9.2. Fermi's Golden Rule

for time independent H' or very slowly varying

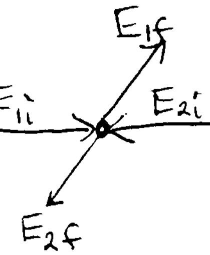
As time increases, we certainly expect the probability of transition to grow as well. Indeed from our Dirac perturbation formula (page -9.15-) we might expect that for approximately equal initial and final energies that $P_{fi}(t,t_0)$ may grow like $|t-t_0|^2$ so that the rate of transition

$$R_{fi} \equiv \frac{P_{fi}(t,t_0)}{|t-t_0|} \quad \text{grows in time!}$$

However, it is precisely the transitions to the states $E_f \simeq E_i$ that can occur below the energy resolution limits that temper this growth of the rate so that in fact the transition rate R_{fi} is constant in time. This property is known as Fermi's Golden Rule for time-dependent perturbation theory.

(Time indep. H' first then sinusoidally varying H')

To be more precise, as well as to determine the conditions for the validity of the rule, let's first consider the example of scattering from a time-independent potential i.e. H' is constant in time: $H' = H'(\vec{r})$. We will consider the



$E_{\substack{i\\f}} = E_{1\substack{i\\f}} + E_{2\substack{i\\f}}$

$E_i = E_f$ conservation of energy as we shall see.

transition probability to scatter from some initial energy eigenstate to some final energy eigenstate. $|\varphi_i\rangle$ is the sum of 2 plane waves, say $|\varphi_f\rangle$ also, but now of different energy for the "incoming" particle E_i and E_f. That is $|\varphi_i\rangle$ describes the incoming particle as well as target and $|\varphi_f\rangle$ the scattered particle, now with different energy and the different state of the target particle. So the incoming particle interacts with the target, exchanges energy and scatters. The transition probability is given to lowest order by the Dirac perturbation theory expression

$$P_{fi}(t,t_0) = \left| \frac{1}{i\hbar} \int_{t_0}^{t} dt_1 \langle \varphi_f | H' | \varphi_i \rangle e^{\frac{i}{\hbar}(E_f - E_i)t_1} \right|^2,$$

just as previously.

Defining the transition frequency

$$\omega_{fi} = \frac{E_f - E_i}{\hbar},$$

and using the fact that H' is <u>time independent</u> yields

$$P_{fi}(t,t_0)=\frac{1}{\hbar^2}\,|\langle\varphi_f|H'|\varphi_i\rangle|^2\left|\frac{\sin\left(\frac{1}{2}\omega_{fi}(t-t_0)\right)}{\frac{1}{2}\omega_{fi}}\right|^2$$

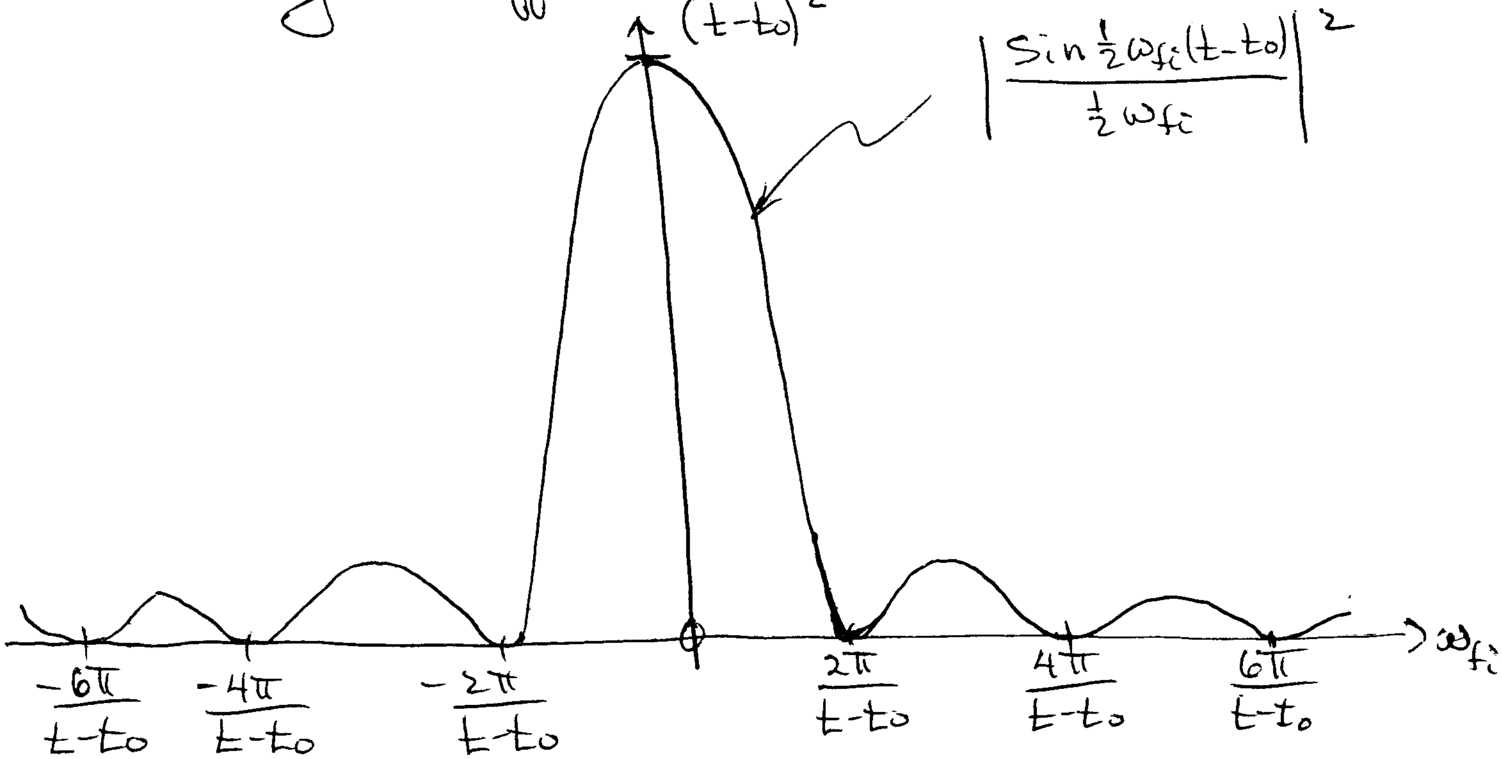
since

$$\int_{t_0}^{t}dt_1\,e^{i\omega_{fi}t_1}=\frac{e^{i\omega_{fi}t}-e^{i\omega_{fi}t_0}}{i\omega_{fi}}$$

$$=\frac{e^{i\omega_{fi}\left(\frac{t+t_0}{2}\right)}}{\frac{1}{2}\omega_{fi}}\left[\frac{e^{i\omega_{fi}\left(\frac{t-t_0}{2}\right)}-e^{-i\omega_{fi}\left(\frac{t-t_0}{2}\right)}}{2i}\right]$$

$$=e^{i\omega_{fi}\left(\frac{t+t_0}{2}\right)}\,\frac{\sin\omega_{fi}\left(\frac{t-t_0}{2}\right)}{\frac{1}{2}\omega_{fi}}\,.$$

Plotting this diffraction function



we see that it is strongly peaked about $\omega_{fi}=0$, i.e. $E_f=E_i$, the larger $t-t_0$ the more sharply peaked. It decreases rapidly for increasing $|\omega_{fi}|$, vanishing at frequencies $\omega_{fi}=\pm\frac{2\pi n}{t-t_0}$ with $n=1,2,3,\ldots$, typical of diffraction patterns.

So defining the total time T over which the interaction occurs as

$$T\equiv t-t_0,$$

The <u>transition rate</u> is given by

$$R_{fi} \equiv \frac{P_{fi}(t,t_0)}{|t-t_0|} = \frac{P_{fi}(t,t_0)}{T}$$

$$= \frac{1}{\hbar^2}|\langle\varphi_f|H'|\varphi_i\rangle|^2 T\left[\frac{\sin(\frac{1}{2}\omega_{fi}T)}{(\frac{1}{2}\omega_{fi}T)}\right]^2$$

$$\equiv \frac{1}{\hbar^2}|\langle\varphi_f|H'|\varphi_i\rangle|^2 f(T,\omega_{fi}) .$$

The function $f(T,\omega_{fi}) \equiv T\left[\frac{\sin(\frac{1}{2}\omega_{fi}T)}{(\frac{1}{2}\omega_{fi}T)}\right]^2$ has the properties

1) $f(T,0) = T$

2) $f(T,\omega_{fi}) \longrightarrow 0$ as $T \to \infty$ for any $\omega_{fi} \neq 0$.

This implies that $f(T,\omega_{fi})$ for large T is very peaked at $\omega_{fi} = 0$.

3) $$\int_{-\infty}^{+\infty} d\omega_{fi} f(T,\omega_{fi}) = T\int_{-\infty}^{+\infty} d\omega_{fi}\left[\frac{\sin(\frac{1}{2}\omega_{fi}T)}{(\frac{1}{2}\omega_{fi}T)}\right]^2$$

(let $s \equiv \frac{1}{2}\omega_{fi}T$) $$= 2\int_{-\infty}^{+\infty} ds\, \frac{\sin^2 s}{s^2} = 2\pi$$

Thus, these properties imply that

$$f(T, \omega_{fi}) \xrightarrow[T\to\infty]{} 2\pi\,\delta(\omega_{fi}) = 2\pi\hbar\,\delta(E_f - E_i),$$

expressing the conservation of energy in a scattering process. Thus ~~for~~ sufficiently large T (to be quantified next) we have that

$$R_{fi} = \frac{2\pi}{\hbar}|\langle\varphi_f|H'|\varphi_i\rangle|^2\,\delta(E_f - E_i),$$

a form of <u>Fermi's Golden Rule</u>, the transition rate is a constant in time.

In this discrete energy case, we have that

$$P_{fi}(t,t_0)\Big|_{E_i=E_f} = T R_{fi}\Big|_{\omega_{fi}=0}$$

$$= \frac{T^2}{\hbar^2}|\langle\varphi_f|H'|\varphi_i\rangle|^2 .$$

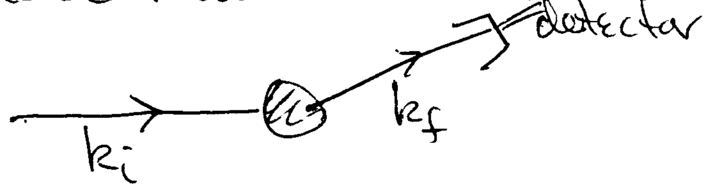
Thus, for $P_{fi}(t,t_0)\Big|_{E_i=E_f} < 1$, we must have that $\frac{1}{\hbar^2}|\langle\varphi_f|H'|\varphi_i\rangle|^2 < \frac{1}{T^2}$.

However, we must also have $(\omega_{fi} T)$ large so that $f(T, \omega_{fi})$ can be approximated by $2\pi\delta(\omega_{fi})$ in order to obtain Fermi's Golden Rule. Hence the region of validity for Fermi's Golden Rule is when the transition rates are small. That is just the condition for the validity for the use of first order perturbation theory. This was just equivalent to approximating $b_i(t)$ for $t > t_0$ by $b_i(t_0)$ in the Schrödinger equation for $b_i(t)$. Thus there is very little depletion in the initial state, i.e. the transition rates are small.

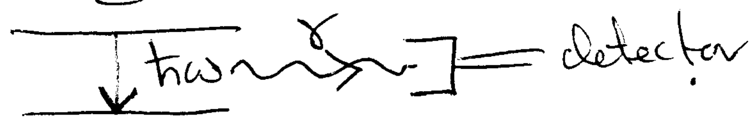
When one of the particles in the final state has energy in the continuum, we can observe more carefully the cancellation of the T^2 growth of the $\omega_{fi} = 0$ channel against the oscillation of the $\omega_{fi} \neq 0$ but E_f in the continuum, channels, resulting in a constant R_{fi} rate and Fermi's Golden Rule. As well this continuum case describes very important realistic examples of actual scattering experiments.

In particular we will be able to apply the golden rule to the cases of

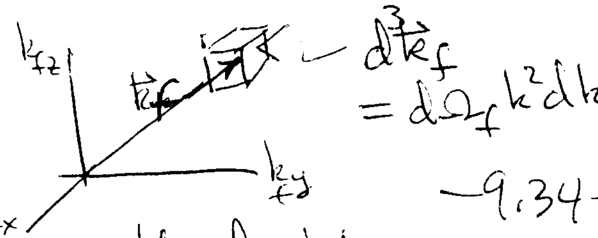
1) Elastic & Inelastic Scattering in which the detector observes a scattered particle with energy $\frac{\hbar^2 k_f^2}{2m}$ which is in the continuum



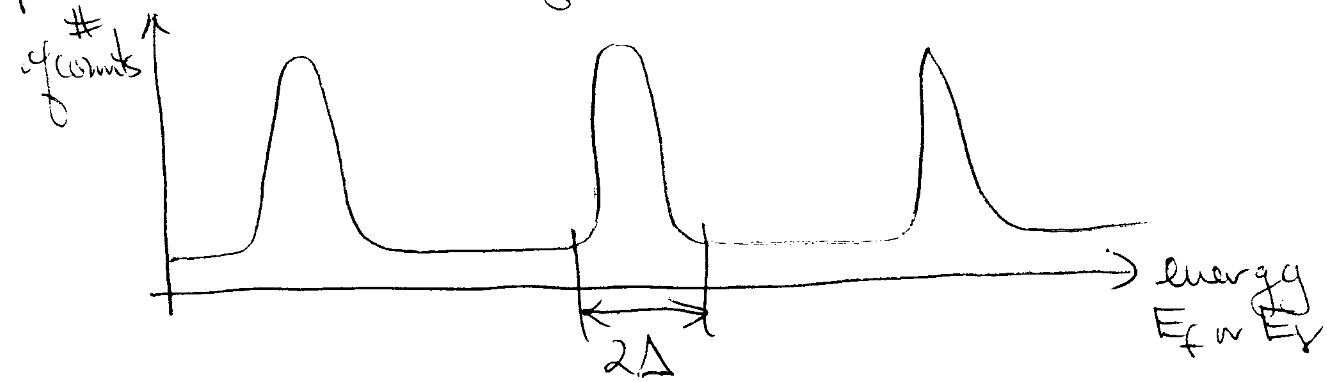
2) Photoemission in which the detector observes an emitted photon with energy $\hbar\omega$ which is in the continuum



In reality all detectors have a finite energy resolution Δ, hence we cannot measure the probability of detecting the system in the distinct state $|\psi_f\rangle$ if the energy of this state lies in the continuum. Thus all physical predictions involve an integration over a group of final states depending on the particular measurement being made. The detector signals when it detects a particle with momentum in the domain D_f of momentum space



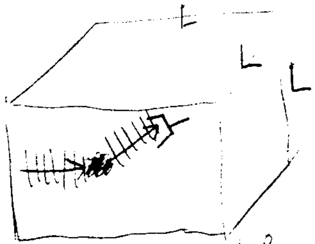
centered about $\vec{p}_f = \hbar \vec{k}_f$ so that the energy is in the interval Δ centered about $E_f = \frac{\hbar^2 k_f^2}{2m}$ for a massive (non-relativistic) particle or $E_\gamma = \hbar k_f c$ for a photon. Hence the detector records a series of peaks (excitations of the target)



where the detector energy resolution Δ is small enough to resolve the individual peaks while large enough to contain an entire peak. Thus we need to count the number of states in $\mathcal{D}_f$. This is easily done by placing the system in a box of side length L and volume $\Omega = L^3$. The normalized final state plane wave is then

$$\varphi_{\vec{k}}(\vec{r}) = \langle \vec{r} | \varphi_f \rangle = \frac{1}{\sqrt{\Omega}} e^{i\vec{k}\cdot\vec{r}}$$

with energy $E_f = \frac{\hbar^2 k^2}{2m}$. Imposing periodic boundary conditions on the plane



wave, the components of $\mathbf{k}$ are given by

$$k_i = \frac{2\pi n_i}{L} \quad , n_i = 0, \pm 1, \pm 2, \ldots \quad i = 1,2,3.$$

Thus $\Delta n_i = \frac{L}{2\pi}\Delta k_i$ and the number of states in D_f is simply

$$\Delta n = \Delta n_x \Delta n_y \Delta n_z = \frac{L^3}{(2\pi)^3}\Delta k_x \Delta k_y \Delta k_z$$
$$= \frac{\Omega}{(2\pi)^3}\Delta k_x \Delta k_y \Delta k_z \quad .$$

As $L \to \infty$, the energy levels become continuously close and $\Delta n \to \infty$, but for L large we write this as

$$dn = \frac{\Omega}{(2\pi)^3} d^3k \quad .$$

In addition there may be additional degeneracy associated with each energy eigenvalue labelled by other quantum numbers (besides the momentum). As usual, denote the degree of degeneracy by $g(E)$ (for photons for example, there are 2 polarization states for each energy), Thus

$$dn = \frac{\Omega}{(2\pi)^3} g(E) d^3k$$ is the

number of states in D_f when $\vec{k}$ is restricted to lie in the domain D_f centered about $\vec{k}_f$. (i.e. $\int_{D_f} dn = n$)

The transition rate to any of these states is given by (recall they are centered about E_f)

$$\delta R_{fi}(E_f) = \int_{D_f} dn\, R_{fi}(E)$$

where $$R_{fi}(E) = \frac{1}{\hbar^2} |\langle\psi_f|H'|\psi_i\rangle|^2 f\left(T, \frac{E-E_i}{\hbar}\right)$$

is the transition rate to final states $|\psi_f\rangle$ with energy E instead of E_f. Of course $E_f - \Delta \leq E \leq E_f + \Delta$ in the integral. Hence we have

$$\delta R_{fi}(E_f) = \frac{\Omega}{(2\pi)^3} \int_{E_f - \Delta \leq \frac{\hbar^2 k^2}{2m} \leq E_f + \Delta} d^3k\; g(E) R_{fi}(E)$$

$$= \frac{\Omega}{(2\pi)^3} \int_{\delta\Omega_f} d\Omega_k \int_{E_f - \Delta \leq E \leq E_f + \Delta} dk\, k^2 g(E) R_{fi}(E)$$

This we write as

$$\delta R_{fi}(E_f) = \int_{E_f-\Delta}^{E_f+\Delta} dE \left(\frac{\partial n(E)}{\partial E}\right) R_{fi}(E)$$

with $\frac{\partial n(E)}{\partial E}$ the density of final states (sometimes $\rho(E) = n(E)$ is used)

$$\boxed{\frac{\partial n(E)}{\partial E} \equiv \delta\Omega_f \, g(E) \frac{\Omega}{(2\pi)^3} k^2 \frac{dk}{dE}}.$$

Thus we find

$$\delta R_{fi}(E_f) = \int_{E_f-\Delta}^{E_f+\Delta} dE \frac{\partial n(E)}{\partial E} \frac{1}{\hbar^2} |\langle\varphi_f|H'|\varphi_i\rangle|^2 f(T, \tfrac{E-E_i}{\hbar}).$$

$f(T, \frac{E-E_i}{\hbar})$ is sharply peaked about $E = E_i$, and since $E_f - \Delta \le E \le E_f + \Delta$, the integral is dominated by the contribution from $E \approx E_i \approx E_f$. Since $\frac{\partial n(E)}{\partial E}$ and $\langle\varphi_f|H'|\varphi_i\rangle$ are slowly varying functions of E, we can evaluate them at $E = E_f = E_i$, and so pull them out of the integral

$$\delta R_{fi}(E_f) = \left[\frac{\partial n(E_f)}{\partial E_f} \frac{1}{\hbar^2} |\langle \varphi_f | H' | \varphi_i \rangle|^2 \right] \Bigg|_{E_f = E_i} \times$$

$$\times \int_{E_f - \Delta}^{E_f + \Delta} dE \, f\left(T, \frac{E - E_i}{\hbar}\right) .$$

The integral can be performed, recall

$$\int_{E_f-\Delta}^{E_f+\Delta} dE \, f\left(T, \frac{E-E_i}{\hbar}\right) = \int_{E_f-\Delta}^{E_f+\Delta} dE \; T \left[\frac{\sin \frac{1}{2\hbar}(E - E_i)T}{\frac{1}{2\hbar}(E-E_i)T} \right]^2$$

(letting $s = \frac{1}{2\hbar}(E - E_i)T$)

$$= 2\hbar \int_{\frac{(E_f - E_i)T}{2\hbar} - \frac{\Delta T}{2\hbar}}^{\frac{(E_f - E_i)T}{2\hbar} + \frac{\Delta T}{2\hbar}} ds \, \frac{\sin^2 s}{s^2}$$

Now if $\Delta T \gg \hbar$ this becomes

$$= 2\hbar \int_{-\infty}^{+\infty} ds \, \frac{\sin^2 s}{s^2} = 2\pi\hbar .$$

Thus we find

$$\begin{pmatrix} E_i = \text{total initial energy} \\ E_f = \text{total final energy} \end{pmatrix}$$

for $T\Delta \gg \hbar$

$$\delta R_{fi}(E_f) = \left[\frac{2\pi}{\hbar} \frac{\partial n(E_f)}{\partial E_f} |\langle \varphi_f | H' | \varphi_i \rangle|^2 \right] \Bigg|_{E_i = E_f},$$

this is Fermi's Golden Rule for the transition rate when one particle is in the final state continuum. Note, this is independent of T and Δ. We can obtain this result directly from our previous form of Fermi's Golden Rule by summing over all states in D_f:

$$\delta R_{fi}(E_f) = \int_{D_f} dn\, R_{fi}(E)$$

$$= \int_{E_f-\Delta}^{E_f+\Delta} dE \frac{\partial n(E)}{\partial E} R_{fi}(E)$$

$$= \int_{E_f-\Delta}^{E_f+\Delta} dE \frac{\partial n(E)}{\partial E} \frac{2\pi}{\hbar} |\langle \varphi_f | H' | \varphi_i \rangle|^2 \delta(E - E_i)$$

$$= \frac{2\pi}{\hbar} \left[\frac{\partial n(E_f)}{\partial E_f} |\langle \varphi_f | H' | \varphi_i \rangle|^2 \right] \Bigg|_{E_f = E_i}.$$

cannot prepare system & immediately probe it — uncertainty principle.

Fermi's Golden Rule is valid for $T\Delta \gg \hbar$ while in order to insure that probability was less than one we had

$$\frac{1}{\hbar}|\langle \psi_f | H' | \psi_i \rangle| \ll \frac{1}{T}.$$

Combining these conditions results in

$$\frac{\hbar}{\Delta} \ll T \ll \frac{\hbar}{|\langle \psi_f | H' | \psi_i \rangle|}.$$

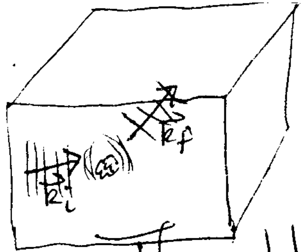
These inequalities can be satisfied for some T provided $|\langle \psi_f | H' | \psi_i \rangle|$ is sufficiently small. Thus again Fermi's Golden Rule is valid for calculating small transition rates.

Example: Elastic Scattering of a spinless particle from a potential $V(\vec{R})$

The full (time-independent) Hamiltonian is

$$H = \frac{1}{2m}\vec{P}^2 + V(\vec{R}) \equiv H_0 + H'$$

with $H_0 = \frac{1}{2m}\vec{P}^2$ and $H' = V(\vec{R})$.



The H_0 eigenstates are given by the normalized plane waves:

initial state: $\varphi_i(\vec{r}) = \langle \vec{r} | \varphi_i \rangle = \frac{1}{\sqrt{\Omega}} e^{i\vec{k}_i \cdot \vec{r}}$

final state: $\varphi_f(\vec{r}) = \langle \vec{r} | \varphi_f \rangle = \frac{1}{\sqrt{\Omega}} e^{i\vec{k}_f \cdot \vec{r}}$

with $|\vec{k}_i| = |\vec{k}_f| = k$ for elastic scattering, with initial and final energies

$$E_i = \frac{\hbar^2 k^2}{2m} = E_f .$$

Since $k \in \mathbb{R}^+$, the final state plane wave has its energy in the continuum. Applying Fermi's Golden Rule we first evaluate the matrix element

$$\langle \varphi_f | H' | \varphi_i \rangle = \frac{1}{\Omega} \int d^3r \, e^{-i\vec{k}_f \cdot \vec{r}} V(\vec{r}) e^{+i\vec{k}_i \cdot \vec{r}}$$

$$= \frac{1}{\Omega} \int d^3r \, e^{-i\vec{q} \cdot \vec{r}} V(\vec{r})$$

where $\vec{q} \equiv \vec{k}_f - \vec{k}_i$ is the momentum transfer. Next we must evaluate the density of states from $E_f = \frac{\hbar^2 k^2}{2m}$

$\Rightarrow$

$= \frac{2m}{2\hbar^2 k} = \frac{m}{\hbar^2 k}$

$$\frac{\partial n(E_f)}{\partial E_f} = \delta\Omega_f \underbrace{g(E_f)}_{=1} \frac{\Omega}{(2\pi)^3} k^2 \underbrace{\frac{dk}{dE_f}}$$

$$= \delta\Omega_f \frac{\Omega}{(2\pi)^3} \frac{mk}{\hbar^2}$$

$$= \delta\Omega_f \frac{\Omega}{(2\pi\hbar)^3} m\hbar k$$

Hence we find from page -116>-

$$\delta R_{fi}(E_f) = \left(\delta\Omega_f \frac{\Omega}{(2\pi\hbar)^3} m\hbar k\right) \frac{2\pi}{\hbar} \left|\frac{1}{\Omega}\int d^3r\, e^{-i\vec{q}\cdot\vec{r}} \delta V(\vec{r})\right|^2.$$

Recall that the elastic cross-section is defined as

$$J_{in}\delta\sigma_{el}(\theta,\varphi) = \frac{\#\text{ of transitions } |\psi_i\rangle \to |\psi_f\rangle}{\text{unit time}}\Bigg|_{E_f=E_i}$$

$$= \text{transition Rate for } |\psi_i\rangle \to |\psi_f\rangle \text{ with } E_f = E_i$$

$$\Rightarrow \delta\sigma_{el}(\theta,\varphi) = \frac{\delta R_{fi}(E_f)}{J_{in}}.$$

Letting $\vec{k}_i = k\hat{z}$, the incident flux is simply

$$J_{in} = \vec{J}_{in} \cdot \hat{z}$$

$$= \frac{\hbar}{2mi}\left[\varphi_i^*(\vec{r})\vec{\nabla}\varphi_i(\vec{r}) - (\vec{\nabla}\varphi_i(\vec{r}))^*\varphi_i(\vec{r})\right]\cdot\hat{z}$$

$$= \frac{1}{\Omega}\frac{\hbar k}{m}.$$

Thus

$$\delta\sigma_{el}(\theta,\varphi) = \frac{1}{\frac{1}{\Omega}\frac{\hbar k}{m}}\left(\delta\Omega_f \frac{\Omega}{(2\pi\hbar)^3} m\hbar k\right)\frac{2\pi}{\hbar}\frac{1}{\Omega^2} \times$$

$$\times\left|\int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\, V(\vec{r})\right|^2$$

$\Rightarrow$

$$\sigma_{el}(\theta,\varphi) \equiv \boxed{\frac{d\sigma_{el}}{d\Omega_f} = \left|\frac{-1}{4\pi}\int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\underbrace{\frac{2m}{\hbar^2}V(\vec{r})}_{=U(\vec{r})}\right|^2}$$

$f(\vec{q}) = -\frac{1}{4\pi}\hat{U}(\vec{q})$ is called the scattering amplitude (in the Born approx.)

Precisely the same result we obtain for elastic scattering in the Born approximation.

Examples of Scattering from Potentials

1) Scattering from a potential barrier

$$V(\vec{r}) = V\,\Theta(a-r)$$

$$V > 0\,, \quad r \geq 0.$$

So $\quad U(\vec{r}) = \frac{2m}{\hbar^2} V\,\Theta(a-r)$ and

the Fourier transform is simply that of the step function

$$\tilde{U}(\vec{q}) = \int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\, U(\vec{r})$$

$$= \frac{2mV}{\hbar^2}\int_0^a dr\, r^2 \int_0^{2\pi} d\varphi \int_{-1}^{+1} d(\cos\theta)\, e^{-iqr\cos\theta}$$

$$= \frac{2mV}{\hbar^2}\int_0^a dr\, r^2 (2\pi)\left(\frac{1}{-iqr}\right)\underbrace{\left(e^{-iqr} - e^{+iqr}\right)}_{=-2i\sin(qr)}$$

$$= \frac{8\pi mV}{\hbar^2 q}\underbrace{\int_0^a dr\, r\sin(qr)}_{=-\frac{d}{dq}\int_0^a dr\cos(qr)}$$

$$= -\frac{8\pi mV}{\hbar^2 q}\frac{d}{dq}\left(\frac{\sin(qa)}{q}\right) \quad \left(\begin{matrix}\sim\text{Spherical}\\ \text{Bessel}\\ \text{function}\end{matrix}\right)$$

$$= \left(\frac{2m}{\hbar^2}\right)\left(\frac{4}{3}\pi a^3\right)V\left[\frac{3}{(qa)}\underbrace{\left(\frac{\sin(qa)}{(qa)^2} - \frac{\cos(qa)}{(qa)}\right)}_{=j_1(qa)}\right]$$

So $\boxed{\tilde{U}(\vec{q}) = \left(\frac{2mV}{\hbar^2}\right)\left(\frac{4}{3}\pi a^3\right)\left[\frac{3}{(qa)}j_1(qa)\right]}$

The scattering amplitude is given by, in the Born approximation,

$$f^{(+)}_{Born}(\vec{k},\vec{k}') = f_{Born}(\vec{q})$$
$$= -\frac{1}{4\pi}\,\tilde{U}(\vec{q})$$

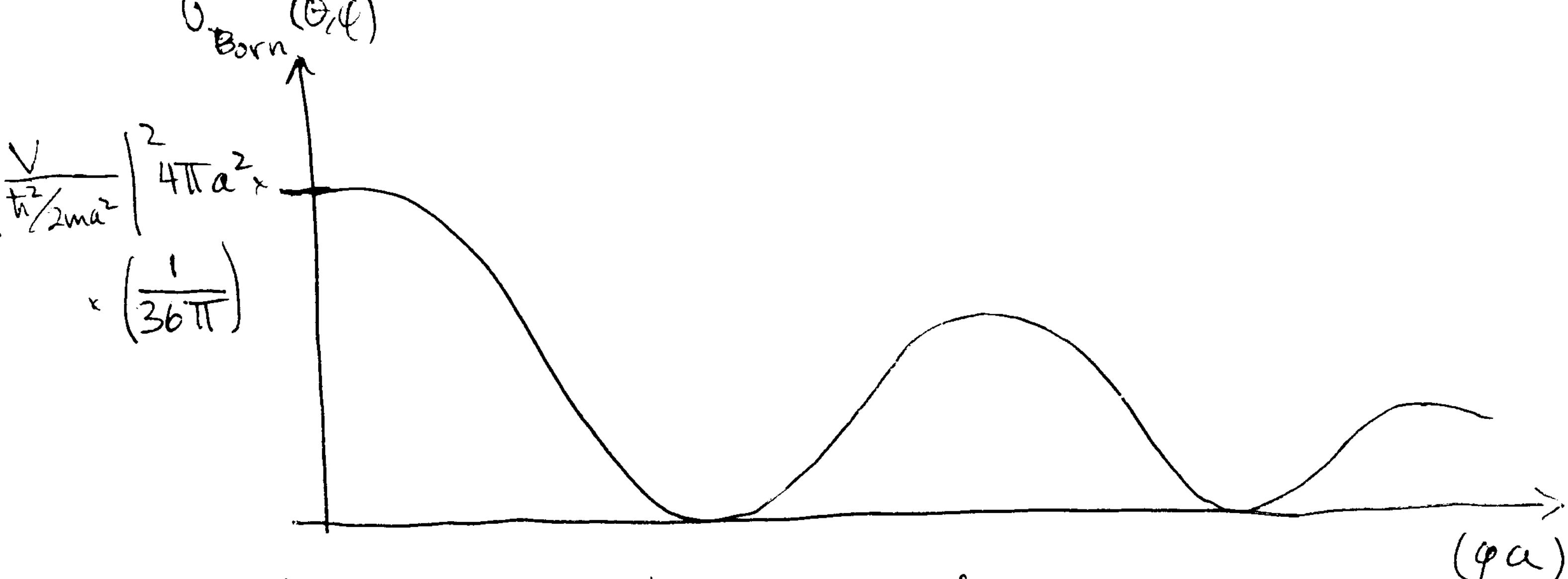
$$\boxed{f_{Born}(\vec{q}) = -\frac{1}{4\pi}\left(\frac{2mV}{\hbar^2}\right)\left(\frac{4}{3}\pi a^3\right)\left[\frac{3}{(qa)}\,j_1(qa)\right]}$$

The factor $\left[\frac{3}{(qa)}\,j_1(qa)\right]$ is called the <u>form factor</u>; it equals $+1$ at $q=0$ and oscillates as a function of q.

The differential cross section for scattering into the solid angle $d\Omega$ is

$$\sigma_{Born}(\theta,\varphi) = |f_{Born}(\vec{q})|^2$$

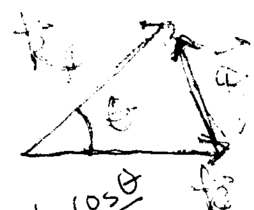
$$= \underbrace{\left|\frac{V}{(\hbar^2/2ma^2)}\right|^2}_{\substack{\text{(dimensionless)} \\ \text{energy ratio}}} \underbrace{4\pi a^2}_{\substack{\text{area} \\ \text{broad side of} \\ \text{a barn}}} \underbrace{\left|\frac{3\,j_1(qa)}{6\sqrt{\pi}\,(qa)}\right|^2}_{\substack{\text{form} \\ \text{factor} \\ \text{(dimensionless)}}}$$



The cross-section has the form of a diffraction pattern due to that abrupt change of $V(\vec{r})$ with r.

Recall $q = 2k|\sin\frac{\theta}{2}|$, so as the particle detector is moved to different angles θ, the number of scattering events per unit time oscillates. At the zeroes of $j_1(qa)$ the number of events measured for that θ vanishes.

$|\vec{k}_i| = |\vec{k}_f| = k$



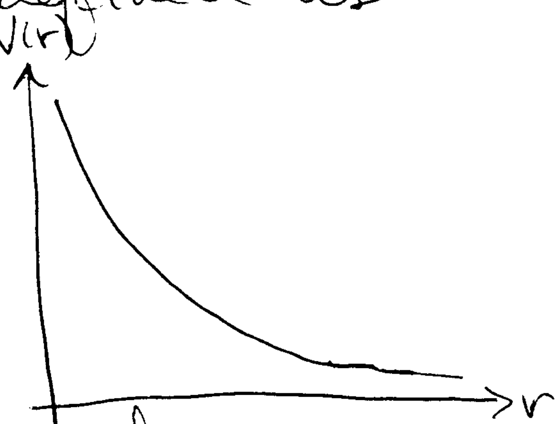
$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$

$$q^2 = k_i^2 + k_f^2 - 2k^2\cos\theta$$
$$= 2k^2(1-\cos\theta)$$
$$= 4k^2\sin^2\theta/2$$
$$\Rightarrow q = 2k|\sin\theta/2|$$

2) Scattering from a Yukawa potential:

The Yukawa potential is defined as

$$V(\vec{r}) = V \frac{a}{r} e^{-r/a}$$



with $V>0$, $a>0$ being the units of length.

As usual the Born approximation scattering amplitude is given by the Fourier transform of $V(\vec{r})$

$$f_{Born}(\vec{q}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\tilde{V}(\vec{q})$$

with

$$\tilde{V}(\vec{q}) = \int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\, V(\vec{r})$$

$$= Va\int_0^\infty dr\, r^2 \frac{e^{-r/a}}{r}\int_0^{2\pi} d\varphi \int_{-1}^{+1} d(\cos\theta)\, e^{-iqr\cos\theta}$$

$$= Va\int_0^\infty dr\, r\, e^{-r/a}(2\pi)\left(\frac{1}{-iqr}\right)\left(e^{-iqr} - e^{+iqr}\right)$$

$$= \frac{2\pi i}{q} Va \int_0^\infty dr \left[e^{-(\frac{1}{a}+iq)r} - e^{-(\frac{1}{a}-iq)r} \right]$$

$$= \frac{2\pi i}{q} Va \left[\frac{1}{\frac{1}{a}+iq} - \frac{1}{\frac{1}{a}-iq} \right]$$

$$= \frac{2\pi i}{q} Va \left[\frac{-2iqa^2}{1+(qa)^2} \right]$$

So

$$\boxed{\tilde{V}(\vec{q}) = \frac{4\pi a^2 (Va)}{1+(qa)^2}}$$

Thus the scattering amplitude is

$$f_{Born}(\vec{q}) = -\left(\frac{Va}{\hbar^2/2ma}\right) \frac{a}{1+(qa)^2}$$

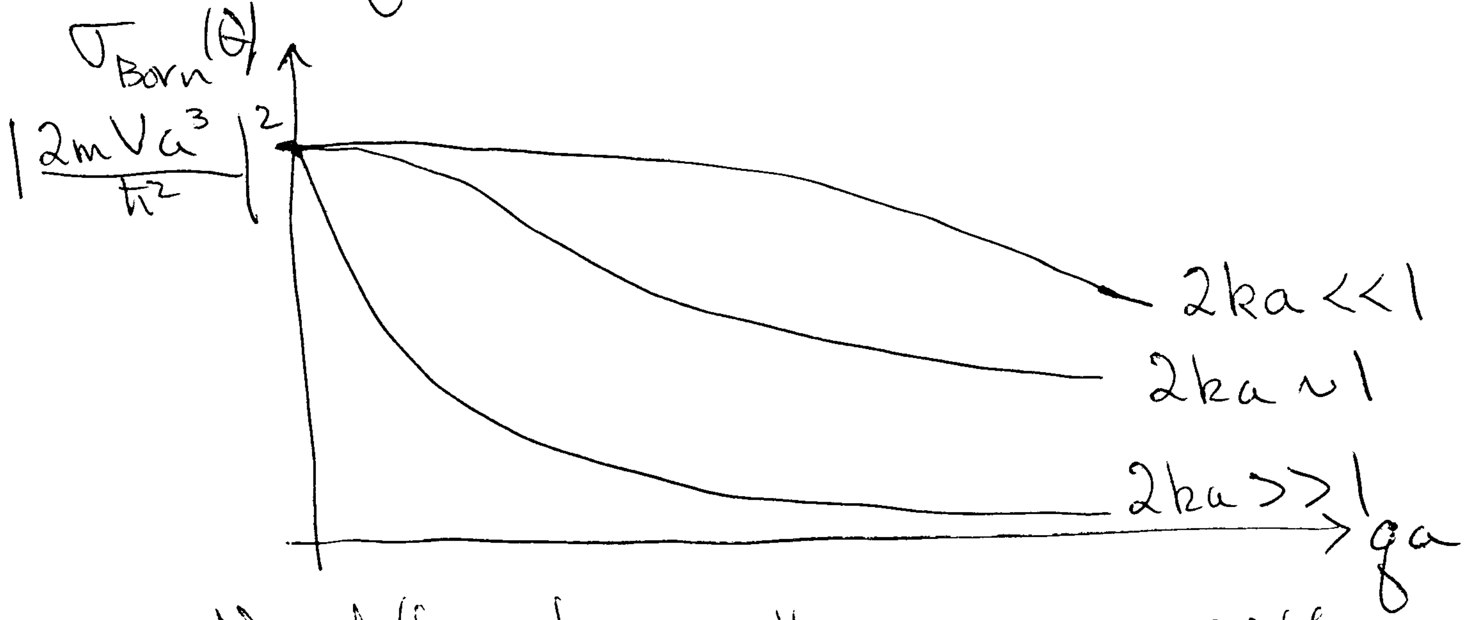
and so the Born approximation differential cross-section is

$$\sigma_{Born}(\theta,\varphi) = |f_{Born}(\vec{q})|^2$$

$$\boxed{\sigma_{Born}(\theta,\varphi) = \left|\frac{V}{\hbar^2/2ma^2}\right|^2 \frac{a^2}{\left(1+(qa)^2\right)^2}}$$

Again we have that $q^2 = 4k^2 \sin^2\frac{\theta}{2}$ and the C-M energy is $E = \frac{\hbar^2 k^2}{2m}$

So $q^2 = \frac{8mE}{\hbar^2}\sin^2\frac{\theta}{2}$.



No diffraction pattern occurs in this case since the Yukawa potential is a smoothly varying function of r. Also σ_{Born} is not zero for any finite (qa).

The cross-section for Coulomb potential scattering can be obtained

by letting $a \to \infty$ but keeping
$V a = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0}$, fixed then
the Yukawa potential goes over to
the Coulomb potential

$$V(\vec{r}) = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r}.$$

In this limit we obtain the
<u>Rutherford scattering formula</u>

$$\boxed{\begin{aligned}\sigma_{Born}(\theta) &= \left|\frac{2m Z_1 Z_2 e^2}{4\pi\epsilon_0 \hbar^2}\right|^2 \frac{1}{q^4} \\ &= \frac{(Z_1 Z_2 e^2 m)^2}{(4\pi\epsilon_0)^2 \hbar^4} \frac{1}{4k^4 \sin^4\frac{\theta}{2}}\end{aligned}}$$

$$= \left(\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0}\right)^2 \frac{1}{16 E^2 \sin^4\frac{\theta}{2}}$$