Review of Poincaré transformations and SL(2,C) spinor representations of the Lorentz group.

Poincaré transformations consist of a Lorentz transformation $\Lambda^{\mu}_{\cdot\nu}$ and a space-time translation by a^{μ}:

$$x'^{\mu} = \Lambda^{\mu}_{\cdot\nu} x^{\nu} + a^{\mu}$$ for which

the interval $ds^2 = dx^{\mu} g_{\mu\nu} dx^{\nu}$ $\left(g_{\mu\nu} = (+1,-1,-1,-1)\right)$ is left invariant:

$$ds'^2 = dx'^{\mu} g_{\mu\nu} dx'^{\nu} = ds^2 = dx^{\alpha} g_{\alpha\beta} dx^{\beta}$$

$$= dx^{\alpha} \Lambda^{\mu}_{\cdot\alpha} g_{\mu\nu} \Lambda^{\nu}_{\cdot\beta} dx^{\beta}$$

$$\Rightarrow \boxed{g_{\alpha\beta} = \Lambda^{\mu}_{\cdot\alpha} g_{\mu\nu} \Lambda^{\nu}_{\cdot\beta}}$$ with metric

$$g_{\mu\nu} = \begin{bmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}_{\mu\nu}.$$

determines $\Lambda^{\mu}_{\cdot\nu}$.

The Poincaré transformations form a group, ISO(1,3), with the product law for 2-transformations $(\Lambda_1, a_1)(\Lambda_2, a_2)$ being equal to another Poincaré transformation (Λ, a) with

$$x'^{\mu} = \Lambda^{\mu}_{2\cdot\rho} x^{\rho} + a_2^{\mu}$$

$$x''^{\mu} = \Lambda^{\mu}_{1\cdot\rho} x'^{\rho} + a_1^{\mu} = \Lambda^{\mu}_{1\cdot\rho}\left[\Lambda^{\rho}_{2\cdot\nu} x^{\nu} + a_2^{\rho}\right] + a_1^{\mu}$$

$$= \left(\Lambda^{\mu}_{1\cdot\rho}\Lambda^{\rho}_{2\,\nu}\right) x^{\nu} + \left[\Lambda^{\mu}_{1\cdot\rho} a_2^{\rho} + a_1^{\mu}\right]$$

$$\Rightarrow \quad = \Lambda^{\mu}_{\cdot\nu} x^{\nu} + a^{\mu}$$

So $\Lambda^{\mu}_{\cdot\nu} = \Lambda^{\mu}_{1\cdot\rho}\Lambda^{\rho}_{2\,\nu}$

$$a^{\mu} = \Lambda^{\mu}_{1\cdot\nu} a_2^{\nu} + a_1^{\mu}$$

As usual we can define contravariant & covariant tensor fields according to their Lorentz transformation properties

Contravariant rank-n tensor field

$$T'^{\mu_1\cdots\mu_n}(x') = \Lambda^{\mu_1}_{\cdot\nu_1}\Lambda^{\mu_2}_{\cdot\nu_2}\cdots\Lambda^{\mu_n}_{\cdot\nu_n} T^{\nu_1\cdots\nu_n}(x)$$

Covariant rank n tensor field

$$T'_{\mu_1\cdots\mu_n}(x') = \Lambda^{-1\nu_1}{}_{\mu_1}\cdots\Lambda^{-1\nu_n}{}_{\mu_n} T_{\nu_1\cdots\nu_n}(x)$$

$$= T_{\nu_1\cdots\nu_n}(x)\Lambda^{-1\nu_1}{}_{\cdot\mu_1}\cdots\Lambda^{-1\nu_n}{}_{\cdot\mu_n}$$

with

$$\Lambda^{\mu}{}_{\cdot\nu}\Lambda^{-1\nu}{}_{\cdot\rho} = \delta^{\mu}{}_{\cdot\rho} = \Lambda^{-1\mu}{}_{\cdot\nu}\Lambda^{\nu}{}_{\cdot\rho}$$

Mixed (m,n) tensor

$$T'^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}(x') = \Lambda^{\mu_1}{}_{\cdot\alpha_1}\cdots\Lambda^{\mu_m}{}_{\cdot\alpha_m} T^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}(x)\cdot$$
$$\cdot\Lambda^{-1\beta_1}{}_{\cdot\nu_1}\cdots\Lambda^{-1\beta_n}{}_{\cdot\nu_n}$$

Note: $V'^{\mu}W'_{\mu} = \Lambda^{\mu}{}_{\cdot\nu}\Lambda^{-1\rho}{}_{\cdot\mu}V^{\nu}W_{\rho}$

$$= \delta^{\rho}{}_{\cdot\nu}V^{\nu}W_{\rho} = V^{\nu}W_{\nu}$$ a Lorentz invariant

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Define the contravariant metric tensor $g^{\mu\nu}$ as the inverse of $g_{\mu\nu}$

$$g^{\mu\nu}g_{\nu\rho} = \delta^{\mu}{}_{\cdot\rho}.$$

So $g = \Lambda^T g \Lambda$ implies that g is invariant

$$g'_{\mu\nu} = g_{\alpha\beta}\Lambda^{-1\alpha}{}_{\cdot\mu}\Lambda^{-1\beta}{}_{\cdot\nu} = g_{\mu\nu}$$

and similarly $g'^{\mu\nu} = g^{\mu\nu}$: the metric tensor is invariant.

For every contravariant vector there is an associated covariant vector, and vice versa, obtained by lowering or raising an index with $g_{\mu\nu}$ or $g^{\mu\nu}$. That is if

$$V'^{\mu} = \Lambda^{\mu}{}_{\cdot\nu} V^{\nu} \quad \text{then}$$

$$V_{\mu} \equiv g_{\mu\nu} V^{\nu} \quad \text{is covariant since}$$

$$V'_{\mu} = g_{\mu\nu} V'^{\nu} = g_{\mu\nu} \Lambda^{\nu}{}_{\cdot\rho} V^{\rho}$$

but

$$g_{\mu\nu} \Lambda^{\nu}{}_{\cdot\rho} = \Lambda^{-1\beta}{}_{\cdot\mu}\, g_{\beta\rho}$$

$$\Rightarrow \quad V'_{\mu} = \Lambda^{-1\beta}{}_{\cdot\mu}\, g_{\beta\rho} V^{\rho}$$

$$= V_{\beta} \Lambda^{-1\beta}{}_{\cdot\mu} .$$

✓ the transformation law for covariant vectors.

Using this notation to define $\Lambda_{\mu\nu} = g_{\mu\alpha} \Lambda^{\alpha}{}_{\cdot\nu}$, etc., we find

$$\delta^{\alpha}{}_{\cdot\beta} = \Lambda^{\mu\alpha} \Lambda_{\mu\beta} \quad \text{that is } \Lambda^{-1\mu\nu} = \Lambda^{T\mu\nu}$$

$$= \Lambda^{\nu\mu} .$$

For infinitesimal Lorentz transformations

$$\Lambda^{\mu}_{\cdot\nu} \doteq \delta^{\mu}_{\cdot\nu} + \omega^{\mu}_{\cdot\nu} \quad . \quad \text{Hence}$$

$$g_{\alpha\beta} = \Lambda^{\mu}_{\cdot\alpha}\, g_{\mu\nu}\, \Lambda^{\nu}_{\cdot\beta}$$

$$= (\delta^{\mu}_{\cdot\alpha} + \omega^{\mu}_{\cdot\alpha})\, g_{\mu\nu}\, (\delta^{\nu}_{\cdot\beta} + \omega^{\nu}_{\cdot\beta})$$

$$= g_{\alpha\beta} + \omega_{\beta\alpha} + \omega_{\alpha\beta}$$

$$\Rightarrow \boxed{\omega_{\mu\nu} + \omega_{\nu\mu} = 0}$$ $\omega_{\mu\nu}$ is anti-symmetric.

The fundamental vector representation of the Lorentz group is given by the coordinate transformation

$$x'^{\mu} = \Lambda^{\mu}_{\cdot\nu} x^{\nu} = x^{\mu} + \omega^{\mu}_{\cdot\nu} x^{\nu}$$

$$\equiv x^{\mu} - \tfrac{1}{2}\omega_{\alpha\beta} [D^{\alpha\beta}]^{\mu}_{\cdot\nu} x^{\nu}$$

$$= x^{\mu} + \tfrac{1}{2}\omega^{\alpha\beta} [\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}] x_{\nu}$$

$$\Rightarrow$$

$$[D^{\alpha\beta}]_{\mu\nu} = -[\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\beta}_{\mu}\delta^{\alpha}_{\nu}]$$ the fundamental representation matrix for the Lorentz group

Note: Here we use the notation $(D^{\alpha\beta})_{\mu\nu}$ for the representation where for SU(3), SU(2), U(1) we used $(T^{i})_{\alpha\beta}$ i.e. $(T^{\alpha\beta})_{\mu\nu} = (D^{\alpha\beta})_{\mu\nu}$

Also note $[D^{\alpha\beta}]_{\mu\nu} = -[D^{\alpha\beta}]_{\nu\mu} = -[D^{\beta\alpha}]_{\mu\nu}$.

From this vector representation we can obtain the commutation relations that all representation matrices $[D^{\alpha\beta}]$ of the Lorentz group must obey

$$[D^{\mu\nu}, D^{\rho\sigma}]_{\alpha\beta} = (D^{\mu\nu})_{\alpha}{}^{\gamma}(D^{\rho\sigma})_{\gamma\beta} - (D^{\rho\sigma})_{\alpha}{}^{\gamma}(D^{\mu\nu})_{\gamma\beta}$$

$$= (\delta^{\nu}_{\alpha} g^{\mu\gamma} - \delta^{\mu}_{\alpha} g^{\nu\gamma})(\delta^{\sigma}_{\gamma}\delta^{\rho}_{\beta} - \delta^{\rho}_{\gamma}\delta^{\sigma}_{\beta})$$

$$-(\delta^{\sigma}_{\alpha} g^{\rho\gamma} - \delta^{\rho}_{\alpha} g^{\sigma\gamma})(\delta^{\nu}_{\gamma}\delta^{\mu}_{\beta} - \delta^{\mu}_{\gamma}\delta^{\nu}_{\beta})$$

$$= g^{\mu\sigma}[\delta^{\nu}_{\alpha}\delta^{\rho}_{\beta} - \delta^{\rho}_{\alpha}\delta^{\nu}_{\beta}]$$

$$-g^{\mu\rho}[\delta^{\nu}_{\alpha}\delta^{\sigma}_{\beta} - \delta^{\sigma}_{\alpha}\delta^{\nu}_{\beta}]$$

$$-g^{\nu\sigma}[\delta^{\mu}_{\alpha}\delta^{\rho}_{\beta} - \delta^{\rho}_{\alpha}\delta^{\mu}_{\beta}] + g^{\nu\rho}[\delta^{\mu}_{\alpha}\delta^{\sigma}_{\beta} - \delta^{\sigma}_{\alpha}\delta^{\mu}_{\beta}]$$

$$= [g^{\mu\sigma}D^{\rho\nu} - g^{\mu\rho}D^{\sigma\nu} - g^{\nu\sigma}D^{\rho\mu} + g^{\nu\rho}D^{\sigma\mu}]_{\alpha\beta}$$

$$= [D^{\mu\nu}, D^{\rho\sigma}]_{\alpha\beta}$$

Hence for a general tensor field, its Lorentz representation matrix can be found in a similar way since it transforms like a product of vector representations:

$$T'^{\mu_1\cdots\mu_n}(x') = \Lambda^{\mu_1\nu_1}\cdots\Lambda^{\mu_n\nu_n}\,T_{\nu_1\cdots\nu_n}(x)$$

$$= [g^{\mu_1\nu_1}+\omega^{\mu_1\nu_1}]\cdots[g^{\mu_n\nu_n}+\omega^{\mu_n\nu_n}]\,T_{\nu_1\cdots\nu_n}(x)$$

$$= T^{\mu_1\cdots\mu_n}(x) + \sum_{i=1}^{n}\omega^{\mu_i\nu_i}\,T^{\mu_1\cdots\underset{\nu_i}{\bullet}\cdots\mu_n}(x)$$

$$= T^{\mu_1\cdots\mu_n}(x) + \frac{1}{2}\omega^{\alpha\beta}\sum_{i=1}^{n}[\delta^{\mu_i}_{\alpha}\delta^{\nu_i}_{\beta}-\delta^{\nu_i}_{\alpha}\delta^{\mu_i}_{\beta}]\times$$

$$\times\, g^{\mu_1\nu_1}\cdots \cancel{g^{\mu_i\nu_i}}\cdots g^{\mu_n\nu_n}\,T_{\nu_1\cdots\nu_n}(x)$$

Using the notation $(\mu) = \mu_1\cdots\mu_n$, etc., this can be written as

$$T'^{(\mu)}(x') \equiv T^{(\mu)}(x) - \frac{1}{2}\omega^{\alpha\beta}[D_{\alpha\beta}]^{(\mu)(\nu)}\,T_{(\nu)}(x)$$

where

$$[D_{\alpha\beta}]^{(\mu)(\nu)} \equiv -\sum_{i=1}^{n} g^{\mu_1\nu_1}\cdots[\delta^{\mu_i}_{\alpha}\delta^{\nu_i}_{\beta}-\delta^{\nu_i}_{\alpha}\delta^{\mu_i}_{\beta}]\cdots g^{\mu_n\nu_n}$$

As in the fundamental vector representation case

The tensorial representation matrices $[D^{\alpha\beta}]_{(\mu)(\nu)}$ obey the Lorentz algebra commutation relations

$$[D^{\mu\nu}, D^{\rho\sigma}]_{(\alpha)(\beta)} = [g^{\mu\rho} D^{\nu\sigma} - g^{\mu\sigma} D^{\nu\rho} + g^{\nu\sigma} D^{\mu\rho} - g^{\nu\rho} D^{\mu\sigma}]_{(\alpha)(\beta)}$$

In general, the difference

$$\boxed{\delta T(x) \equiv T'(x') - T(x)}$$

is called the <u>total</u> <u>variation</u> of T.

The <u>intrinsic</u> (<u>Lie</u>) variation of T is defined as

$$\boxed{\bar{\delta} T(x) \equiv T'(x) - T(x)}$$

$$= [T'(x') - T(x)] - [T'(x') - T'(x)]$$

$$= \delta T(x) - [T'(x') - T'(x)]$$

Since these are infinitesimal variations

$$\boxed{x'^{\mu} = x^{\mu} + \delta x^{\mu}}$$

and so $T'(x') = T'(x) + \delta x^\mu \partial_\mu T(x)$

Hence
$$\boxed{\bar{\delta} T(x) = \delta T(x) - \delta x^\mu \partial_\mu T(x)}$$

For a Lorentz transformation

$$x'^\mu = x^\mu + \omega^{\mu\nu} x_\nu = x^\mu + \tfrac{1}{2}\omega_{\alpha\beta}\left[g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}\right]x_\nu$$

$$\Rightarrow \quad \delta x^\mu = x'^\mu - x^\mu = \tfrac{1}{2}\omega_{\alpha\beta}\left[g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}\right]x_\nu$$

So for a rank n tensor $T^{(\mu)}$ the intrinsic variation is

$$\bar{\delta} T^{(\mu)}_{(x)} = -\tfrac{1}{2}\omega_{\alpha\beta}\Big\{\left[D^{\alpha\beta}\right]^{(\mu)(\nu)} - g^{(\mu)(\nu)}\left[x^\alpha \partial^\beta - x^\beta \partial^\alpha\right]\Big\} T_{(\nu)}(x)$$

Thus we define the differential operator

$$\boxed{\begin{aligned} M^{\mu\nu} T^{(\alpha)}_{(x)} &\equiv i\Big[\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) g^{(\alpha)(\beta)} - \left[D^{\mu\nu}\right]^{(\alpha)(\beta)}\Big] T_{(\beta)}(x) \\ &\equiv \left[M^{\mu\nu}\right]^{(\alpha)(\beta)} T_{(\beta)}(x) \end{aligned}}$$

The $M^{\mu\nu}$ operators obey the Lorentz algebra (with a i factor)

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\left[g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho} - g^{\nu\rho}M^{\mu\sigma}\right]$$

(Note: $J^{\mu\nu} \equiv i(x^\mu \partial^\nu - x^\nu \partial^\mu)$ obeys the algebra as a differential operator and of course

$$x'^\mu = x^\mu + \left(\tfrac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta}\right)x^\mu$$
$$= e^{\frac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta}}\, x^\mu \quad)$$

Hence we can represent the Lorentz group by finite matrice $[D^{\mu\nu}]$ and by space-time differential operators, $J^{\mu\nu}$, acting on tensor fields $T^{(\alpha)}$.

Similarly we can consider infinitesimal space-time translations

$$x'^\mu = x^\mu + \epsilon^\mu \quad , \quad \text{i.e. } \delta x^\mu = \epsilon^\mu$$

For translation invariant fields

$$T'(x') = T(x) \quad \text{i.e. } \delta T(x) = 0$$

we have the intrinsic variation

$$\bar{\delta} T(x) = -\epsilon^{\mu} \partial_{\mu} T(x) \equiv i \epsilon^{\mu} P_{\mu} T(x)$$

That is $P_{\mu} \equiv i \partial_{\mu}$ represents the translation subgroup on tensor fields

For any tensor representation of the Lorentz group $M^{\mu\nu}$, we can calculate the commutator of P^{λ} and $M^{\mu\nu}$

$$[M^{\mu\nu}, P^{\lambda}] = +i [P^{\mu} g^{\nu\lambda} - P^{\nu} g^{\mu\lambda}]$$

Along with $[P_{\mu}, P_{\nu}] = 0$ this set of 3 commutators defines the action of the Poincaré group on tensor fields $T^{(\mu)}$.

To summarize, the Poincaré group of transformations is defined by the algebra its generators obey, P_μ the energy-momentum operator and generator of space-time translations and $M_{\mu\nu}$, the angular momentum operator and generator of Lorentz transformations and space rotations.

Poincaré Algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho} M_{\nu\sigma} - g_{\mu\sigma} M_{\nu\rho} + g_{\nu\sigma} M_{\mu\rho} - g_{\nu\rho} M_{\mu\sigma})$$

$$[P_\mu, P_\nu] = 0$$

$$[M_{\mu\nu}, P_\lambda] = i(P_\mu g_{\nu\lambda} - P_\nu g_{\mu\lambda})$$

For the tensor field representation of the Poincaré group this algebra is represented by matrix-differential operators acting on the tensor fields

$$P_\mu T^{(\alpha)}(x) = i\partial_\mu T^{(\alpha)}(x)$$

$$M_{\mu\nu} T^{(\alpha)}(x) = i\left[(x_\mu \partial_\nu - x_\nu \partial_\mu) g^{(\alpha)(\beta)} - [D_{\mu\nu}]^{(\alpha)(\beta)}\right] T_{(\beta)}(x)$$

where $(\alpha) = \alpha_1 \alpha_2 \cdots \alpha_n$ for a n^{th} rank tensor and

$$g^{(\alpha)(\beta)} = g^{\alpha_1\beta_1} \cdots g^{\alpha_n\beta_n}$$ while

$$[D_{\mu\nu}]^{(\alpha)(\beta)} = \sum_{i=1}^{n} g^{\alpha_1\beta_1} \cdots [\delta^{\alpha_i}_{\mu} \delta^{\beta_i}_{\nu} - \delta^{\beta_i}_{\mu} \delta^{\alpha_i}_{\nu}] \cdots g^{\alpha_n\beta_n}$$

is the matrix for the (reducible) finite dimensional tensor representation of the Lorentz group where $[D_{\mu\nu}]$ obeys the Lorentz algebra

$$[D_{\mu\nu}, D_{\rho\sigma}] = (g_{\mu\rho} D_{\nu\sigma} - g_{\mu\sigma} D_{\nu\rho} + g_{\nu\sigma} D_{\mu\rho} - g_{\nu\rho} D_{\mu\sigma})$$

The transformations induced in $T^{(\alpha)}_{(x)}$ when finite Poincaré transformations are made,

$$x'^{\mu} = \Lambda^{\mu}_{\cdot\nu} x^{\nu} + a^{\mu},$$

are obtained by exponentiating the generators P_μ & $M_{\mu\nu}$ with their corresponding transformation parameters

$$T'^{\alpha_1\cdots\alpha_n}(x) = \Lambda^{\alpha_1}_{\cdot\,\beta_1}\cdots\Lambda^{\alpha_n}_{\cdot\,\beta_n}\, T^{\beta_1\cdots\beta_n}(\Lambda^{-1}(x-a))$$

$$= \left[e^{+ia^\mu P_\mu}\, e^{-\frac{i}{2}\omega^{\mu\nu}(\Lambda) M_{\mu\nu}}\right]^{(\alpha)(\beta)} T_{(\beta)}(x)$$

with $\omega^{\mu\nu}(\Lambda)$ the finite angles of rotation defining $\Lambda^{\mu\nu}$:

$$e^{-\frac{1}{2}\omega^{\mu\nu}(\Lambda) J_{\mu\nu}}\, x^\rho = e^{\frac{1}{2}\omega^{\mu\nu}(\Lambda)[x_\mu \partial_\nu - x_\nu \partial_\mu]}\, x^\rho$$

$$= (\Lambda^{-1})^{\rho}_{\cdot\,\sigma}\, x^\sigma \, .$$

$$= \left[e^{\frac{1}{2}\omega^{\mu\nu}(\Lambda) D_{\mu\nu}}\right]^{\rho}_{\sigma} x^\sigma$$

$$= \left[e^{(-\omega)}\right]^{\rho}_{\sigma} x^\sigma$$

The tensor representations are not the only representations of the algebra — there are also <u>spinor representations</u>

Recall that the total angular momentum operator corresponds to

$$J^i \equiv -\frac{1}{2}\epsilon_{ijk} M_{jk} = (M_{32}, M_{13}, M_{21})$$

and the Lorentz boost operator

$$K^i \equiv M^{i0} = (M^{10}, M^{20}, M^{30})$$

The $M^{\mu\nu}$ Lorentz algebra $\Rightarrow$

$$[J_i, J_j] = +i\epsilon_{ijk} J_k$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$

$$[J_i, K_j] = +i\epsilon_{ijk} K_k \quad .$$

The complex (non-Hermitian) generators

$$\vec{N} \equiv \frac{1}{2}(\vec{J} + i\vec{K})$$

$$\vec{N}^\dagger \equiv \frac{1}{2}(\vec{J} - i\vec{K})$$

can be defined to disentangle the commutation relations:

$$[N_i, N_j] = +i\epsilon_{ijk} N_k$$

$$[N_i^\dagger, N_j^\dagger] = +i\epsilon_{ijk} N_k^\dagger$$

$$[N_i, N_j^\dagger] = 0.$$

Each set $\vec{N}$, $\vec{N}^\dagger$ obey SU(2) commutation relations. Hence the eigenvalues of $\vec{N}\cdot\vec{N}$ & $\vec{N}^\dagger\cdot\vec{N}^\dagger$ label the representation

These eigenvalues have $m(m+1)$ for $\vec{N}\cdot\vec{N}$ and $n(n+1)$ for $\vec{N}^\dagger\cdot\vec{N}^\dagger$ with $m, n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

The pair (m,n) labels the representation of the Lorentz group under consideration. The functions within each representation are distinguished by the eigenvalue of one of the components of each operator $\vec{N}$ & $\vec{N}^\dagger$. For instance, for the third component N_3 eigenvalues have the range $-m, -m+1, \ldots, m-1, +m$ and those of $N_3^\dagger$ are $-n, -n+1, \ldots, n-1, +n$. The (m,n) representation has $(2m+1)(2n+1)$ components.

ex. $(0,0) \leftrightarrow T(x)$ one totally invariant (scalar) function $\delta T = 0$

$(\frac{1}{2},\frac{1}{2}) \leftrightarrow T^\mu(x)$ the 4 functions of the vector representation; $\delta T^\mu = \omega^{\mu\nu} T_\nu$

$(1,0) \leftrightarrow F^{\mu\nu} = \tilde{F}^{\mu\nu}$ with $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ the self-dual, anti-symmetric tensor.

Since $\vec{J} = \vec{N} + \vec{N}^\dagger$, $(m+n)$ will denote the total spin of the representation

It can be shown that all integer values of $(m+n)$ ($(m+n) = 0, 1, 2, 3, \ldots$) can be described by the tensor representations we discussed.

However, we can also have representations for

$$(m+n) = \frac{(2k+1)}{2}, \quad k = 0, 1, 2, \ldots$$

These all can be built up from products of the basic spinor representations (as we built up the ~~tensor~~ (reducible) representations from the fundamental vector representation)

The $(\frac{1}{2},0)$ representation = Left-handed spinors

$(0,\frac{1}{2})$ " = Right-handed spinors

In this way we will obtain all possible (finite dimensional) representations of the Lorentz group.

In order to obtain the transformation law for spinors we will consider the set of 2×2 complex matrices with determinant =1. These form the group called SL(2,C). We will represent the Lorentz group by the action of these matrices on two component complex spinors. Equivalently we could build the spinor transformation law from the spin $\frac{1}{2}$ angular momentum representation matrices familiar from quantum mechanics, the Pauli matrices. Once we know the action of $\vec{N}$ and $\vec{N}^\dagger$ on the spinors we can reconstruct that of $M^{\mu\nu}$. However, it is more useful and to the point to proceed by considering directly SL(2,C). To obtain the relation of the Lorentz group to SL(2,C) we must first recall that there exists a one to one correspondence between 2×2 Hermitian matrices and space-time points. The Pauli matrices

$$
\begin{aligned}
(\sigma^0)_{\alpha\dot{\alpha}} &\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot{\alpha}} \\
(\sigma^1)_{\alpha\dot{\alpha}} &\equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\alpha\dot{\alpha}} \\
(\sigma^2)_{\alpha\dot{\alpha}} &\equiv \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}_{\alpha\dot{\alpha}} \\
(\sigma^3)_{\alpha\dot{\alpha}} &\equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\alpha\dot{\alpha}},
\end{aligned}
\tag{II.1}
$$

where $\alpha = 1, 2$ labels the two rows and $\dot{\alpha} = 1, 2$ labels the two columns, form a basis for 2×2 Hermitian matrices. Let $X_{\alpha\dot{\alpha}}$ be a Hermitian matrix, that is,

$$
\begin{aligned}
X^\dagger &= X \\
(X^*)_{\dot{\alpha}\alpha} &= (X)_{\alpha\dot{\alpha}}.
\end{aligned}
\tag{II.2}
$$

It has the general form

$$
\begin{aligned}
X_{\alpha\dot{\alpha}} &= \begin{pmatrix} (x_0 + x_3) & (x_1 - ix_2) \\ (x_1 + ix_2) & (x_0 - x_3) \end{pmatrix}_{\alpha\dot{\alpha}} \\
&= x_\mu (\sigma^\mu)_{\alpha\dot{\alpha}} \equiv \not{x}_{\alpha\dot{\alpha}}
\end{aligned}
\tag{II.3}
$$

for x_μ real with $\not{x}$ called "x slash". Thus corresponding to any 4 vector x^μ we associate a 2×2 Hermitian matrix $X_{\alpha\dot{\alpha}}$ by equation (II.3). Using the trace relation for the product of two Pauli matrices

$$
(\sigma^\mu)_{\alpha\dot{\alpha}} (i\sigma^2)_{\dot{\alpha}\dot{\beta}} (\sigma^{\nu\ T})_{\dot{\beta}\beta} (i\sigma^2)_{\beta\alpha} = -2g^{\mu\nu},
$$

or more succinctly written

$$Tr[\sigma^\mu (i\sigma^2) \sigma^{\nu\ T} (i\sigma^2)] = -2g^{\mu\nu}, \qquad (II.4)$$

we have for every Hermitian matrix $X_{\alpha\dot{\alpha}}$ an associated four vector x^μ

$$x^\mu = -\frac{1}{2} Tr[X (i\sigma^2) \sigma^{\mu\ T} (i\sigma^2)]. \qquad (II.5)$$

This correspondence is one to one (we will use $X = \not{x}$ in what follows to underscore this correspondence).

Simplifying the notation, since we would like to keep our dotted and undotted indices separate that is when we sum over indices we would like them to be of the same type in order to avoid extra confusion, we introduce an antisymmetric tensor $\epsilon^{\alpha\beta}$, that is, $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ with $\epsilon^{12} = +1$ and with lowered indices

$$\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta} = -\epsilon_{\beta\alpha},$$

that is $\epsilon_{12} = -\epsilon^{12} = -1$. Note that the matrix is the same when we use dotted indices, that is,

$$\epsilon^{\alpha\beta} = (i\sigma^2)_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{\alpha\beta}$$

$$\epsilon^{\dot{\alpha}\dot{\beta}} = (i\sigma^2)_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{\dot{\alpha}\dot{\beta}}. \qquad (II.6)$$

Also note that

$$\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^\alpha_{\ \gamma}$$

$$\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\beta}\dot{\gamma}} = \delta^{\dot{\alpha}}_{\ \dot{\gamma}}.$$

Then we can define the Pauli matrices with upper indices

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \equiv \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} (\sigma^\mu)_{\beta\dot{\beta}}$$

$$= -(i\sigma^2)_{\dot{\alpha}\dot{\beta}} (\sigma^{\mu\ T})_{\dot{\beta}\beta} (i\sigma^2)_{\beta\alpha}. \qquad (II.7)$$

We can write the trace condition as

$$(\sigma^\mu)_{\alpha\dot{\alpha}} (\bar{\sigma}^\nu)^{\dot{\alpha}\alpha} = +2g^{\mu\nu} \qquad (II.8)$$

and equation (II.5) has the simple form

$$x^\mu = +\frac{1}{2}(\not x)_{\alpha\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\alpha} = +\frac{1}{2}\mathrm{Tr}[\not x\bar\sigma^\mu]. \qquad (II.9)$$

The $\bar\sigma^\mu$ matrices are given by

$$\begin{aligned}
(\bar\sigma^0)^{\dot\alpha\alpha} &\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\dot\alpha\alpha} = +(\sigma^0)_{\dot\alpha\alpha} \\
(\bar\sigma^1)^{\dot\alpha\alpha} &\equiv \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}_{\dot\alpha\alpha} = -(\sigma^1)_{\dot\alpha\alpha} \\
(\bar\sigma^2)^{\dot\alpha\alpha} &\equiv \begin{pmatrix} 0 & +i \\ -i & 0 \end{pmatrix}_{\dot\alpha\alpha} = -(\sigma^2)_{\dot\alpha\alpha} \\
(\bar\sigma^3)^{\dot\alpha\alpha} &\equiv \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}_{\dot\alpha\alpha} = -(\sigma^3)_{\dot\alpha\alpha}. \qquad (II.10)
\end{aligned}$$

We can readily derive the completeness properties of the Pauli matrices

$$(\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha\alpha} = +2g^{\mu\nu} \qquad (II.11)$$

$$(\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma_\mu)^{\dot\beta\beta} = +2\delta_\alpha^{\ \beta}\delta^{\dot\beta}_{\ \dot\alpha}. \qquad (II.12)$$

Further products of two yield

$$\begin{aligned}
(\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha\beta} + (\sigma^\nu)_{\alpha\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\beta} &= 2g^{\mu\nu}\delta_\alpha^{\ \beta} \\
(\bar\sigma^\mu)^{\dot\alpha\alpha}(\sigma^\nu)_{\alpha\dot\beta} + (\bar\sigma^\nu)^{\dot\alpha\alpha}(\sigma^\mu)_{\alpha\dot\beta} &= 2g^{\mu\nu}\delta^{\dot\alpha}_{\ \dot\beta}. \qquad (II.13)
\end{aligned}$$

If S is an element of SL(2,C) (that is 2×2 complex matrices with determinant equal to one) with matrix elements $S_\alpha^{\ \beta}$, where α labels the rows and β labels the columns, and $\not x$ is a Hermitian matrix, then we can define the transformed matrix $\not x'$ as

$$(\not x')_{\alpha\dot\alpha} = S_\alpha^{\ \beta}(\not x)_{\beta\dot\beta}S^{*\dot\beta}_{\dot\alpha} \qquad (II.14)$$

with S^* the complex conjugate of S, again with $\dot\alpha$ labelling the rows and $\dot\beta$ labelling the columns, or taking the transpose we have $(S^\dagger)^{\dot\beta}_{\ \dot\alpha} = (S^*)_{\dot\alpha}^{\ \dot\beta}$, with $\dot\beta$ labelling the rows and $\dot\alpha$ labeling the columns of $S^\dagger$. Since $\det S = S_1^{\ 1}S_2^{\ 2} - S_1^{\ 2}S_2^{\ 1} = 1$ we have

$$\det \not x' = \det \not x. \qquad (II.15)$$

Calculating the determinant we find

$$\det \not{x} = (x_0+x_3)(x_0-x_3)-(x_1-ix_2)(x_1+ix_2) = (x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2$$
$$= x_\mu x^\mu$$
$$= \det \not{x}' = x'_\mu x'^\mu; \qquad (II.16)$$

the determinant is the Minkowski interval and is invariant. Thus, the transformation

$$\not{x}' = S \not{x} S^\dagger \qquad (II.17)$$

corresponds to a Lorentz transformation, $\Lambda^{\mu\nu}$, of the coordinates. In order to determine it in terms of the SL(2,C) matrix S consider

$$x'^\mu = \frac{1}{2}(\not{x}')_{\alpha\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}$$
$$= \frac{1}{2}S_\alpha{}^\beta(\not{x})_{\beta\dot{\beta}}S^{*\dot{\beta}}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \frac{1}{2}S_\alpha{}^\beta S^{*\dot{\beta}}_{\dot{\alpha}}(\sigma^\nu)_{\beta\dot{\beta}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}x_\nu$$
$$\equiv \Lambda^{\mu\nu}x_\nu \qquad (II.18)$$

where we identify

$$\Lambda^{\mu\nu} \equiv \frac{1}{2}Tr[S\sigma^\nu S^\dagger\bar{\sigma}^\mu] \qquad (II.19)$$

that is,

$$\Lambda^{\mu\nu}(\sigma_\mu)_{\alpha\dot{\alpha}} = \frac{1}{2}S_\beta{}^\gamma S^{*\dot{\gamma}}_{\dot{\beta}}(\sigma^\nu)_{\gamma\dot{\gamma}}(\bar{\sigma}^\mu)^{\dot{\beta}\beta}(\sigma_\mu)_{\alpha\dot{\alpha}} = S_\beta{}^\gamma S^{*\dot{\gamma}}_{\dot{\beta}}(\sigma^\nu)_{\gamma\dot{\gamma}}\delta_\alpha{}^\beta\delta_{\dot{\alpha}}{}^{\dot{\beta}}$$
$$= S_\alpha{}^\gamma S^{*\dot{\gamma}}_{\dot{\alpha}}(\sigma^\nu)_{\gamma\dot{\gamma}},$$

or more simply written

$$\Lambda^{\mu\nu}\sigma_\mu = S\sigma^\nu S^\dagger. \qquad (II.20)$$

So for every element $\pm S$ of SL(2,C) there is an element Λ of the Lorentz group, the mapping of SL(2,C) into $L^\uparrow_+$ is 2 to 1 since $\pm S \to \Lambda$.

We can use the SL(2,C) matrices to define the spinor representations of the Lorentz group. The spinor transformation laws are given by

$$\psi'_\alpha(x') \equiv S_\alpha{}^\beta\psi_\beta(x)$$
$$\psi'^\alpha(x') \equiv \psi^\beta(x)(S^{-1})_\beta{}^\alpha \qquad (II.21)$$

where $S_\alpha^{\ \beta}(S^{-1})_\beta^{\ \gamma} = \delta_\alpha^{\ \gamma}$ and ψ_α and ψ^α are two different two component complex spinors transforming (as we will see) as the $(\frac{1}{2}, 0)$ representation of the Lorentz group, the ψ are called Weyl spinors. Similarly, we can use $S^\dagger$ and $(S^\dagger)^{-1}$ to define two more different Weyl spinors, the complex conjugates of ψ, denoted $\bar{\psi}$, which transform as $(0, \frac{1}{2})$ representations of the Lorentz group

$$\bar{\psi}'_{\dot{\alpha}}(x') \equiv \bar{\psi}_{\dot{\beta}}(x)(S^\dagger)^{\dot{\beta}}_{\ \dot{\alpha}}$$

$$\bar{\psi}'^{\dot{\alpha}}(x') \equiv (S^{\dagger -1})^{\dot{\alpha}}_{\ \dot{\beta}}\bar{\psi}^{\dot{\beta}}(x) \qquad (II.22)$$

where, for the adjoint matrices $(S^\dagger)^{\dot{\alpha}}_{\ \dot{\beta}}$, $\dot{\alpha}$ labels the rows and $\dot{\beta}$ labels the columns. As with tensors, higher rank spinors transform just like products of the basic rank 1 spinors, for example,

$$\psi'_{\alpha_1\cdots\alpha_n}(x') = S_{\alpha_1}^{\ \ \beta_1}\cdots S_{\alpha_n}^{\ \ \beta_n}\psi_{\beta_1\cdots\beta_n}(x)$$

$$\psi'_{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_m}(x') = S_{\alpha_1}^{\ \ \beta_1}\cdots S_{\alpha_1}^{\ \ \beta_m}\psi_{\beta_1\cdots\beta_n\dot{\beta}_1\cdots\dot{\beta}_m}(x)(S^\dagger)^{\dot{\beta}_1}_{\ \ \dot{\alpha}_1}\cdots(S^\dagger)^{\dot{\beta}_m}_{\ \ \dot{\alpha}_m}. \qquad (II.23)$$

Since S is special, i.e. det S = 1, we have

$$(S^{-1})_\alpha^{\ \beta} = \begin{pmatrix} S_2^{\ 2} & -S_1^{\ 2} \\ -S_2^{\ 1} & S_1^{\ 1} \end{pmatrix}_{\alpha\beta}$$

$$= \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}S_\delta^{\ \gamma} \qquad (II.24)$$

or in matrix notation

$$S^{-1} = -\epsilon S^T\epsilon. \qquad (II.25)$$

Further ϵ is an anti-symmetric invariant second rank spinor, that is

$$\epsilon'_{\alpha\beta} = S_\alpha^{\ \gamma}S_\beta^{\ \delta}\epsilon_{\gamma\delta}$$

or again in matrix form

$$\epsilon' = S\epsilon S^T = SS^{-1}\epsilon = \epsilon.$$

Since the indices can be confusing, let's write this out explicitly

$$\epsilon'_{12} = S_1^{\ 1}S_2^{\ 2}\epsilon_{12} + S_1^{\ 2}S_2^{\ 1}\epsilon_{21} = -\det S = -1 = \epsilon_{12}.$$

So indeed $\epsilon'_{\alpha\beta} = \epsilon_{\alpha\beta}$, ϵ is an invariant second rank spinor. Hence, we can use ϵ to lower and raise indices of the spinors analogous to the invariant metric tensor $g^{\mu\nu}$ which lowers and raises vector indices

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$$

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$$

$$\bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}$$

$$\bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}. \qquad (II.26)$$

As a consequence of (II.26) the transformation law for ψ^α, for instance, follows from that of ψ_α

$$\psi'^\alpha(x') = \epsilon^{\alpha\beta}\psi'_\beta(x') = \epsilon^{\alpha\beta}S_\beta{}^\gamma\psi_\gamma(x)$$

$$= \epsilon^{\alpha\beta}S_\beta{}^\gamma\epsilon_{\gamma\delta}\psi^\delta(x) = -\epsilon_{\delta\gamma}S_\beta{}^\gamma\epsilon_{\beta\alpha}\psi^\delta(x) = \psi^\delta(x)(S^{-1})_\delta{}^\alpha. \qquad (II.27)$$

Thus, we can contract similar spinor indices to make Lorentz scalars

$$\psi'^\alpha(x')\psi'_\alpha(x') = (S^{-1})_\beta{}^\alpha S_\alpha{}^\gamma\psi^\beta(x)\psi_\gamma(x)$$

$$= \delta_\beta{}^\gamma\psi^\beta(x)\psi_\gamma(x) = \psi^\alpha(x)\psi_\alpha(x) \qquad (II.28)$$

and similarly for $\bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}$. Also using the properties of the Pauli matrices we can make a four vector object whose vector index then contracts with another four vector index in order to make a scalar, for example

$$\psi'^\alpha(x')(\sigma^\mu)_{\alpha\dot{\alpha}}\partial'_\mu\bar{\psi}'^{\dot{\alpha}}(x') = (S^{-1})_\beta{}^\alpha(S^{\dagger-1})^{\dot{\alpha}}{}_{\dot{\beta}}(\Lambda^{-1})^\nu{}_\mu\psi^\beta(x)(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\nu\bar{\psi}^{\dot{\beta}}(x).$$

But

$$(S^{-1})_\beta{}^\alpha(\sigma_\mu)_{\alpha\dot{\alpha}}(S^{\dagger-1})^{\dot{\alpha}}{}_{\dot{\beta}}\Lambda^{\mu\nu} = (S^{-1})_\beta{}^\alpha(\sigma_\mu)_{\alpha\dot{\alpha}}(S^{\dagger-1})^{\dot{\alpha}}{}_{\dot{\beta}}\left(\frac{1}{2}Tr[S\sigma^\nu S^\dagger\bar{\sigma}^\mu]\right)$$

$$= \frac{1}{2}(S^{-1})_\beta{}^\alpha(S^{\dagger-1})^{\dot{\alpha}}{}_{\dot{\beta}}(S\sigma^\nu S^\dagger)_{\delta\dot{\delta}}(\sigma_\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\delta}\delta}$$

$$= (S^{-1})_\beta{}^\alpha(S^{\dagger-1})^{\dot{\alpha}}{}_{\dot{\beta}}(S\sigma^\nu S^\dagger)_{\delta\dot{\delta}}\delta_\alpha{}^\delta\delta_{\dot{\alpha}}{}^{\dot{\delta}} = (S^{-1})_\beta{}^\alpha(S^{\dagger-1})^{\dot{\alpha}}{}_{\dot{\beta}}(S\sigma^\nu S^\dagger)_{\alpha\dot{\alpha}}$$

$$= (\sigma^\nu)_{\beta\dot{\beta}}, \qquad (II.29)$$

hence

$$\psi'^\alpha(x')(\sigma^\mu)_{\alpha\dot{\alpha}}\partial'_\mu\bar{\psi}'^{\dot{\alpha}}(x') = \psi^\alpha(x)(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu\bar{\psi}^{\dot{\alpha}}(x). \qquad (II.30)$$

As stated, $\psi \not{\partial}\bar{\psi}$ is a Lorentz invariant.

Finally, let's consider infinitesimal Lorentz transformations

$$x'^{\mu} = x^{\mu} + \omega^{\mu\nu} x_{\nu} \tag{II.31}$$

where now, for infinitesimal transformations, S differs from the identity by an infinitesimal matrix Σ

$$\begin{aligned} S_{\alpha}^{\ \beta} &= \delta_{\alpha}^{\ \beta} + \Sigma_{\alpha}^{\ \beta} \\ S_{\dot{\alpha}}^{*\dot{\beta}} &= \delta_{\dot{\alpha}}^{\ \dot{\beta}} + \Sigma_{\dot{\alpha}}^{*\dot{\beta}}. \end{aligned} \tag{II.32}$$

Since ϵ is invariant we have that

$$\epsilon_{\alpha\beta} = S_{\alpha}^{\ \gamma} S_{\beta}^{\ \delta} \epsilon_{\gamma\delta} = (\delta_{\alpha}^{\ \gamma} + \Sigma_{\alpha}^{\ \gamma})(\delta_{\beta}^{\ \delta} + \Sigma_{\beta}^{\ \delta})\epsilon_{\gamma\delta}$$

$$= \left[\delta_{\alpha}^{\ \gamma}\delta_{\beta}^{\ \delta} + \Sigma_{\alpha}^{\ \gamma}\delta_{\beta}^{\ \delta} + \Sigma_{\beta}^{\ \delta}\delta_{\alpha}^{\ \gamma}\right]\epsilon_{\gamma\delta} = \epsilon_{\alpha\beta} + \epsilon_{\gamma\beta}\Sigma_{\alpha}^{\ \gamma} + \epsilon_{\alpha\gamma}\Sigma_{\beta}^{\ \gamma} \tag{II.33}$$

which implies that Σ is symmetric. With lowered indices we have

$$\Sigma_{\beta\alpha} - \Sigma_{\alpha\beta} = 0. \tag{II.34}$$

Now given $\omega^{\mu\nu}$ we desire $\Sigma_{\alpha\beta}$; using $\Lambda^{\mu\nu} = \frac{1}{2} Tr[S\sigma^{\nu} S^{\dagger}\bar{\sigma}^{\mu}]$ we find

$$g^{\mu\nu} + \omega^{\mu\nu} = \frac{1}{2}\left(\delta_{\alpha}^{\ \beta} + \Sigma_{\alpha}^{\ \beta}\right)(\sigma^{\nu})_{\beta\dot{\beta}}\left(\delta_{\dot{\alpha}}^{\ \dot{\beta}} + \Sigma_{\dot{\alpha}}^{*\dot{\beta}}\right)(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}$$

$$= \frac{1}{2}(\sigma^{\nu})_{\alpha\dot{\alpha}}(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} + \frac{1}{2}\Sigma_{\alpha}^{\ \beta}(\sigma^{\nu})_{\beta\dot{\alpha}}(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} + \frac{1}{2}(\sigma^{\nu})_{\alpha\dot{\beta}}\Sigma_{\dot{\alpha}}^{*\dot{\beta}}(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}$$

$$= g^{\mu\nu} + \frac{1}{2}\Sigma_{\alpha}^{\ \beta}(\sigma^{\nu})_{\beta\dot{\alpha}}(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} + \frac{1}{2}\Sigma_{\dot{\alpha}}^{*\dot{\beta}}(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}(\sigma^{\nu})_{\alpha\dot{\beta}}. \tag{II.35}$$

Thus, we must find a solution for

$$\omega^{\mu\nu} = \frac{1}{2}\Sigma_{\alpha}^{\ \beta}(\sigma^{\nu})_{\beta\dot{\alpha}}(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} + \frac{1}{2}\Sigma_{\dot{\alpha}}^{*\dot{\beta}}(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}(\sigma^{\nu})_{\alpha\dot{\beta}}$$

$$= \frac{1}{2} Tr\left[\Sigma\sigma^{\nu}\bar{\sigma}^{\mu} + \Sigma^{\dagger}\bar{\sigma}^{\mu}\sigma^{\nu}\right]. \tag{II.36}$$

Multiplying by σ_{μ} and $\bar{\sigma}_{\nu}$ we have

$$(\sigma_{\mu})_{\gamma\dot{\gamma}}(\bar{\sigma}_{\nu})^{\dot{\delta}\delta}\omega^{\mu\nu} = \frac{1}{2}\left[(\sigma_{\mu})_{\gamma\dot{\gamma}}(\bar{\sigma}_{\nu})^{\dot{\delta}\delta} - (\sigma_{\nu})_{\gamma\dot{\gamma}}(\bar{\sigma}_{\mu})^{\dot{\delta}\delta}\right]\omega^{\mu\nu}$$

$$= 2\Sigma_{\gamma}{}^{\delta}\delta_{\dot{\gamma}}{}^{\dot{\delta}} + 2\Sigma^{*\dot{\delta}}_{\dot{\gamma}}\delta_{\gamma}{}^{\delta}. \qquad (II.37)$$

Using $\Sigma_{\alpha}{}^{\alpha} = 0 = \Sigma^{*\dot{\alpha}}_{\dot{\alpha}}$, we find

$$\Sigma_{\gamma}{}^{\delta} = \frac{1}{8}\left[(\sigma_{\mu})_{\gamma\dot{\gamma}}(\bar{\sigma}_{\nu})^{\dot{\gamma}\delta} - (\sigma_{\nu})_{\gamma\dot{\gamma}}(\bar{\sigma}_{\nu})^{\dot{\gamma}\delta}\right]\omega^{\mu\nu} \qquad (II.38)$$

and similarly

$$\Sigma^{*\dot{\delta}}_{\dot{\gamma}} = \frac{1}{8}\left[(\bar{\sigma}_{\nu})^{\dot{\delta}\gamma}(\sigma_{\mu})_{\gamma\dot{\gamma}} - (\bar{\sigma}_{\mu})^{\dot{\delta}\gamma}(\sigma_{\nu})_{\gamma\dot{\gamma}}\right]\omega^{\mu\nu}. \qquad (II.39)$$

The above commutators of the Pauli matrices arise frequently and so we define the matrices

$$(\sigma^{\mu\nu})_{\alpha}{}^{\beta} \equiv \frac{i}{2}\left[(\sigma^{\mu})_{\alpha\dot{\alpha}}(\bar{\sigma}^{\nu})^{\dot{\alpha}\beta} - (\sigma^{\nu})_{\alpha\dot{\alpha}}(\bar{\sigma}^{\mu})^{\dot{\alpha}\beta}\right] = \frac{i}{2}(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu})_{\alpha}{}^{\beta}$$

$$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \equiv \frac{i}{2}\left[(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}(\sigma^{\nu})_{\alpha\dot{\beta}} - (\bar{\sigma}^{\nu})^{\dot{\alpha}\alpha}(\sigma^{\mu})_{\alpha\dot{\beta}}\right] = \frac{i}{2}(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu})^{\dot{\alpha}}{}_{\dot{\beta}}. \qquad (II.40)$$

Thus we secure

$$\Sigma_{\alpha}{}^{\beta} = \frac{-i}{4}\omega_{\mu\nu}(\sigma^{\mu\nu})_{\alpha}{}^{\beta}$$

$$(\Sigma^{\dagger})^{\dot{\beta}}{}_{\dot{\alpha}} = \frac{+i}{4}\omega_{\mu\nu}(\bar{\sigma}^{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}. \qquad (II.41)$$

From our definitions of $(\sigma^{\mu})_{\alpha\dot{\alpha}}$ we see that

$$(\sigma^{\mu})_{\alpha\dot{\alpha}}(\bar{\sigma}^{\nu})^{\dot{\alpha}\beta} = g^{\mu\nu}\delta_{\alpha}{}^{\beta} - i(\sigma^{\mu\nu})_{\alpha}{}^{\beta}$$

$$(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}(\sigma^{\nu})_{\alpha\dot{\beta}} = g^{\mu\nu}\delta^{\dot{\alpha}}{}_{\dot{\beta}} - i(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}. \qquad (II.42)$$

The infinitesimal spinor transformations can now be obtained

$$\psi'_{\alpha}(x') = S_{\alpha}{}^{\beta}\psi_{\beta}(x) = \psi_{\alpha}(x) - \frac{i}{4}\omega_{\mu\nu}(\sigma^{\mu\nu})_{\alpha}{}^{\beta}\psi_{\beta}(x)$$

$$\equiv \psi_{\alpha}(x) - \frac{1}{2}\omega_{\mu\nu}(D^{\mu\nu})_{\alpha}{}^{\beta}\psi_{\beta}(x) \qquad (II.43)$$

and

$$\bar{\psi}'_{\dot{\alpha}}(x') = \bar{\psi}_{\dot{\beta}}(x)(S^{\dagger})^{\dot{\beta}}{}_{\dot{\alpha}} = \bar{\psi}_{\dot{\alpha}}(x) + \frac{i}{4}\omega_{\mu\nu}\bar{\psi}_{\dot{\beta}}(x)(\bar{\sigma}^{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}$$

$$\equiv \bar{\psi}_{\dot{\alpha}}(x) - \frac{1}{2}\omega_{\mu\nu}(\bar{D}^{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}. \qquad (II.44)$$

Hence, the spinor representations are given by

$$(D^{\mu\nu})_\alpha^{\ \beta} \equiv \frac{+i}{2}(\sigma^{\mu\nu})_\alpha^{\ \beta}$$

$$(\bar{D}^{\mu\nu})^{\dot{\beta}}_{\ \dot{\alpha}} \equiv \frac{-i}{2}(\bar{\sigma}^{\mu\nu})^{\dot{\beta}}_{\ \dot{\alpha}}. \qquad (II.45)$$

We must check that these matrices indeed obey the Lorentz algebra as did the tensor representation matrices. After some tedious Pauli matrix algebra we find

$$\begin{aligned}[\sigma^{\mu\nu}, \sigma^{\rho\sigma}]_\alpha^{\ \beta} &= \frac{-1}{4}\left[(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu), (\sigma^\rho\bar{\sigma}^\sigma - \sigma^\sigma\bar{\sigma}^\rho)\right]_\alpha^{\ \beta} \\ &= -2i\left(g^{\mu\rho}\sigma^{\nu\sigma} - g^{\mu\sigma}\sigma^{\nu\rho} + g^{\nu\sigma}\sigma^{\mu\rho} - g^{\nu\rho}\sigma^{\mu\sigma}\right)_\alpha^{\ \beta}. \end{aligned} \qquad (II.46)$$

Thus the spinor representation obeys the Lorentz algebra

$$[D^{\mu\nu}, D^{\rho\sigma}]_\alpha^{\ \beta} = \left(g^{\mu\rho}D^{\nu\sigma} - g^{\mu\sigma}D^{\nu\rho} + g^{\nu\sigma}D^{\mu\rho} - g^{\nu\rho}D^{\mu\sigma}\right)_\alpha^{\ \beta}, \qquad (II.47)$$

and ψ_α is the $(\frac{1}{2}, 0)$ spinor representation of the Lorentz group. Similarly the commutation relation for $\bar{\sigma}^{\mu\nu}$ can be worked out and we find that the complex conjugate dotted spinors $\bar{\psi}_{\dot{\alpha}}$ are the $(0, \frac{1}{2})$ representation of the Lorentz group with the $\bar{D}^{\mu\nu}$ obeying the Lorentz algebra.

As with tensors, we find the intrinsic variations of the spinor fields are given by

$$\begin{aligned}\bar{\delta}\psi_\alpha &= \psi'_\alpha(x) - \psi_\alpha(x) = \delta\psi_\alpha - \delta x^\mu\partial_\mu\psi_\alpha \\ \bar{\delta}\bar{\psi}_{\dot{\alpha}} &= \bar{\psi}'_{\dot{\alpha}}(x) - \bar{\psi}_{\dot{\alpha}}(x) = \delta\bar{\psi}_{\dot{\alpha}} - \delta x^\mu\partial_\mu\bar{\psi}_{\dot{\alpha}}.\end{aligned} \qquad (II.48)$$

For Poincare' transformations

$$x'^\mu = x^\mu + \omega^{\mu\nu}x_\nu + \epsilon^\mu$$

we find that

$$\begin{aligned}\bar{\delta}\psi_\alpha &= \frac{1}{2}\omega_{\mu\nu}\left[(x^\mu\partial^\nu - x^\nu\partial^\mu)\delta_\alpha^{\ \beta} - (D^{\mu\nu})_\alpha^{\ \beta}\right]\psi_\beta(x) - \epsilon^\mu\partial_\mu\psi_\alpha(x) \\ &\equiv -\frac{i}{2}\omega_{\mu\nu}(M^{\mu\nu})_\alpha^{\ \beta}\psi_\beta(x) + i\epsilon^\mu P_\mu\psi_\alpha(x)\end{aligned} \qquad (II.49)$$

and

$$\bar{\delta}\bar{\psi}_{\dot{\alpha}} = \frac{1}{2}\omega_{\mu\nu}\left[(x^\mu\partial^\nu - x^\nu\partial^\mu)\delta^{\dot{\beta}}_{\ \dot{\alpha}} - (\bar{D}^{\mu\nu})^{\dot{\beta}}_{\ \dot{\alpha}}\right]\psi_{\dot{\beta}}(x) - \epsilon^\mu\partial_\mu\bar{\psi}_{\dot{\alpha}}(x)$$

$$\equiv -\frac{i}{2}\omega_{\mu\nu}(M^{\mu\nu})^{\dot\beta}{}_{\dot\alpha}\bar\psi_{\dot\beta}(x) + i\epsilon^\mu P_\mu \bar\psi_{\dot\alpha}(x). \qquad (II.50)$$

As with tensors, the P^μ and $M^{\mu\nu}$ obey the defining commutation relations of the Poincare' group. Note that in the rest frame for $\psi(x)$, assuming it describes a massive particle with rest frame four momentum $p^\mu = (m, 0, 0, 0)$,

$$J^i \equiv \frac{1}{2}\epsilon_{ijk}M^{jk} = \frac{i}{2}\epsilon_{ijk}D^{jk} = -\frac{1}{4}\epsilon_{ijk}\sigma^{jk}$$
$$= +\frac{1}{2}\sigma^i, \qquad (II.51)$$

hence the third component of the intrinsic angular momentum J^3 has eigenvalues $\pm\frac{1}{2}$ and the particle has spin $\frac{1}{2}$. Similarly finding $\vec{K}$ we have that $\vec{N}\cdot\vec{N} = \frac{1}{2}(\frac{1}{2}+1)$ and $\vec{N}^\dagger\cdot\vec{N}^\dagger = 0$, so ψ_α is the $(\frac{1}{2}, 0)$ representation of the Lorentz group. Similarly we find that $\bar\psi_{\dot\alpha}$ is the $(0, \frac{1}{2})$ representation of the Lorentz group.

For finite Poincare' transformations

$$x'^\mu = \Lambda^{\mu\nu}x_\nu + a^\mu \qquad (II.52)$$

we again exponentiate the generators to obtain

$$\psi'_\alpha(x) = S_\alpha{}^\beta\psi_\beta(\Lambda^{-1}(x-a))$$
$$= \left[e^{+ia^\mu P_\mu}e^{-\frac{i}{2}\omega_{\mu\nu}(\Lambda)M^{\mu\nu}}\right]_\alpha{}^\beta \psi_\beta(x) \qquad (II.53)$$

and

$$\bar\psi'_{\dot\alpha}(x) = \bar\psi_{\dot\beta}(\Lambda^{-1}(x-a))(S^\dagger)^{\dot\beta}{}_{\dot\alpha}$$
$$= \left[e^{+ia^\mu P_\mu}e^{-\frac{i}{2}\omega_{\mu\nu}(\Lambda)M^{\mu\nu}}\right]^{\dot\beta}{}_{\dot\alpha} \bar\psi_{\dot\beta}(x) \qquad (II.54)$$

with

$$\Lambda^{\mu\nu} = \frac{1}{2}Tr[S\sigma^\nu S^\dagger\bar\sigma^\mu] \qquad (II.55)$$

and $\omega^{\mu\nu}(\Lambda)$ as given earlier

$$e^{\frac{1}{2}\omega_{\mu\nu}(\Lambda)(x^\mu\partial^\nu - x^\nu\partial^\mu)}x^\rho = (\Lambda^{-1})^\rho{}_\sigma x^\sigma. \qquad (II.56)$$

Thus we have found all representations of the Poincare' group.

Finally, let's relate our two-component Weyl spinors to the usual Dirac four-component spinors. We can realize the Clifford algebra defining the

4×4 Dirac matrices $\gamma^{\mu a}{}_{b}$ by using the Pauli matrices, this representation being referred to as the Weyl basis (or representation) or the chiral basis (or representation). Defining the Dirac matrices as

$$\gamma^{\mu} \equiv \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}, \tag{II.57}$$

that is,

$$\gamma^{0a}{}_{b} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}_{ab}$$

$$\gamma^{1a}{}_{b} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}_{ab}$$

$$\gamma^{2a}{}_{b} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \\ 0 & +i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}_{ab}$$

$$\gamma^{3a}{}_{b} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}_{ab}. \tag{II.58}$$

Thus, the γ^{μ} obey the defining Dirac anti-commutation relations

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}\mathbf{1}. \tag{II.59}$$

Also, we can define an additional matrix γ_5

$$\gamma_5 \equiv +i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$= \begin{pmatrix} -\sigma^0 & 0 \\ 0 & +\bar{\sigma}^0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}. \tag{II.60}$$

In this basis the 4 component complex Dirac spinor, denoted ψ_D^a, is given in terms of two Weyl spinors ψ_α and $\bar{\chi}^{\dot{\alpha}}$

$$\psi_D^a \equiv \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \bar{\chi}^1 \\ \bar{\chi}^2 \end{pmatrix}_a . \qquad (II.61)$$

Under a Lorentz transformation the Dirac spinor transforms as

$$\psi_D'^a(x') = L^a{}_b \psi_D^b(x) \qquad (II.62)$$

where

$$L^a{}_b = \begin{pmatrix} S_\alpha{}^\beta & 0 \\ 0 & (S^{\dagger -1})^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix}_{ab} . \qquad (II.63)$$

Further since $\Lambda^{\mu\nu}\sigma_\mu = S\sigma^\nu S^\dagger$ and $\Lambda^{\mu\nu}\bar{\sigma}_\mu = S^{\dagger -1}\bar{\sigma}^\nu S^{-1}$ we have

$$\Lambda^{\mu\nu}\gamma^a_{\mu b} = L^a{}_c \gamma^{\nu c}{}_d (L^{-1})^d{}_b . \qquad (II.64)$$

For left, ψ_{DL}, and right, ψ_{DR}, handed spinors we have

$$\begin{aligned}\psi_{DL} &\equiv \frac{1}{2}(1-\gamma_5)\psi_D \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \bar{\chi}^1 \\ \bar{\chi}^2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix}\end{aligned} \qquad (II.65)$$

and

$$\begin{aligned}\psi_{DR} &\equiv \frac{1}{2}(1+\gamma_5)\psi_D \\ &= \begin{pmatrix} 0 \\ 0 \\ \bar{\chi}^1 \\ \bar{\chi}^2 \end{pmatrix} .\end{aligned} \qquad (II.66)$$

Thus, we have that ψ_{DL} corresponds to our $(\frac{1}{2}, 0)$ spinor ψ_α while ψ_{DR} corresponds to our $(0, \frac{1}{2})$ spinor $\bar{\chi}^{\dot{\alpha}}$. If the Dirac spinor is a Majorana spinor, ψ_M, that is ψ_D is self charge conjugate, then

$$\psi_M^C = C\bar{\psi}_M^T \equiv \psi_M \qquad (II.67)$$

with the charge conjugation matrix C given by

$$C = i\gamma^2\gamma^0$$
$$= \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\bar{\sigma}^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{II.68}$$

with $C\gamma^\mu C^{-1} = -\gamma^{\mu T}$ and the conjugate Dirac spinor is defined as

$$\bar{\psi}_D \equiv \psi_D^\dagger \gamma^0. \tag{II.69}$$

The Majorana condition (II.67) implies that

$$C\bar{\psi}_M^T = i\gamma^2\gamma^0\gamma^0\psi_M^*$$
$$= \begin{pmatrix} 0 & i\sigma^2 \\ i\bar{\sigma}^2 & 0 \end{pmatrix} \begin{pmatrix} \bar{\psi} \\ \chi \end{pmatrix}$$
$$= \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \equiv \psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}, \tag{II.70}$$

or $\psi = \chi, \bar{\psi} = \bar{\chi}$. Hence we find that a 4 component Majorana spinor is made up of a 2 component Weyl spinor and its complex conjugate

$$\psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}. \tag{II.71}$$

Needless to say the Weyl representation for the Dirac γ matrices is not the only way we could have reralized the Dirac algebra

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbf{1}. \tag{II.72}$$

After all, this remains invariant under unitary transformations U, $U^\dagger = U^{-1}$

$$\hat{\gamma}^\mu \equiv U^\dagger\gamma^\mu U,$$

and so (II.72) becomes

$$\hat{\gamma}^\mu\hat{\gamma}^\nu + \hat{\gamma}^\nu\hat{\gamma}^\mu = 2g^{\mu\nu}\mathbf{1}. \tag{II.73}$$

Further we can use these unitary transformations to define linear combinations of the Weyl spinor components to form a four-component complex spinor

$$\hat{\psi}_D \equiv U^\dagger \psi_D. \tag{II.74}$$

Under a Poincare' transformation we have

$$\begin{aligned}\hat{\psi}'_D(x') = U^\dagger \psi'_D(x') &= U^\dagger L \psi_D(x) = U^\dagger L U U^\dagger \psi_D(x) \\ &= U^\dagger L U \psi_D(x) = \hat{L}\hat{\psi}_D(x)\end{aligned} \tag{II.75}$$

where $\hat{L} \equiv U^\dagger L U$. As before we have for the hatted transformations

$$\begin{aligned}\Lambda^{\mu\nu}\hat{\gamma}_\mu = \Lambda^{\mu\nu} U^\dagger \gamma_\mu U &= U^\dagger L \gamma^\nu L^{-1} U = U^\dagger L U U^\dagger \gamma^\nu U U^\dagger L^{-1} U \\ &= \hat{L}\hat{\gamma}^\nu \hat{L}^{-1}.\end{aligned} \tag{II.76}$$

Thus, all relations go through as before with all quantities replaced by their hatted values.

There are several common choices for the four- component Dirac quantities. We have first defined the Weyl (or chiral) representation, in brief review in obvious notation

$$\gamma^\mu_{Weyl} \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{II.77}$$

That is

$$\begin{aligned}\gamma^0_{Weyl} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \gamma^i_{Weyl} &= \begin{pmatrix} o & \sigma^i \\ \bar{\sigma}^i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.\end{aligned} \tag{II.78}$$

The Weyl basis Dirac spinor, now denoted ψ_{Weyl}, is given as

$$\psi_{Weyl} \equiv \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \bar{\chi}^1 \\ \bar{\chi}^2 \end{pmatrix}. \tag{II.79}$$

Under Poincare' transformations

$$\psi'_{Weyl}(x') = L\psi_{Weyl}(x) \tag{II.80}$$

with

$$L = \begin{pmatrix} S & 0 \\ 0 & S^{\dagger -1} \end{pmatrix}. \tag{II.81}$$

Left handed and right handed chiral spinors are defined by

$$\psi_{Weyl\ L} \equiv \frac{1}{2}(1-\gamma_{5\ Weyl})\psi_{Weyl}(x) = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}$$

$$\psi_{Weyl\ R} \equiv \frac{1}{2}(1+\gamma_{5\ Weyl})\psi_{Weyl}(x) = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \qquad (II.82)$$

with

$$\gamma_{5\ Weyl} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}.$$

Another common representation is that of Dirac

$$\gamma^\mu_{Dirac} \equiv U^\dagger \gamma^\mu_{Weyl} U \qquad (II.83)$$

with

$$U \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$U^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \qquad (II.84)$$

Hence

$$\gamma^\mu_{Dirac} = \frac{1}{2}\begin{pmatrix} (\sigma+\bar\sigma)^\mu & (\sigma-\bar\sigma)^\mu \\ (\bar\sigma-\sigma)^\mu & -(\sigma+\bar\sigma)^\mu \end{pmatrix} \qquad (II.85)$$

that is

$$\gamma^0_{Dirac} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^i_{Dirac} = \begin{pmatrix} 0 & \sigma^i \\ \bar\sigma^i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \qquad (II.86)$$

Writing out all the components in order to be explicit, we have

$$\gamma^0_{Dirac} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^1_{Dirac} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2_{Dirac} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^3_{Dirac} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \qquad (II.87)$$

The γ_5 matrix becomes

$$\gamma_{5\ Dirac} \equiv U^\dagger \gamma_{5\ Weyl} U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad (II.88)$$

Note the γ matrices in all representations obey $\gamma^{0\dagger} = \gamma^0, \gamma^{i\dagger} = -\gamma^i, \gamma_5^\dagger = \gamma_5$. The Dirac four component spinors (or bi-spinors as they are sometimes called) in the Dirac representation are

$$\psi_{Dirac} \equiv U^\dagger \psi_{Weyl} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} (\psi + \bar{\chi}) \\ (-\psi + \bar{\chi}) \end{pmatrix}. \qquad (II.89)$$

Hence, the chiral spinors are given by

$$\psi_{Dirac\ L} = \frac{1}{2}(1 - \gamma_{5\ Dirac})\psi_{Dirac} = U^\dagger \psi_{Weyl\ L} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi \\ -\psi \end{pmatrix}$$

$$\psi_{Dirac\ R} = \frac{1}{2}(1 + \gamma_{5\ Dirac})\psi_{Dirac} = U^\dagger \psi_{Weyl\ R} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{\chi} \\ \bar{\chi} \end{pmatrix}. \qquad (II.90)$$

Another common representation is the Majorana representation in which all the γ matrices have pure imaginary matrix elements. In this basis we have

$$\gamma^\mu_{Majorana} \equiv U^\dagger \gamma^\mu_{Dirac} U \qquad (II.91)$$

with the unitary transformation matrix also being hermitian and given as

$$U \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \sigma^2 \\ \sigma^2 & -1 \end{pmatrix} = U^\dagger. \qquad (II.92)$$

Let us connect this classical discussion with the treatment of the fields as quantum mechanical operators.

Theorem: The Quantum Mechanical Poincare' Group.

Every continuous unitary representation up to a phase of $\mathcal{P}_+^\uparrow$ can be brought, by an appropriate choice of phase factor, into the form of a continuous representation (a,S)→ U(a,S) of the inhomogeneous SL(2,C). The multiplication law becoming

$$U(a,S) = U(a_1,S_1)U(a_2,S_2) \qquad (II.93)$$

for

$$(a,S) = (a_1 + S_1 a_2 S_1^\dagger, S_1 S_2). \qquad (II.94)$$

Recall that the inhomogeneous SL(2,C) transformations are defined by

$$\not{x}' = S\ \not{x} S^\dagger + \not{a} \qquad (II.95)$$

where $\not{a}$ is a two-by-two Hermitian matrix corresponding to the space-time translation by the four vector a^μ. It is understood that the SL(2,C) transformations are performed before the translation. So

$$\not{x}_2 = S_2\ \not{x} S_2^\dagger + \not{a}_2 \qquad (II.96)$$

and

$$\begin{aligned}
\not{x}_1 &= S_1\ \not{x}_2 S_1^\dagger + \not{a}_1 \\
&= S_1 S_2\ \not{x} S_2^\dagger S_1^\dagger + \not{a}_1 + S_1\ \not{a}_2 S_1^\dagger \\
&= S_1 S_2\ \not{x} (S_1 S_2)^\dagger + \not{a}_1 + S_1\ \not{a}_2 S_1^\dagger \\
&\equiv S\ \not{x} S^\dagger + \not{a}.
\end{aligned} \qquad (1)$$

Hence, $(a,S) = (a_1,S_1)(a_2,S_2) = (a_1 + S_1 a_2 S_2^\dagger, S_1 S_2)$ gives the composition law for ISL(2,C).

Finally, let's just point out that if we have an operator, perhaps depending on space-time, $A(x)$, an observer in another frame describes the operator in the same way, it is only translated, rotated or boosted as compared to the original frame. That is, an observer S' uses $A(x')$ to study states Ψ' while an observer S uses $A(x)$ to study states Ψ. Since the theory is to be relativistically invariant the corresponding matrix elements, the experimental observations, should transform covariantly like a tensor or a spinor, hence,

$$< \phi_{(a,S)}|A^{(\alpha)}(x')|\psi_{(a,s)} >= D^{(\alpha)}{}_{(\beta)}(S) < \phi|A^{(\beta)}(x)|\psi > \qquad (II.97)$$

where $x'^{\mu} = \Lambda^{\mu}{}_{\nu}(S)x^{\nu} + a^{\mu}$ and $|\psi_{(a,S)} >= U(a,S)|\psi >$ so that

$$U^{-1}(a,S)A^{(\alpha)}(x')U(a,S) = D^{(\alpha)}{}_{(\beta)}(S)A^{(\beta)}(x) \qquad (II.98)$$

is the corresponding transformation law for operators; in particular our field operators will transform thusly. Note that $\varphi'^{(\alpha)}(x') \equiv < \phi_{(a,S)}|A^{(\alpha)}(x')|\psi_{(a,S)} >$ is like the classical field $\varphi'^{(\alpha)}(x')$ with the original field $\varphi^{(\alpha)}(x) \equiv < \phi|A^{(\alpha)}(x)|\psi >$ so that they transform as

$$\varphi'^{(\alpha)}(x') = D^{(\alpha)}{}_{(\beta)}(S)\varphi^{(\beta)}(x). \qquad (II.99)$$

Thus quantum mechanical operators transform as

$$U(a,S)A^{(\alpha)}(x)U^{-1}(a,S) = D^{-1(\alpha)}{}_{(\beta)}(S)A^{(\beta)}(x'). \qquad (II.100)$$

We are now ready to study the quantum mechanical representations of the inhomogeneous SL(2,C). Since U(a,S) is unitary we can always write it as the exponential map. In addition, we have that

$$U(a,S) = U(a,\mathbf{1})U(0,S)$$

where

$$U(a,\mathbf{1}) \equiv e^{ia_{\mu}\mathcal{P}^{\mu}}$$
$$U(0,S) \equiv e^{\frac{-i}{2}\omega_{\mu\nu}(S)\mathcal{M}^{\mu\nu}} \qquad (II.101)$$

where the Hermitian (since U is unitary) operators are $\mathcal{P}^{\mu}$ the space-time translation generators identified with the energy and momentum operators and $\mathcal{M}^{\mu\nu}$ are the Lorentz transformation (rotation) generators identified with the angular momentum operators. a_{μ} is just the translation vector $a^{\mu} = \frac{1}{2}Tr[\not{a}\bar{\sigma}^{\mu}]$ while $\omega_{\mu\nu}(S)$ are just the angles of rotation in the $x_{\mu} - x_{\nu}$ plane parameterizing the finite SL(2,C) transformation S, that is

$$S = e^{-\frac{i}{4}\omega_{\mu\nu}(S)\sigma^{\mu\nu}}, \qquad (II.102)$$

this is related to $\Lambda^{\mu\nu}$ by

$$\Lambda^{\mu\nu} = \Lambda^{\mu\nu}(S) = \frac{1}{2}Tr\left[S\sigma^{\nu}S^{\dagger}\bar{\sigma}^{\mu}\right]. \qquad (II.103)$$

For infinitesimal transformations

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu} + \omega^{\mu}{}_{\nu} x^{\nu}, \qquad (II.104)$$

with ϵ^{μ} and $\omega^{\mu\nu}$ infinitesimal parameters, we can expand the unitary operators to first order

$$U(a, S) = \mathbf{1} + i\epsilon_{\mu}\mathcal{P}^{\mu} - \frac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu} \qquad (II.105)$$

where now $\omega_{\mu\nu}(S) = \omega_{\mu\nu}$, the infinitesimal rotation angles. Recall that since

$$\begin{aligned} g_{\mu\nu} &= (g_{\alpha\mu} + \omega_{\alpha\mu})g^{\alpha\beta}(g_{\beta\nu} + \omega_{\beta\nu}) \\ &= g_{\nu\mu} + \omega_{\nu\mu} + \omega_{\mu\nu}, \end{aligned} \qquad (2)$$

we find $\omega_{\mu\nu} = -\omega_{\nu\mu}$ is antisymmetric and hence so is the generator $\mathcal{M}_{\mu\nu} = -\mathcal{M}_{\mu\nu}$.

To make more explicit the identification of $\mathcal{M}_{\mu\nu}$ with rotations consider the transformation $U = \mathbf{1} - i\omega_{12}\mathcal{M}^{12}$ describing the change in the state vector. The corresponding infinitesimal coordinate change is

$$\begin{aligned} x'^{0} &= x^{0} \\ x'^{1} &= x^{1} - \omega^{12}x^{2} \\ x'^{2} &= x^{2} + w^{12}x^{1} \\ x'^{2} &= x^{3}. \end{aligned} \qquad (II.106)$$

This is a rotation in the $x_1 - x_2$ plane. Hence, $-i\mathcal{M}_{ij}$ is the generator of rotations in the $x_i - x_j$ plane for the state vectors and corresponds to the total angular momentum operator

$$\mathcal{J}^{i} \equiv \frac{1}{2}\epsilon_{ijk}\mathcal{M}_{jk} = (\mathcal{M}_{23}, \mathcal{M}_{31}, \mathcal{M}_{12}). \qquad (II.107)$$

For an infinitesimal Lorentz boost along the x^1 direction

$$\begin{aligned} x'^{0} &= x^{0} - x^{1}\omega^{01} \\ x'^{1} &= x^{1} - x^{0}\omega^{01} \\ x'^{2} &= x^{2} \end{aligned}$$

$$x'^3 = x^3, \qquad (II.108)$$

the state vector is transformed by

$$U = 1 - i\omega_{01}\mathcal{M}^{01}. \qquad (II.109)$$

Hence, $-i\mathcal{M}_{0i}$ generates Lorentz boosts along the i^{th} axis for the state vectors. We write $\mathcal{K}^i$ for the three-vector $\mathcal{K}^i \equiv \mathcal{M}^{0i}$.

Since $\mathcal{P}^\mu, \mathcal{M}^{\mu\nu}$ are the genrators of the Poincare' or SL(2,C) group they obey commutation relations which characterize their group multiplication law (the commutation relations for $\mathcal{P}_+^\uparrow$ and SL(2,C) generators are the same)

$$U(a_1, S_1)U(a_2, S_2) = U(a_1 + S_1 a_2 S_1^\dagger, S_1 S_2)$$

$$U(a, S)^{-1} = U(-S^{-1}aS^{-1\dagger}, S^{-1}). \qquad (II.110)$$

Using the above laws we find

$$U(a_1, 1)U(a_2, 1) = U(a_1 + a_2, 1) \qquad (II.111)$$

which implies $[\mathcal{P}_\mu, \mathcal{P}_\nu] = 0$. Further, we have

$$U(0, S^{-1})U(a, 1)U(0, S) = U(S^{-1}aS^{-1\dagger}, 1) \qquad (II.112)$$

that is

$$U(0, S^{-1})e^{ia_\mu \mathcal{P}^\mu}U(0, S) = e^{(\Lambda^{-1}(S)a)_\mu \mathcal{P}^\mu}. \qquad (II.113)$$

For infinitesimal a^μ this yields

$$U^{-1}(0, S)a_\mu \mathcal{P}^\mu U(0, S) = \Lambda_\mu^{-1\nu}(S)a_\nu \mathcal{P}^\mu \qquad (II.114)$$

or

$$e^{\frac{i}{2}\omega_{\mu\nu}(S)\mathcal{M}^{\mu\nu}}a_\lambda \mathcal{P}^\lambda e^{\frac{-i}{2}\omega_{\mu\nu}(S)\mathcal{M}^{\mu\nu}} = \Lambda_\mu^{-1\nu}(S)a_\nu \mathcal{P}^\mu. \qquad (II.115)$$

For infinitesimal S we have

$$\begin{aligned} a_\lambda \mathcal{P}^\lambda + a_\lambda \frac{i}{2}\omega_{\mu\nu}[\mathcal{M}^{\mu\nu}, \mathcal{P}^\lambda] &= (\delta_\mu{}^\nu - \omega_\mu{}^\nu)a_\nu \mathcal{P}^\mu \\ &= a_\lambda \mathcal{P}^\lambda - \frac{a^\lambda}{2}(\omega_{\mu\lambda} - \omega_{\lambda\mu})\mathcal{P}^\mu. \qquad (3) \end{aligned}$$

Thus we obtain the commutator

$$[\mathcal{M}^{\mu\nu}, \mathcal{P}^\lambda] = i[g^{\lambda\nu}\mathcal{P}^\mu - g^{\lambda\mu}\mathcal{P}^\nu]. \qquad (II.116)$$

Finally we obtain the angular momentum commutation relations by considering the infinitesimal S' transformations

$$U(0,S^{-1})U(0,S')U(0,S)=U(0,S^{-1}S'S) \qquad (II.117)$$

or

$$e^{\frac{i}{2}\omega_{\mu\nu}(S)\mathcal{M}^{\mu\nu}}e^{\frac{-i}{2}\omega'_{\rho\lambda}\mathcal{M}^{\rho\lambda}}e^{\frac{-i}{2}\omega_{\alpha\beta}(S)\mathcal{M}^{\alpha\beta}}$$
$$=e^{\frac{-i}{2}\omega_{\mu\nu}(S^{-1}S'S)\mathcal{M}^{\mu\nu}}. \qquad (II.118)$$

Now the parameter describing the product of Lorentz transformations $S^{-1}S'S$ is found by considering the action of the three successive transformations on x^μ . First we transform to

$$x_1^\mu=\Lambda^{\mu\nu}(S)x_\nu, \qquad (II.119)$$

then to

$$x_2^\alpha=\Lambda^{\alpha\beta}(S')\Lambda_{\beta\nu}(S)x^\nu, \qquad (II.120)$$

and finally back by

$$x_3^\mu=\Lambda^{-1\mu\alpha}(S)\Lambda_{\alpha\beta}(S')\Lambda^{\beta\nu}(S)x_\nu. \qquad (II.121)$$

For S' infinitesimal we have

$$\Lambda_{\alpha\beta}(S')=g_{\alpha\beta}+\omega'_{\alpha\beta} \qquad (II.122)$$

so

$$\begin{aligned} x_3^\mu &= \Lambda^{-1\mu\alpha}(S)g_{\alpha\beta}\Lambda^{\beta\nu}(S)x_\nu+\Lambda^{-1\mu\alpha}(S)\omega'_{\alpha\beta}\Lambda^{\beta\nu}(S)x_\nu \\ &= (g^{\mu\nu}+\Lambda^{-1\mu\alpha}(S)\omega'_{\alpha\beta}\Lambda^{\beta\nu}(S))x_\nu. \end{aligned} \qquad (4)$$

Hence, we have that

$$\omega_{\mu\nu}(S^{-1}S'S)=\Lambda^{-1}_{\mu\alpha}(S)\omega'^{\alpha\beta}\Lambda_{\beta\nu}(S) \qquad (II.123)$$

and thus,

$$\begin{aligned} U(0,S^{-1})\mathcal{M}^{\mu\nu}U(0,S) &= \Lambda^{-1\alpha\mu}(S)\Lambda^{\nu\beta}(S)\mathcal{M}_{\alpha\beta} \\ &= \Lambda^{\mu\alpha}(S)\Lambda^{\nu\beta}(S)\mathcal{M}_{\alpha\beta}. \end{aligned} \qquad (5)$$

Taking S to be infinitesimal also, we find

$$\omega_{\rho\lambda}\frac{i}{2}[\mathcal{M}^{\rho\lambda},\mathcal{M}^{\mu\nu}] = (\omega^{\mu\alpha}g^{\nu\beta}+g^{\mu\alpha}\omega^{\nu\beta})\mathcal{M}_{\alpha\beta}$$

$$
\begin{aligned}
&= \frac{1}{2}\omega_{\rho\lambda}[g^{\rho\mu}g^{\lambda\alpha}g^{\nu\beta} - g^{\lambda\mu}g^{\rho\alpha}g^{\nu\beta} + g^{\mu\alpha}g^{\rho\nu}g^{\lambda\beta} \\
&\qquad - g^{\mu\alpha}g^{\lambda\nu}g^{\rho\beta}]\mathcal{M}_{\alpha\beta} \\
&= \frac{1}{2}\omega_{\rho\lambda}[g^{\rho\mu}\mathcal{M}^{\lambda\nu} - g^{\lambda\mu}\mathcal{M}^{\rho\nu} + g^{\rho\nu}\mathcal{M}^{\mu\lambda} - g^{\lambda\nu}\mathcal{M}^{\mu\rho}]. \quad (6)
\end{aligned}
$$

We finally secure the angular momentum commutation relations

$$
[\mathcal{M}^{\mu\nu}, \mathcal{M}^{\rho\lambda}] = i(g^{\mu\lambda}\mathcal{M}^{\nu\rho} - g^{\mu\rho}\mathcal{M}^{\nu\lambda} + g^{\nu\rho}\mathcal{M}^{\mu\lambda} - g^{\nu\lambda}\mathcal{M}^{\mu\rho}). \qquad (II.124)
$$

As before with the space-time differential operators we define

$$
\begin{aligned}
\mathcal{J}^i &\equiv \frac{1}{2}\epsilon_{ijk}\mathcal{M}_{jk} \\
\mathcal{K}^i &\equiv \mathcal{M}^{0i}
\end{aligned}
\qquad (II.125)
$$

and see that they obey the algebra

$$
\begin{aligned}
[\mathcal{J}_i, \mathcal{J}_j] &= +i\epsilon_{ijk}\mathcal{J}_k \\
[\mathcal{K}_i, \mathcal{K}_j] &= -i\epsilon_{ijk}\mathcal{J}_k \\
[\mathcal{J}_i, \mathcal{K}_j] &= +i\epsilon_{ijk}\mathcal{K}_k.
\end{aligned}
\qquad (II.126)
$$

Hence $\vec{\mathcal{J}}$ are the angular momentum operators, $\vec{\mathcal{K}}$ the boost operators and $\mathcal{P}_\mu$ the translation operators.

In particular let's consider the action of the space-time translations further. In the Heisenberg representation the states are independent of time while the operators depend on time. Thus the space-time translation of our operators is determined by the action of $\mathcal{P}^\mu$. Recall Poincare' invariance implies

$$
< \phi_{(a,S)}|A^{(\alpha)}(x')|\psi_{(a,S)} >= D^{(\alpha)}{}_{(\beta)}(S) < \phi|A^{(\beta)}(x)|\psi > \qquad (II.127)
$$

or equivalently

$$
U(a,S)A^{(\alpha)}(x)U^{-1}(a,S) = D^{-1(\alpha)}{}_{(\beta)}(S)A^{(\beta)}(x'). \qquad (II.128)
$$

For $\not{x}' = \not{x} + \not{a}$ we find

$$
e^{ia_\mu\mathcal{P}^\mu}A^{(\alpha)}(x)e^{-ia_\mu\mathcal{P}^\mu} = A^{(\alpha)}(x+a). \qquad (II.129)
$$

For a_μ infinitesimal we expand the exponentials and Taylor expand the operator

$$(1 + ia_\mu \mathcal{P}^\mu) A^{(\alpha)}(x)(1 - ia_\mu \mathcal{P}^\mu) = A^{(\alpha)}(x) + a^\mu \partial_\mu A^{(\alpha)}(x), \qquad (II.130)$$

which implies

$$ia_\mu [\mathcal{P}^\mu, A^{(\alpha)}(x)] = a^\mu \partial_\mu A^{(\alpha)}(x). \qquad (II.131)$$

Thus for the translation operator we find

$$[\mathcal{P}^\mu, A^{(\alpha)}(x)] = -i\partial^\mu A^{(\alpha)}(x) = -P^\mu A^{(\alpha)}(x) \qquad (II.132)$$

with $P^\mu \equiv i\partial^\mu$, as earlier. Likewise, we can consider Lorentz transformations $x'^\mu = \Lambda^{\mu\nu}(S) x_\nu$

$$e^{\frac{i}{2}\omega_{\mu\nu}(S)\mathcal{M}^{\mu\nu}} A^{(\alpha)}(x) e^{\frac{-i}{2}\omega_{\mu\nu}(S)\mathcal{M}^{\mu\nu}} = D^{-1(\alpha)}{}_{(\beta)}(S) A^{(\beta)}(x'). \qquad (7)$$

For infinitesimal Lorentz transformations $x'^\mu = x^\mu + \omega^{\mu\nu} x_\nu$ and $D^{-1(\alpha)}{}_{(\beta)}(S) = \delta^{(\alpha)}{}_{(\beta)} + \frac{1}{2}\omega^{\mu\nu}[D_{\mu\nu}]^{(\alpha)}{}_{(\beta)}$ this becomes

$$\begin{aligned}(1 - \frac{i}{2}\omega^{\mu\nu}\mathcal{M}_{\mu\nu}) A^{(\alpha)}(x)(1 + \frac{i}{2}\omega^{\mu\nu}\mathcal{M}_{\mu\nu}) &= A^{(\alpha)}(x) \\ &\quad -\frac{1}{2}\omega^{\mu\nu}\left[(x_\mu\partial_\nu - x_\nu\partial_\mu)\delta^{(\alpha)}{}_{(\beta)} - [D_{\mu\nu}]^{(\alpha)}{}_{(\beta)}\right] A^{(\beta)}(x).\end{aligned} \qquad (8)$$

Hence

$$\begin{aligned}[\mathcal{M}_{\mu\nu}, A^{(\alpha)}(x)] &= -i\left[(x_\mu\partial_\nu - x_\nu\partial_\mu)\delta^{(\alpha)}{}_{(\beta)} - [D_{\mu\nu}]^{(\alpha)}{}_{(\beta)}\right] A^{(\beta)}(x) \\ &= -M_{\mu\nu} A^{(\alpha)}(x),\end{aligned} \qquad (9)$$

with $M_{\mu\nu}$ as we found earlier for the classical fields.