

So

$$|J=1, M=0\rangle = \frac{1}{\sqrt{2}}\left[|\tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle + |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle \right]$$

The symmetric spin up - spin down combination.

Applying J_- again, we have

$$|J=1, M=-1\rangle = \frac{1}{\hbar\sqrt{2}} J_- |J=1, M=0\rangle$$

$$= \frac{1}{2\hbar}(J_{1-}+J_{2-})\Big[|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle + |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle\Big]$$

$$= \frac{\hbar}{2\hbar}\Big[|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle + |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle\Big]$$

$$= |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$= |\tfrac{1}{2},\tfrac{1}{2};-\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

Thus we have the basis vectors for $\mathcal{H}(J=1)$.

To find the next lower $J=0$ basis vector, $|J=0, M=0\rangle$, we consider

$M=j_1+j_2-1=0$. Then $g_{j_1j_2}(M=j_1+j_2-1)=2$, this means that $M=j_1+j_2-1$ is two-fold degenerate. We have already constructed one of the $M=0$ eigenstates belonging to $J=j_1+j_2=1$. The other vector must be orthogonal to it and have norm 1. Of course it is orthogonal to the other 2 $J=1$ states hence it must be a linear combination of the $|\frac{1}{2},\pm\frac{1}{2}\rangle \otimes |\frac{1}{2},\mp\frac{1}{2}\rangle$ states.

$$|J=0, M=0\rangle = \alpha\,|\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}\rangle + \beta\,|\tfrac{1}{2},\tfrac{1}{2};-\tfrac{1}{2},\tfrac{1}{2}\rangle$$

i) Norm $=1 \Rightarrow$

$$1 = \langle 0,0|0,0\rangle = |\alpha|^2 + |\beta|^2$$

2) Orthogonal to $|1,0\rangle$

$$0 = \langle 1,0|0,0\rangle = \frac{1}{\sqrt{2}}(\alpha+\beta)$$

$$\Rightarrow \boxed{\alpha = -\beta = \frac{1}{\sqrt{2}}e^{i\varphi}} \quad ; \varphi \in \mathbb{R}.$$

α, β are fixed to within a phase. By convention we take $\varphi = 0$.

Thus $$|J=0, M=0\rangle = \frac{1}{\sqrt{2}}\left[|\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}\rangle - |\tfrac{1}{2},\tfrac{1}{2};-\tfrac{1}{2},\tfrac{1}{2}\rangle\right].$$

Just as we found earlier.

This example shows us the general procedure. (suppress k_1, k_2) for $j_1 j_2$ fixed we have

$$\mathcal{H}(j_1, j_2) = \mathcal{H}(j_1+j_2=J) \oplus \mathcal{H}(J=j_1+j_2-1) \oplus \cdots \oplus \mathcal{H}(J=|j_1-j_2|).$$

1) We start by constructing the basis vectors for $\mathcal{H}(J=j_1+j_2)$

Only one vector corresponds to $M=j_1+j_2$; the vector with $m_1=j_1$, $m_2=j_2$.

Since $M=j_1+j_2$ is not degenerate, the vector

$$|j_1, j_2; m_1=j_1, m_2=j_2\rangle \text{ must}$$

also be an eigenvector of $\vec{J}^2$. From the decomposition it can only correspond to $J=j_1+j_2$.

(i.e. $$\vec{J}^2 |j_1 j_2; j_1 j_2\rangle = (\vec{J}_1^2 + \vec{J}_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}) |j_1 j_2; j_1 j_2\rangle$$

$$= \hbar^2 [j_1(j_1+1) + j_2(j_2+1) + 2j_1 j_2] |j_1 j_2; j_1 j_2\rangle$$

Since $J_{2+}|j_1,j_2;j_1,j_2\rangle = 0$. But this $= \hbar^2[(j_1+j_2)(j_1+j_2+1)]|j_1,j_2;j_1,j_2\rangle$ as stated).

We choose the phase to be such that

$$|j_1,j_2;J=j_1+j_2, M=j_1+j_2\rangle \equiv |j_1,j_2;\underset{m_1}{j_1},\underset{m_2}{j_2}\rangle.$$

We can now act on this repeatedly with $J_- = J_{1-} + J_{2-}$ to form the remaining $J=j_1+j_2$ eigenvectors. For example

$$J_-|j_1,j_2;J=j_1+j_2, M=j_1+j_2\rangle = \hbar\sqrt{2(j_1+j_2)}\,|j_1,j_2;J=j_1+j_2, M=j_1+j_2-1\rangle$$

thus

$$|j_1,j_2;J=j_1+j_2, M=j_1+j_2-1\rangle = \frac{1}{\hbar\sqrt{2(j_1+j_2)}}\times$$

$$\times(J_{1-}+J_{2-})|j_1,j_2;m_1=j_1, m_2=j_2\rangle$$

$$= \frac{1}{\hbar\sqrt{2(j_1+j_2)}}\Big[\hbar\sqrt{2j_1}\,|j_1,j_2;m_1=j_1-1,m_2=j_2\rangle$$
$$+\hbar\sqrt{2j_2}\,|j_1,j_2;m_1=j_1,m_2=j_2-1\rangle\Big]$$

$\Rightarrow$

$$|j_1,j_2;J=j_1+j_2,M=j_1+j_2-1\rangle$$
$$=\sqrt{\frac{j_1}{j_1+j_2}}\,|j_1,j_2;m_1=j_1-1,m_2=j_2\rangle$$
$$+\sqrt{\frac{j_2}{j_1+j_2}}\,|j_1,j_2;m_1=j_1,m_2=j_2-1\rangle.$$

Thus we can continue this procedure until all $M=j_1+j_2, j_1+j_2-1, \ldots, -(j_1+j_2)$ eigenvectors are determined, the sequence ending at

$$|j_1,j_2;J=j_1+j_2,M=-(j_1+j_2)\rangle$$
$$=|j_1,j_2;m_1=-j_1,m_2=-j_2\rangle,$$

the only such vector.

Hence the total angular momentum standard basis in $\mathcal{H}(J=j_1+j_2)$ is determined.

Next consider $\mathcal{H}(J=j_1+j_2-1)$; the highest value of M in this subspace is

$M=j_1+j_2-1$. We know that

this M is 2-fold degenerate; $g_{j_1 j_2}(M=j_1+j_2-1)=2$,

We ~~have~~ already constructed the one eigenvector in the $\mathcal{H}(J=j_1+j_2)$ space. The remaining vector belongs in this $\mathcal{H}(J=j_1+j_2-1)$ subspace. It is a linear combination of the $m_1=j_1;\, m_2=j_2-1$ and $m_1=j_1-1;\, m_2=j_2$ eigenvectors

$$|j_1,j_2;J=j_1+j_2-1,M=j_1+j_2-1\rangle$$

$$=\alpha|j_1,j_2;m_1=j_1,m_2=j_2-1\rangle$$

$$+\beta|j_1,j_2;m_1=j_1-1,m_2=j_2\rangle$$

We now choose α, β so that

1) $|j_1, j_2; J=j_1+j_2-1, M=j_1+j_2-1\rangle$ is orthogonal to $|j_1, j_2; J=j_1+j_2, M=j_1+j_2-1\rangle$

$$= \sqrt{\frac{j_1}{j_1+j_2}}\, |j_1, j_2; m_1=j_1-1, m_2=j_2\rangle$$

$$+ \sqrt{\frac{j_2}{j_1+j_2}}\, |j_1, j_2; m_1=j_1, m_2=j_2-1\rangle .$$

Thus

$$0 = \langle j_1, j_2; J=j_1+j_2, M=j_1+j_2-1 |$$
$$j_1, j_2; J=j_1+j_2-1, M=j_1+j_2-1\rangle$$

$$\Rightarrow \quad 0 = \alpha\sqrt{\frac{j_2}{j_1+j_2}} + \beta\sqrt{\frac{j_1}{j_1+j_2}}$$

$$\Rightarrow \quad \boxed{\alpha = -\beta\sqrt{\frac{j_1}{j_2}}}$$

2) $|j_1, j_2; J = j_1+j_2-1, M = j_1+j_2-1\rangle$ is normalized to 1. $\Rightarrow$

$$\boxed{|\alpha|^2 + |\beta|^2 = 1}$$

This determines α, β to within an overall phase.

3) Choose α, β to be real and α positive.

$$\Rightarrow \quad \boxed{\alpha = \sqrt{\frac{j_1}{j_1+j_2}} \; ; \; \beta = -\sqrt{\frac{j_2}{j_1+j_2}}}$$

Thus we obtain

$$|j_1, j_2; J = j_1+j_2-1, M = j_1+j_2-1\rangle$$

$$= \sqrt{\frac{j_1}{j_1+j_2}}\, |j_1, j_2; m_1 = j_1, m_2 = j_2-1\rangle$$

$$- \sqrt{\frac{j_2}{j_1+j_2}}\, |j_1, j_2; m_1 = j_1-1, m_2 = j_2\rangle$$

Next we can apply J_- to $|j_1 j_2; J=j_1+j_2-1, M=j_1+j_2-1\rangle$ to obtain the remaining $J=j_1+j_2-1$ eigenvectors for $M=j_1+j_2-1, \ldots, -(j_1+j_2-1)$.

Hence the $\mathcal{H}(J=j_1+j_2-1)$ total angular momentum standard basis is constructed in terms of the $|j_1 j_2; m_1, m_2\rangle$ vectors.

Next we consider the subspace $\mathcal{H}(J=j_1+j_2-2)$. The largest value of M in it is $M=j_1+j_2-2$. But this value of M is 3-fold degenerate; $g_{j_1 j_2}(M=j_1+j_2-2)=3$. Two of the eigenvectors have already been constructed:

$|j_1 j_2; J=j_1+j_2, M=j_1+j_2-2\rangle$ and

$|j_1, j_2; J=j_1+j_2-1, M=j_1+j_2-2\rangle$. The third vector $|j_1, j_2; J=j_1+j_2-2, M=j_1+j_2-2\rangle$

is a linear combination of the $\mathcal{H}(j_1, j_2)$ standard basis vectors with $m_1 + m_2 = M = j_1 + j_2 - 2$; these are

$$|j_1 j_2; m_1 = j_1, m_2 = j_2 - 2\rangle$$

$$|j_1, j_2; m_1 = j_1 - 1, m_2 = j_2 - 1\rangle$$

$$|j_1, j_2; m_1 = j_1 - 2, m_2 = j_2\rangle.$$

The coefficients in the expansion are determined by

1) $|j_1, j_2; J = j_1 + j_2 - 2, M = j_1 + j_2 - 2\rangle$ is orthogonal to $|j_1, j_2; J = j_1 + j_2, M = j_1 + j_2 - 2\rangle$ <u>and</u> to $|j_1, j_2; J = j_1 + j_2 - 1, M = j_1 + j_2 - 2\rangle$.

2) $|j_1, j_2; J = j_1 + j_2 - 2; M = j_1 + j_2 - 2\rangle$ has norm $= 1$.

These 3 conditions fix the 3 coefficients to within an overall phase.

3) The phase is chosen so that the coefficients are real and that of the $m_1 = j_1$ vector is positive.

Once this highest M vector for $J = j_1 + j_2 - 2$ is found, the other basis vectors can be determined by repeated application of J_-. Thus we have constructed the $\mathcal{H}(J = j_1 + j_2 - 2)$ total angular momentum standard basis.

We can continue this procedure until all vectors for $j_1 + j_2 \geq J \geq |j_1 - j_2|$ are constructed. Hence we have determined in this manner the Clebsch-Gordon coefficients in the expansion

$$|k_1, k_2; j_1, j_2; J, M\rangle = \sum_{m_1=-j_1}^{+j_1} \sum_{m_2=-j_2}^{+j_2} |k_1, k_2; j_1, j_2; m_1, m_2\rangle$$

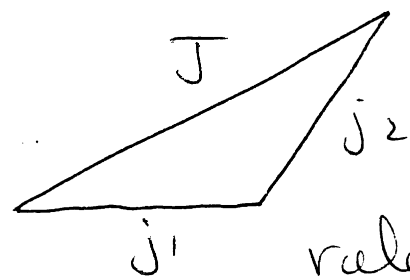
$$\times \langle j_1, j_2; m_1, m_2 | J, M\rangle.$$

Clearly $\langle j_1, j_2; m_1, m_2 | J, M\rangle \neq 0$ only for $M = m_1 + m_2$ and

$$|j_1 - j_2| \leq J \leq j_1 + j_2 .$$

(note J is integral or half-integral according to whether $j_1 + j_2$ and $|j_1 - j_2|$ is integral or half-integral)

This last inequality is called the "triangle rule," we must be able to form a triangle with the 3 lines of length j_1, j_2, J



So if $|j_1 - j_2| \le J \le j_1 + j_2$ we also have the symmetrical rules $|J - j_1| \le j_2 \le J + j_1$

$|J - j_2| \le j_1 \le J + j_2$.

So the double sum reduces to a single sum since $M = m_1 + m_2$ only. Also orthonormality implies

$$\langle j_1, j_2; J', M' | j_1, j_2; J, M\rangle = \delta_{J'J}\delta_{M'M}$$

$$= \sum_{m_1=-j_1}^{+j_1} \sum_{m_2=-j_2}^{+j_2} \langle j_1, j_2; m_1, m_2 | J', M'\rangle^* \langle j_1, j_2; m_1, m_2 | J, M\rangle$$

The ($\langle j_1, j_2, m_1 = j_1, m_2 = J - j_1 | J, J\rangle \ge 0$) have phase convention we have chosen for constructing the eigenvectors $|j_1, j_2; J, J\rangle$ implies the Clebsch's are real

$$\langle j_1, j_2; m_1, m_2 | J, M\rangle^* = \langle j_1, j_2; m_1, m_2 | J, M\rangle$$

Hence orthonormality becomes

$$\delta_{J'J}\delta_{M'M} = \sum_{m_1=-j_1}^{+j_1}\sum_{m_2=-j_2}^{+j_2} \langle j_1, j_2; m_1, m_2 | J', M' \rangle \times$$

$$\times \langle j_1, j_2; m_1, m_2 | J, M \rangle.$$

As well, we can invert the unitary relation between basis vectors using the orthonormality to obtain

$$|k_1, k_2; j_1, j_2; m_1, m_2\rangle = \sum_{J=|j_1-j_2|}^{j_1+j_2}\sum_{M=-J}^{+J} |k_1, k_2; j_1, j_2; J, M\rangle$$

$$\langle j_1, j_2; m_1, m_2 | J, M\rangle = \underbrace{\times \langle J, M | j_1, j_2; m_1, m_2 \rangle}.$$

Thus $\langle j_1, j_2; m_1', m_2' | j_1, j_2; m_1, m_2 \rangle = \delta_{m_1' m_1}\delta_{m_2' m_2}$

implies

$$\delta_{m_1' m_1}\delta_{m_2' m_2} = \sum_{J=|j_1-j_2|}^{j_1+j_2}\sum_{M=-J}^{+J} \langle j_1, j_2; m_1', m_2' | J, M \rangle \times$$

$$\times \langle J, M | j_1, j_2; m_1, m_2 \rangle$$

and using reality of the Clebsch's

yields

$$\delta_{m_1'm_1}\delta_{m_2'm_2} = \sum_{J=|j_1-j_2|}^{j_1+j_2} \sum_{M=-J}^{+J} \langle j_1,j_2;m_1',m_2'|J,M\rangle \times$$
$$\times \langle j_1,j_2;m_1,m_2|J,M\rangle.$$

(Again $M=m_1+m_2$, so this double sum collapses to a single sum; i.e.
$\langle j_1,j_2;m_1,m_2|J,M\rangle \propto \delta_{M,m_1+m_2}$)

Lastly we can derive recursion relations for the C-G coefficients by applying the J_- operator. Thus we will be mimicking our explicit construction of the transformation between the 2 basis sets. (surpressing k_1,k_2) Starting with

$$|j_1,j_2;J,M\rangle = \sum_{m_1=-j_1}^{+j_1} \sum_{m_2=-j_2}^{+j_2} |j_1,j_2;m_1,m_2\rangle\langle j_1,j_2;m_1,m_2|J,M\rangle$$

apply $J_- = J_{1-} + J_{2-}$. On the LHS we have

$$J_-|j_1,j_2;JM\rangle = \hbar\sqrt{J(J+1)-M(M-1)}\,|j_1,j_2;J,M-1\rangle.$$

On the RHS we use $J_- = J_{1-} + J_{2-}$ to find

$$\sqrt{J(J+1) - M(M-1)}\ |j_1, j_2; J, M-1\rangle$$

$$= \sum_{m_1=-j_1}^{+j_1} \sum_{m_2=-j_2}^{+j_2} \langle j_1, j_2; m_1, m_2 | J, M\rangle \times$$

$$\times \left\{ \sqrt{j_1(j_1+1) - m_1(m_1-1)}\ |j_1, j_2; m_1-1, m_2\rangle \right.$$

$$\left. + \sqrt{j_2(j_2+1) - m_2(m_2-1)}\ |j_1, j_2; m_1, m_2-1\rangle \right\}$$

Taking the inner product with $|j_1, j_2; m_1, m_2\rangle$ and using orthonormality we find the sum reduces to one term:

$$\boxed{\begin{aligned} &\sqrt{J(J+1) - M(M-1)}\ \langle j_1, j_2; m_1, m_2 | J, M-1\rangle \\ &= \sqrt{j_1(j_1+1) - m_1(m_1+1)}\ \langle j_1, j_2; m_1+1, m_2 | J, M\rangle \\ &\quad + \sqrt{j_2(j_2+1) - m_2(m_2+1)}\ \langle j_1, j_2; m_1, m_2+1 | J, M\rangle \end{aligned}}$$

Applying $J_+ = J_{1+} + J_{2+}$ and proceeding in analogous fashion also implies

$$\boxed{\begin{aligned}&\sqrt{J(J+1)-M(M+1)}\,\langle j_1,j_2;m_1,m_2|J,M+1\rangle\\ &=\sqrt{j_1(j_1+1)-m_1(m_1-1)}\,\langle j_1,j_2;m_1-1,m_2|J,M\rangle\\ &\quad+\sqrt{j_2(j_2+1)-m_2(m_2-1)}\,\langle j_1,j_2;m_1,m_2-1|J,M\rangle\end{aligned}}$$

Using the phase convention that

$$\langle j_1,j_2;m_1=j_1,m_2=j_2|J=j_1+j_2,M=j_1+j_2\rangle=1$$

(i.e. $|j_1,j_2;J=j_1+j_2,M=j_1+j_2\rangle$

$$\equiv|j_1,j_2;m_1=j_1,m_2=j_2\rangle$$

The upper-right hand corner of the m_1-m_2 plot and $\langle j_1,j_2;m_1=j_1,m_2=J-j_1|J,J=M\rangle\geq 0$.)
we can obtain all the lower C-G coefficients by applying these recursion formulae.

In fact consider $J=j_1+j_2$, $M=j_1+j_2$
$m_1=j_1-1$; $m_2=j_2$ we find from the J_-

relation

$$\overbrace{\sqrt{(j_1+j_2)(j_1+j_2+1)-(j_1+j_2)(j_1+j_2-1)}}^{=\sqrt{2(j_1+j_2)}}\ \langle j_1,j_2;\ m_1=j_1-1,\ m_2=j_2 \,|\, J=j_1+j_2,\ M=j_1+j_2-1$$

$$= \overbrace{\sqrt{j_1(j_1+1)-j_1(j_1-1)}}^{=\sqrt{2j_1}}\ \overbrace{\langle j_1,j_2;\ j_1,j_2 \,|\, J=j_1+j_2,\ M=j_1+j_2\rangle}^{\equiv 1}$$

$$+\cancelto{0}{\sqrt{j_2(j_2+1)-j_2(j_2+1)}}\ \langle j_1,j_2;\ j_1-1,\ j_2+1 \,|\, J=j_1+j_2,\ M=j_1+j_2\rangle$$

$\Rightarrow$

$$\boxed{\langle j_1,j_2;\ m_1=j_1-1,\ m_2=j_2 \,|\, J=j_1+j_2,\ M=j_1+j_2-1\rangle = \sqrt{\frac{j_1}{j_1+j_2}}}$$

Exactly what we obtained explicitly on page -762-.

In fact Wigner found the closed expression for the Clebsch-Gordon coefficients to be

$$\langle j_1, j_2; m_1, m_2 | J, M\rangle = \delta_{M,(m_1+m_2)} \times$$

$$\times \left[\frac{(2J+1)}{(j_1+j_2+J+1)} (j_2+J-j_1)!\,(J+j_1-j_2)!\,(j_1+j_2-J)! \right.$$
$$\left. \times (j_1+m_1)!\,(j_1-m_1)!\,(j_2+m_2)!\,(j_2-m_2)!\,(J+M)!\,(J-M)! \right]^{1/2} \times$$

$$\times \sum_n{}' \frac{(-1)^n}{n!\,(j_1+j_2-J-n)!\,(n+J-j_1-m_2)!\,(j_2+m_2-n)!\,(n+J-j_2+m_1)! \times (j_1-m_1-n)!}$$

---

where the sum over n is a sum over all non-negative integers n greater than or equal to (m_2+j_1-J) and (j_2-J-m_1) and less than or equal to (j_1+j_2-J) and (j_2+m_2) and (j_1-m_1); thus avoiding the poles in the factorial functions.

(See: Lyubarskii, "The Application of Group Theory in Physics"; Chapter XI; page 232.
Wigner, "Group Theory and its application to the Quantum Mechanics of Atomic Spectra"; Chapter 17, page 191. His notation is
$S^{j_1 j_2}_{J m_1 m_2} \equiv \langle j_1, j_2; m_1, m_2 | J, M \rangle$.)

Indeed we can apply this formula to obtain the C-G coefficients for the $j_1 = \frac{1}{2}, j_2 = \frac{1}{2}$ examples we have considered explicitly

$$\langle j_1 = \tfrac{1}{2}, j_2 = \tfrac{1}{2}; m_1, m_2 | J, M \rangle$$

(m_1, m_2) \ (J, M)	(triplet) $(1,1)$	$(1,0)$	$(1,-1)$	(singlet) $(0,0)$
$(\frac{1}{2}, \frac{1}{2})$	1	0	0	0
$(\frac{1}{2}, -\frac{1}{2})$	0	$\sqrt{\frac{1}{2}}$	0	$\sqrt{\frac{1}{2}}$
$(-\frac{1}{2}, \frac{1}{2})$	0	$\sqrt{\frac{1}{2}}$	0	$-\sqrt{\frac{1}{2}}$
$(-\frac{1}{2}, -\frac{1}{2})$	0	0	1	0

Again recovering our previous results.

As a final example, consider the addition of orbital angular momentum $\vec{L}$ and spin angular momentum $\vec{S}$. $\vec{J} = \vec{L} + \vec{S}$. To be concrete let's consider the case of a spin $\frac{1}{2}$ particle. The eigenvalues of $\vec{L}^2, L_z$ are $l(l+1)\hbar^2$, $m\hbar$ and those of $\vec{S}^2, S_z$ are $s(s+1)\hbar^2$, $m_s\hbar$, $l = 0, 1, 2, \ldots$ and $m = -l, \ldots, +l$ while we will consider $s = \frac{1}{2}$ and $m_s = \pm\frac{1}{2}$. According to our previous notation we call $\vec{J}_1 = \vec{L}$ and $\vec{J}_2 = \vec{S}$; hence $j_1 = l = 0, 1, 2, \ldots$ and $j_2 = s = \frac{1}{2}$. The invariant subspace (surpressing k_1, k_2)

$$\mathcal{H}(j_1 = l, j_2 = \tfrac{1}{2}) \text{ has dimension } (2l+1)(2(\tfrac{1}{2})+1) = (2l+1)2$$

The standard direct product basis

$$\{|j_1, j_2; m_1, m_2\rangle = |l, \tfrac{1}{2}; m, \pm\tfrac{1}{2}\rangle = |l, m\rangle \otimes |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle\}$$

is useful for studying $\vec{L}$ and $\vec{S}$ separately. Many times, as we will see, there is a spin-orbit coupling in the Hamiltonian that is proportional to $\vec{L} \cdot \vec{S}$. Since the above are eigenstates of $\vec{L}^2, L_z, \vec{S}^2, S_z$

They are not eigenstates of $\vec{L}\cdot\vec{S}$, and hence ~~not~~ as useful a basis set as states which are also $\vec{L}\cdot\vec{S}$ eigenstates. Since $\vec{L}\cdot\vec{S}$ $(=\vec{J}_1\cdot\vec{J}_2)$ can be written as

$$\vec{L}\cdot\vec{S} = \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$$ a more

useful basis for studying spin-orbit interaction is one in terms of eigenstates of $\vec{L}^2, \vec{S}^2, \vec{J}^2$, including J_z. This is just our total angular momentum standard basis.

The allowed values of J determine the decomposition of $\mathcal{H}(j_1=l, j_2=\frac{1}{2})$ into fixed J, total angular momentum invariant subspaces $\mathcal{H}(J)$. We found that

$$J = j_1+j_2,\ j_1+j_2-1, \ldots, |j_1-j_2|$$

for $l=0$ we have $J=\frac{1}{2}$ only. In this case the eigenvectors of $\{\vec{L}^2, L_z, \vec{S}^2, S_z\}$ span the 2-dimensional subspace $\mathcal{H}(0,\frac{1}{2}) = \mathcal{H}(\frac{1}{2})$ and $|l=0, s=\frac{1}{2}; m=0; m_s=\pm\frac{1}{2}\rangle$

are also eigenvectors of $\vec{J}^2$ and J_z

since $J_z = L_z + S_z$ clearly $M = m_s = \pm\frac{1}{2}$
and

$$\vec{J}^2 = \vec{L}^2 + \vec{S}^2 + 2\vec{L}\cdot\vec{S} \quad \text{and since}$$

not only does $\vec{L}^2, L_z$ equal zero on this state but
$L\,|l=0, m=0\rangle = 0$ we have

$$\vec{J}^2\,|l=0, \tfrac{1}{2}; m=0; m_s=\pm\tfrac{1}{2}\rangle = \vec{S}^2\,|0, \tfrac{1}{2}; 0, \pm\tfrac{1}{2}\rangle$$
$$= \tfrac{1}{2}(\tfrac{1}{2}+1)\hbar^2\,|0, \tfrac{1}{2}; 0, \pm\tfrac{1}{2}\rangle$$

So $J = \frac{1}{2}$. So

$$|l=0, s=\tfrac{1}{2}; J=\tfrac{1}{2}, M=\pm\tfrac{1}{2}\rangle = |l=0, s=\tfrac{1}{2}; m=0; m_s=\pm\tfrac{1}{2}\rangle$$

For $l > 0$ we have that

$$J = l+\tfrac{1}{2} \text{ and } l-\tfrac{1}{2} \text{ only}$$

hence $\mathcal{H}(l, s=\frac{1}{2}) = \mathcal{H}(J=l+\frac{1}{2}) \oplus \mathcal{H}(J=l-\frac{1}{2})$.

$\mathcal{H}(J=l+\frac{1}{2})$ has dimension $= (2(l+\frac{1}{2})+1) = 2l+2$
$\mathcal{H}(J=l-\frac{1}{2})$ has dimension $= (2(l-\frac{1}{2})+1) = 2l$
for a total of $2(2l+1)$ the dimension of $\mathcal{H}(l, s=\frac{1}{2})$.

We first consider $\mathcal{H}(J=l+\frac{1}{2})$

Since $J = l+\frac{1}{2}$, the highest M value is $M = l+\frac{1}{2}$. The unique eigenvector is, recalling that $M = m + m_s$

$$|l,\tfrac{1}{2};J=l+\tfrac{1}{2},M=l+\tfrac{1}{2}\rangle = |l,m=l\rangle \otimes |\tfrac{1}{2};m_s=\tfrac{1}{2}\rangle$$

Using J_-, the $M = l-\frac{1}{2}$ eigenvector for this $J = l+\frac{1}{2}$ is found

$$J_-|l,\tfrac{1}{2};J=l+\tfrac{1}{2},M=l+\tfrac{1}{2}\rangle\cdot$$

$$= \hbar\sqrt{J(J+1)-M(M-1)}\,|l,\tfrac{1}{2};J=l+\tfrac{1}{2},M'=l+\tfrac{1}{2}-1\rangle$$

using the identity $J(J+1)-M(M-1)$
$$=(J+M)(J-M+1)$$

we have

$$J_-|l,\tfrac{1}{2};J=l+\tfrac{1}{2},M=l+\tfrac{1}{2}\rangle$$
$$=\hbar\sqrt{2l+1}\,|l,\tfrac{1}{2};J=l+\tfrac{1}{2};M'=l-\tfrac{1}{2}\rangle.$$

Also $J_- = L_- + S_-$, applying this to the direct product basis vector we have

$$J_-|l,\tfrac{1}{2};J=l+\tfrac{1}{2},M=l+\tfrac{1}{2}\rangle = L_-|l,m=l\rangle\otimes|\tfrac{1}{2};m_s=\tfrac{1}{2}\rangle$$
$$+|l,m=l\rangle\otimes S_-|\tfrac{1}{2};m_s=\tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{2l}\,|l, m'=l-1\rangle \otimes |\tfrac{1}{2}; m_s=\tfrac{1}{2}\rangle$$
$$+\hbar\,|l, m=l\rangle \otimes |\tfrac{1}{2}; m_s'=-\tfrac{1}{2}\rangle$$

Putting these two expressions equal we have

$$|l, \tfrac{1}{2}; J=l+\tfrac{1}{2}, M=l-\tfrac{1}{2}\rangle$$
$$= \frac{1}{\sqrt{2l+1}}\Big[\sqrt{2l}\,|l, m=l-1\rangle \otimes |\tfrac{1}{2}; m_s=\tfrac{1}{2}\rangle$$
$$+ |l, m=l\rangle \otimes |\tfrac{1}{2}; m_s=-\tfrac{1}{2}\rangle\Big]$$

Applying J_- again, gives the next vector with $M=l-\tfrac{3}{2}$; we find

$$|l, \tfrac{1}{2}; J=l+\tfrac{1}{2}, M=l-\tfrac{3}{2}\rangle$$
$$= \frac{1}{\sqrt{2l+1}}\Big[\sqrt{2l-1}\,|l, m=l-2\rangle \otimes |\tfrac{1}{2}; m_s=\tfrac{1}{2}\rangle$$
$$+\sqrt{2}\,|l, m=l-1\rangle \otimes |\tfrac{1}{2}; m_s=-\tfrac{1}{2}\rangle\Big]$$

In general we see that $|l, \tfrac{1}{2}; J=l+\tfrac{1}{2}, M\rangle$ is the sum of 2 vectors in the $|l, s; m, m_s\rangle$ basis

$$|l, \tfrac{1}{2}; m=M-\tfrac{1}{2}, m_s=+\tfrac{1}{2}\rangle \text{ and } |l, \tfrac{1}{2}; m=M+\tfrac{1}{2}, m_s=-\tfrac{1}{2}\rangle$$

Proceeding we find

$|l,\tfrac{1}{2};J=l+\tfrac{1}{2},M\rangle$

$$=\frac{1}{\sqrt{2l+1}}\Big[\sqrt{l+M+\tfrac{1}{2}}\,|l,m=M-\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},m_s=+\tfrac{1}{2}\rangle$$

$$+\sqrt{l-M+\tfrac{1}{2}}\,|l,m=M+\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},m_s=-\tfrac{1}{2}\rangle\Big]$$

with $M=l+\tfrac{1}{2},\ l-\tfrac{1}{2},\ l-\tfrac{3}{2},\ \dots,\ -(l+\tfrac{1}{2})$.

This can be verified by induction and use of J_-. Assume the above for M and apply J_-

$$J_-|l,\tfrac{1}{2};J=l+\tfrac{1}{2},M\rangle=\hbar\sqrt{(l+M+\tfrac{1}{2})(l-M+\tfrac{3}{2})}\,|l,\tfrac{1}{2};J=l+\tfrac{1}{2},M-1\rangle$$

$$=\frac{1}{\sqrt{2l+1}}\Big[\sqrt{l+M+\tfrac{1}{2}}\Big(\hbar\sqrt{(l+M-\tfrac{1}{2})(l-M+\tfrac{3}{2})}\,|l,m'=M-\tfrac{3}{2}\rangle\otimes|\tfrac{1}{2},m_s=\tfrac{1}{2}\rangle$$

$$+\hbar\,|l,m=M-\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},m_s'=-\tfrac{1}{2}\rangle\Big)$$

$$+\sqrt{l-M+\tfrac{1}{2}}\Big(\hbar\sqrt{(l+M+\tfrac{1}{2})(l-M+\tfrac{1}{2})}\,|l,m'=M-\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},m_s=-\tfrac{1}{2}\rangle$$

$$+|l,m=M+\tfrac{1}{2}\rangle\otimes\overbrace{\cancel{S_-|\tfrac{1}{2},m_s=-\tfrac{1}{2}\rangle}}^{0}\Big)\Big]$$

adding the two $(M-\tfrac{1}{2})$ terms and equating expressions, we have

$$|l,\tfrac{1}{2};J=l+\tfrac{1}{2},M-1\rangle$$

$$=\frac{1}{\sqrt{2l+1}}\Big[\sqrt{l+(M-1)+\tfrac{1}{2}}\,|l,m=(M-1)-\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},m_s=\tfrac{1}{2}\rangle$$

$$+\sqrt{l-(M-1)+\tfrac{1}{2}}\,|l,m=(M-1)+\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},m_s=-\tfrac{1}{2}\rangle\Big]$$

which is the general formula for $(M-1)$. Thus we have constructed the $\mathcal{H}(J=l+\frac{1}{2})$ total angular momentum standard basis vectors.

Next consider $\mathcal{H}(J=l-\frac{1}{2})$. The maximum value of $M=l-\frac{1}{2}$; a 2-fold degenerate value. We have already constructed

$|l,\frac{1}{2};J=l+\frac{1}{2},M=l-\frac{1}{2}\rangle$ and so this

vector must be orthogonal to it as

well as normalized to 1. $|l,\frac{1}{2};J=l-\frac{1}{2},M=l-\frac{1}{2}\rangle$ is a linear combination of

$|l,m=l\rangle\otimes|\frac{1}{2},m_s=-\frac{1}{2}\rangle$ and $|l,m=l-1\rangle\otimes|\frac{1}{2},m_s=\frac{1}{2}\rangle$.

Thus

$$|l,\tfrac{1}{2};J=l-\tfrac{1}{2},M=l-\tfrac{1}{2}\rangle$$
$$=\alpha\,|l,m=l\rangle\otimes|\tfrac{1}{2},m_s=-\tfrac{1}{2}\rangle$$
$$+\beta\,|l,m=l-1\rangle\otimes|\tfrac{1}{2},m_s=+\tfrac{1}{2}\rangle$$

1) Orthogonality $\Rightarrow$

$$0=\langle l,\tfrac{1}{2};J=l+\tfrac{1}{2},M=l-\tfrac{1}{2}|l,\tfrac{1}{2};J=l-\tfrac{1}{2},M=l-\tfrac{1}{2}\rangle$$
$$=\frac{\sqrt{2l}}{\sqrt{2l+1}}\beta+\frac{1}{\sqrt{2l+1}}\alpha$$

$$\Rightarrow\quad\boxed{\beta=-\frac{1}{\sqrt{2l}}\alpha}$$

2) Norm $=1$ $\Rightarrow$ $|\alpha|^2+|\beta|^2=1$

$$\Rightarrow\quad\boxed{\begin{aligned}\alpha&=\frac{\sqrt{2l}}{\sqrt{2l+1}}e^{i\varphi}\\ \beta&=-\frac{1}{\sqrt{2l+1}}e^{i\varphi}\end{aligned}}\quad,\ \varphi\in\mathbb{R}.$$

According to our phase convention, we choose the $\langle j_1, j_2; m_1 = j_1, m_2 = J - j_1 | J, J \rangle \geq 0$ for the Clebsch. This means that $\langle l, \tfrac{1}{2}; m = l, m_s = -\tfrac{1}{2} | J = l - \tfrac{1}{2}, M = l - \tfrac{1}{2} \rangle \geq 0$

hence we take $\varphi = 0$ by convention. Thus we obtain

$$|l, \tfrac{1}{2}; J = l - \tfrac{1}{2}, M = l - \tfrac{1}{2}\rangle$$

$$= \frac{1}{\sqrt{2l+1}} \Big[\sqrt{2l}\, |l, m = l\rangle \otimes |\tfrac{1}{2}, m_s = -\tfrac{1}{2}\rangle$$

$$- |l, m = l-1\rangle \otimes |\tfrac{1}{2}, m_s = +\tfrac{1}{2}\rangle \Big]$$

Since this has a similar structure to the $(J = l + \tfrac{1}{2}, M = l - \tfrac{1}{2})$ case we easily verify that J_- applied to it leads to the general expression for the basis vectors

$$|l, \tfrac{1}{2}; J = l - \tfrac{1}{2}, M\rangle$$

$$= \frac{1}{\sqrt{2l+1}} \Big[\sqrt{l + M + \tfrac{1}{2}}\, |l, m = M + \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, m_s = -\tfrac{1}{2}\rangle$$

$$- \sqrt{l - M + \tfrac{1}{2}}\, |l, m = M - \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, m_s = +\tfrac{1}{2}\rangle \Big]$$

for $M = l-\frac{1}{2}, l-\frac{3}{2}, \ldots, -(l+\frac{1}{2})$.

Hence we have constructed the total angular momentum standard basis for subspace $\mathcal{H}(J=l-\frac{1}{2})$.

The non-vanishing Clebsch-Gordon coefficients are simply read off from our expansions:

$$\langle l, \tfrac{1}{2}; m=M+\tfrac{1}{2}, m_s=-\tfrac{1}{2} | J=l+\tfrac{1}{2}, M\rangle = \sqrt{\frac{l-M+\frac{1}{2}}{2l+1}}$$

$$\langle l, \tfrac{1}{2}; m=M-\tfrac{1}{2}, m_s=+\tfrac{1}{2} | J=l+\tfrac{1}{2}, M\rangle = \sqrt{\frac{l+M+\frac{1}{2}}{2l+1}}$$

$$\langle l, \tfrac{1}{2}; m=M+\tfrac{1}{2}, m_s=-\tfrac{1}{2} | J=l-\tfrac{1}{2}, M\rangle = \sqrt{\frac{l+M+\frac{1}{2}}{2l+1}}$$

$$\langle l, \tfrac{1}{2}; m=M-\tfrac{1}{2}, m_s=+\tfrac{1}{2} | J=l-\tfrac{1}{2}, M\rangle = -\sqrt{\frac{l-M+\frac{1}{2}}{2l+1}}$$

Finally recall that spin $\frac{1}{2}$ particles have wavefunctions (p. -719-) of the form (in the $\vec{S}^2, S_z, |\vec{r}\rangle$ basis)

$$\psi^{(\frac{1}{2})}_{m_s}(\vec{r}) = \langle \vec{r}, s=\tfrac{1}{2}, m_s | \psi \rangle$$

which we can write as a 2-component column vector of wavefunctions

$$\psi^{(\frac{1}{2})}_{m_s}(\vec{r}) = \begin{pmatrix} \psi_\uparrow(\vec{r}) \\ \psi_\downarrow(\vec{r}) \end{pmatrix}_{m_s}$$

Further in the $\{A, \vec{L}^2, L_z\}$ basis each $\psi_{\uparrow\downarrow}(\vec{r})$ can be expanded in terms of spherical harmonics: the eigenfunctions of $\vec{L}^2, L_z$ in the $\{|\vec{r}\rangle\}$ basis; thus each $\{A, \vec{L}^2, L_z, \vec{S}^2, S_z\}$ basis vector

$|k, l, s=\frac{1}{2}; m, m_s = \pm\frac{1}{2}\rangle$ has the 2-component wavefunction representation

$$\psi_{|k,l,s=\frac{1}{2};m,m_s=\frac{1}{2}\rangle}(\vec{r}) = R_{k,l}(r)\, Y_l^m(\theta,\varphi) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\psi_{|k,l,s=\frac{1}{2};m,m_s=-\frac{1}{2}\rangle}(\vec{r}) = R_{kl}(r)\, Y_l^m(\theta,\varphi) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The total angular momentum basis vectors $|k,l,s=\frac{1}{2};J,M\rangle$ have wavefunctions given by the Clebsch-Gordon coefficients times the $|k,l,s;m,m_s\rangle$ wavefunctions. Thus we find (recall J is half-integral & so is M; thus $M\pm\frac{1}{2}$ is integral)

$$\psi_{|k,l,s=\frac{1}{2};J=l+\frac{1}{2},M\rangle}(\vec{r}) = \frac{R_{kl}(r)}{\sqrt{2l+1}} \begin{bmatrix} \sqrt{l+M+\frac{1}{2}}\; Y_l^{M-\frac{1}{2}}(\theta,\varphi) \\ \\ \sqrt{l-M+\frac{1}{2}}\; Y_l^{M+\frac{1}{2}}(\theta,\varphi) \end{bmatrix}$$

and

$$\psi_{|k,l,s=\frac{1}{2};J=l-\frac{1}{2},M\rangle}(\vec{r}) = \frac{R_{kl}(r)}{\sqrt{2l+1}} \begin{bmatrix} -\sqrt{l-M+\frac{1}{2}}\; Y_l^{M-\frac{1}{2}}(\theta,\varphi) \\ \\ \sqrt{l+M+\frac{1}{2}}\; Y_l^{M+\frac{1}{2}}(\theta,\varphi) \end{bmatrix}$$