4.4. Schrödinger and Heisenberg Representations

For an isolated system, the above formalism is such that all operators are time independent while the physical states of the system have time dependence given by the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle,$$

and $$\frac{d}{dt} A = 0.$$

This dynamical description is referred to as the Schrödinger representation or picture.

Such a separation of the time evolution of the system is not unique. In fact we can equivalently use time independent states and time dependent operators. Such a description is referred to as the Heisenberg representation or picture.

The two pictures are related by a unitary transformation.

Let $|\psi\rangle_S$ and A_S denote a state and operator in the Schrödinger representation. The corresponding state and operator in the Heisenberg representation is defined by

$$|\psi\rangle_H \equiv e^{+\frac{i}{\hbar}Ht}\,|\psi(t)\rangle_S$$

$$A_H(t) \equiv e^{+\frac{i}{\hbar}Ht}\,A_S\,e^{-\frac{i}{\hbar}Ht}$$

where H is the Hermitian Hamiltonian operator $H = H^\dagger$ and $H_H = H_S = H$ since

$$H_H = e^{iHt/\hbar}\,\underbrace{H_S}_{\equiv H}\,e^{-iHt/\hbar} \qquad \text{since } [H,H]=0$$

$$= H = H_S.$$

Thus the operator $U(t) \equiv e^{iHt/\hbar}$

is unitary

$$U^\dagger(t) = \left(e^{iHt/\hbar}\right)^\dagger$$

$$= \left[1 + \frac{i}{\hbar}Ht + \frac{1}{2!}\left(\frac{i}{\hbar}\right)^2 H^2 t^2 + \cdots\right]^\dagger$$

$$= \left[1 + \left(\frac{i}{\hbar}\right)^* H^\dagger + \frac{1}{2}\left(\frac{i}{\hbar}\right)^{2*} H^\dagger H^\dagger t^2 + \cdots\right]$$

$$= \left[1 - \frac{i}{\hbar}H + \frac{1}{2}\left(-\frac{i}{\hbar}\right)^2 H^2 t^2 + \cdots\right]$$

$$= e^{-iHt/\hbar} = U^{-1}(t).$$

Hence the scalar product and all linear relations are left unchanged.

$${}_H\langle\psi|\phi\rangle_H = {}_S\langle\psi(t)|e^{-iHt/\hbar}\, e^{+iHt/\hbar}|\phi(t)\rangle_S$$

$$= {}_S\langle\psi(t)|\phi(t)\rangle_S .$$

Further matrix elements of operators are the same in each picture

$${}_H\langle\psi|A_H(t)|\phi\rangle_H$$

$$= {}_S\langle\psi(t)|e^{-iHt/\hbar}\left(e^{+\frac{i}{\hbar}Ht} A_S\, e^{-\frac{i}{\hbar}Ht}\right) \times$$

$$\times\, e^{+\frac{i}{\hbar}Ht}|\phi(t)\rangle_S$$

$$= {}_S\langle\psi(t)|A_S|\phi(t)\rangle_S .$$

Since the states are time independent in the Heisenberg picture the operators carry all the dynamics. So consider

$$\frac{d}{dt} A_H(t) = \frac{d}{dt}\left(e^{\frac{iHt}{\hbar}} A_S e^{-iHt/\hbar}\right)$$

$$= \frac{i}{\hbar} H e^{\frac{iHt}{\hbar}} A_S e^{-iHt/\hbar} - \frac{i}{\hbar} e^{\frac{iHt}{\hbar}} A_S e^{-iHt/\hbar} H$$

$$= \frac{i}{\hbar}\left(H A_H(t) - A_H(t) H\right)$$

$$= \frac{i}{\hbar}[H, A_H(t)]$$

That is

$$\boxed{\frac{d}{dt} A_H(t) = \frac{i}{\hbar}[H, A_H(t)]}$$

This is the <u>Heisenberg equation of motion</u>. (if A_S has explicit time dependence $\left(\frac{\partial A_S}{\partial t}\right)$ we add i.e. $0 \neq \frac{\partial}{\partial t} A_S \Rightarrow \frac{d}{dt} A_H(t) = \frac{\partial}{\partial t} A_H(t) + \frac{i}{\hbar}[H, A_H(t)]$)

On the other hand,

$$\frac{d}{dt} |\psi\rangle_H = \frac{d}{dt}\left(e^{iHt/\hbar} |\psi(t)\rangle_S\right)$$

$$= \frac{i}{\hbar} H e^{iHt/\hbar} |\psi(t)\rangle_S + e^{iHt/\hbar} \underbrace{\frac{d}{dt}|\psi(t)\rangle_S}$$

$$= \frac{i}{\hbar} H|\psi\rangle_H - \frac{i}{\hbar} e^{iHt/\hbar} H|\psi(t)\rangle_S \qquad = -\frac{i}{\hbar}H|\psi(t)\rangle_S$$

$$= \frac{i}{\hbar} H|\psi\rangle_H - \frac{i}{\hbar} H|\psi\rangle_H = 0 .$$

Thus $\boxed{\frac{d}{dt}|\psi\rangle_H = 0}$, as promised.

Since the inner product and matrix elements of operators remains unchanged by the unitary transformation between the pictures, each is equally valid as a description of the time evolution of the system. Clearly any unitary transformation will leave the physics unaltered, hence there are infinitely many pictures all equally valid descriptions of the time evolution.

A particularly useful third picture is one obtained by dividing the Hamiltonian into two parts

$$H = H_0^S + H_I^S$$ here in the Schrödinger picture to start.

The idea of the interaction picture is that the states evolve in time according to the "interaction" Hamiltonian H_I, while the operators evolve according to the unperturbed or "free" Hamiltonian H_0. Hence we begin by removing the free Hamiltonian evolution from the Schrödinger state by defining the interaction picture state as

$$|\psi(t)\rangle_{IP} \equiv e^{+\frac{i}{\hbar}H_0^S t}\,|\psi(t)\rangle_S ,$$

and the interaction picture operators evolve in time according to H_0^S only

$$A_{IP}(t) \equiv e^{+\frac{i}{\hbar}H_0^S t}\,A_S\,e^{-\frac{i}{\hbar}H_0^S t} .$$

First note that $H_0^{IP} = e^{\frac{i}{\hbar}H_0^S t}\,H_0^S\,e^{-\frac{i}{\hbar}H_0^S t}$

$$= H_0^S$$

But since in general $[H_0^S, H_I^S] \neq 0$ we have $H_{IP} \neq H_S$, $H_0^H \neq H_0^S$ and H_I^S, H_I^H, H_I^{IP} all different.

So we have the Heisenberg equation of motion

$$\frac{d}{dt} A_{IP}(t) = \frac{i}{\hbar}\left[H_0^{IP}, A_{IP}(t)\right]$$

for the operators, with just H_0^{IP}.

Next consider,

$$i\hbar \frac{d}{dt}|\psi(t)\rangle_{IP} = -H_0^s |\psi(t)\rangle_{IP} + e^{\frac{i}{\hbar}H_0^s t} \underbrace{\underbrace{i\hbar\frac{d}{dt}|\psi(t)\rangle_s}_{= H^s|\psi(t)\rangle_s}}_{=(H_0^s + H_I^s)|\psi(t)\rangle_s}$$

$$= \cancel{-H_0^s |\psi(t)\rangle_{IP}} + \cancel{H_0^s |\psi(t)\rangle_{IP}} + \underbrace{e^{\frac{i}{\hbar}H_0^s t} H_I^s e^{-\frac{i}{\hbar}H_0^s t}}_{= H_I^{IP}(t)} \underbrace{e^{+\frac{i}{\hbar}H_0^s t} |\psi(t)\rangle_s}_{= |\psi(t)\rangle_{IP}}$$

$$= H_I^{IP}(t)\,|\psi(t)\rangle_{IP}.$$

That is the interaction picture states obey the Schrödinger equation with H_I^{IP},

$$i\hbar \frac{d}{dt} |\psi(t)\rangle_{IP} = H_I^{IP}(t)\, |\psi(t)\rangle_{IP} .$$

Note that $H_I^{IP}(t) = e^{\frac{i}{\hbar} H_0^S t}\, H_I^S\, e^{-\frac{i}{\hbar} H_0^S t}$

depends on time in general even if H_0^S and H_I^S are time independent since $[H_0^S, H_I^S] \neq 0$.

As usual $e^{\frac{i}{\hbar} H_0^S t}$ is unitary, so

$$
\begin{aligned}
{}_{IP}\langle \psi(t) | A^{IP}(t) | \phi(t) \rangle_{IP}
&= {}_S\langle \psi(t) | A_S | \phi(t) \rangle_S \\
&= {}_H\langle \psi | A_H(t) | \phi \rangle_H .
\end{aligned}
$$

Clearly, other pictures are possible.

Finally, note that in the Schrödinger picture, for an isolated system in which H is time independent, it is straightforward to integrate the Schrödinger equation to obtain finite time translations. Suppose

$$i\hbar \frac{d}{dt} |\psi(t)\rangle_s = H\, |\psi(t)\rangle_s$$

with $|\psi(t_0)\rangle_s$ given initially, then

$$|\psi(t)\rangle_s = U(t,t_0)\, |\psi(t_0)\rangle_s$$

where $$U(t,t_0) = e^{-\frac{i}{\hbar} H(t-t_0)} .$$

We can check directly that $|\psi(t)\rangle_s$ satisfies the Schrödinger equation and since $U(t_0,t_0) = 1$, it reduces to $|\psi(t_0)\rangle_s$ at $t = t_0$.

$U(t,t_0)$ is the unitary time translation operator or time evolution operator.

When H depends on time, or in the interaction picture since H_I^{IP} depends on time, such a simple solution does not exist for $U(t,t_0)$. Even so, since we desire the principle of superposition to remain valid at time t as well as t_0, the states $|\psi(t)\rangle_s$ should be related to $|\psi(t_0)\rangle_s$ by a linear operator, So in general we have

$$|\psi(t)\rangle_s = U(t,t_0)\,|\psi(t_0)\rangle_s$$

with $U(t_0,t_0)=1$. The Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle_s = H(t)\,|\psi(t)\rangle_s$$

implies

$$i\hbar \frac{d}{dt}U(t,t_0) = H(t)\,U(t,t_0)$$

along with the initial condition $U(t_0,t_0)=1$.

These can be turned into an integral equation for the time evolution operator

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, H(t')\,U(t',t_0) \,.$$

As we will see next semester, this has a useful iterative solution.

Similar results apply to the IP case.