Time Dependent Perturbation Theory

Up until now we have been dealing with time independent Hamiltonians. Using stationary state techniques we were able to analyze the bound state spectrum (as well as elastic scattering) properties of such Hamiltonians. In such cases the system was in a fixed energy state, we would also like to analyze the case where a particle undertakes a transition from one energy state to another. For instance a hydrogen atom may absorb or emit a photon to make a transition from one energy level to another, such a case can be described by a Hamiltonian involving a time dependent external electro-magnetic field. As well, the Hamiltonian can be time independent but the state of the system does not have to be an energy eigenstate. In inelastic scattering ~~the~~ target is in different states before and after the scattering process. All these examples will be described by means of time-dependent perturbation theory.

9.1) Dirac Perturbation Theory

In general we will consider time dependent Hamiltonian of the form

$$H(t) = H_0 + H'(t)$$

where $H'(t) = 0$ for $t < t_0$. H_0 is time independent and it is assumed we know its spectrum and hence its stationary states

$$H_0 |\varphi_n\rangle = E_n |\varphi_n\rangle .$$

In general $\{E_n\}$ will have discrete as well as continuous values. As usual $\{|\varphi_n\rangle\}$ are a complete set of orthonormal states

$$\langle \varphi_n | \varphi_m \rangle = \delta_{nm}$$

$$\sum_n |\varphi_n\rangle\langle\varphi_n| = 1 .$$

(here we have used discrete notation,

for eigenstates with E_n in the continuum, one should integrate over its values with the appropriate density of states and Dirac delta function normalization. We will continue to use the discrete sum for convenience, note that Schiff uses the sum/integral notation "S_n" for this point.)

Since $H'(t) = 0$ for $t < t_0$, the $|\varphi_n\rangle$ are also eigenstates of $H(t)$ for $t < t_0$. Hence a system initially prepared in state $|\varphi_i\rangle$, any eigenstate of H_0, is also in an eigenstate of $H(t)$. Since $H'(t) \neq 0$ for $t > t_0$, the system is no longer in an eigenstate of $H(t)$ for $t > t_0$, i.e. $|\varphi_i\rangle$ is not an eigenstate of $H(t)$, $t > t_0$. $H'(t)$ will cause transitions between the various eigenstates $|\varphi_n\rangle$ to occur. If the system is initially prepared ($t < t_0$) in the eigenstate $|\varphi_i\rangle$ of H_0 ($= H$ for $t < t_0$), the probability that the system has made a transition and is in the state $|\varphi_f\rangle$ at time $t > t_0$, the transition is due to $H'(t) \neq 0$ for $t > t_0$, is

$$P_{fi}(t, t_0) = |\langle \varphi_f | \psi(t) \rangle|^2$$

where $|\Psi(t)\rangle$ is the state of the system at time t. $|\Psi(t)\rangle$ obeys the Schrödinger equation

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle = H(t)\,|\Psi(t)\rangle$$

with the initial condition

$$|\Psi(t_0)\rangle = e^{-\frac{iE_i t_0}{\hbar}}\,|\varphi_i\rangle$$

since $H(t<t_0) = H_0$ and $H_0|\varphi_i\rangle = E_i|\varphi_i\rangle$ is a stationary state. Since the Schrödinger equation is a first order differential equation in time the initial condition is all that is needed to uniquely specify the state at time t.

Since $\{|\varphi_n\rangle\}$ form a complete set, we can expand $|\Psi(t)\rangle$ at any instant of time in terms of these vectors to

$$|\Psi(t)\rangle = \sum_n{}' C_n(t)\,|\varphi_n\rangle$$

with the <u>time dependent</u> coefficients

$$C_n(t) = \langle\varphi_n|\Psi(t)\rangle.$$

The initial condition,

$$|\psi(t_0)\rangle = e^{-\frac{i}{\hbar}E_i t_0}|\varphi_i\rangle$$

$$= \sum_n' C_n(t_0)|\varphi_n\rangle ,$$

implies that

$$\boxed{C_n(t_0) = e^{-\frac{i}{\hbar}E_i t_0}\delta_{ni},}$$

the initial condition for the coefficients $C_n(t)$. The time evolution of the $C_n(t)$ is found from the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = i\hbar\sum_n' \frac{dC_n(t)}{dt}|\varphi_n\rangle$$

$$= H(t)|\psi(t)\rangle = \sum_n' H(t)C_n(t)|\varphi_n\rangle$$

Now $H(t) = H_0 + H'(t)$ & $H_0|\varphi_n\rangle = E_n|\varphi_n\rangle$

So

$$i\hbar\sum_n' \frac{dC_n(t)}{dt}|\varphi_n\rangle = \sum_n'(E_n + H'(t))C_n(t)|\varphi_n\rangle,$$

Projecting onto the $|\varphi_m\rangle$ direction gives

$$i\hbar \sum_n \frac{dC_n(t)}{dt} \underbrace{\langle \varphi_m | \varphi_n \rangle}_{=\delta_{mn}}$$

$$= \sum_n \Big[E_n C_n(t) \underbrace{\langle \varphi_m | \varphi_n \rangle}_{=\delta_{mn}} + \langle \varphi_m | H'(t) | \varphi_n \rangle C_n(t) \Big]$$

$\Rightarrow$

$$\boxed{i\hbar \frac{dC_m}{dt} = E_m C_m(t) + \sum_n \langle \varphi_m | H'(t) | \varphi_n \rangle C_n(t)}$$

This is a system of coupled (by matrix elements of $H'(t)$) first order in time differential equations with the initial condition

$$\boxed{C_m(t_0) = e^{-\frac{i}{\hbar} E_i t_0} \delta_{mi} .}$$

This system of equations for $C_n(t)$ is equivalent to the original Schrödinger equation. The transition probability

$$P_{fi}(t,t_0) = |\langle \varphi_f | \psi(t) \rangle|^2$$

$$= |\langle \varphi_f | \sum_n C_n(t) \varphi_n \rangle|^2$$

$$= |C_f(t)|^2 .$$

Of course we, in general, cannot solve the exact problem for the $C_n(t)$ but must develop approximation techniques for them. For the special case of $H'(t)=0$, the $C_n(t)$ equations de-couple and are trivially solved to yield

$$i\hbar \frac{dC_n(t)}{dt} = E_n C_n(t)$$

$$\Rightarrow \quad C_n(t) = C_n(0)\, e^{-\frac{i}{\hbar}E_n t}.$$

(From the initial condition $C_n(0)=\delta_{ni}$.)

We can separate this unperturbed time evolution from that of the exact or full $C_n(t)$ since we are interested in the affects of the time dependent Hamiltonian. Thus let

$$C_n(t) \equiv b_n(t)\, e^{-\frac{i}{\hbar}E_n t}.$$

This is not an approximation, it is equivalent to transforming to the interaction (or Dirac) picture

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}H_0 t}\, |\psi(t)\rangle_{IP}$$

Expanding in terms of the $\{|\varphi_n\rangle\}$

$$|\psi(t)\rangle = \sum_n c_n(t)\,|\varphi_n\rangle \quad \text{and}$$

$$|\psi(t)\rangle_{IP} = \sum_n b_n(t)\,|\varphi_n\rangle,$$ this definition becomes

$$|\psi(t)\rangle = \sum_n c_n(t)\,|\varphi_n\rangle = e^{-\frac{i}{\hbar}H_0 t}\,|\psi(t)\rangle_{IP}$$

$$= e^{-\frac{i}{\hbar}H_0 t}\sum_n b_n(t)\,|\varphi_n\rangle$$

$$= \sum_n e^{-\frac{i}{\hbar}E_n t}\,b_n(t)\,|\varphi_n\rangle,$$

projecting on $|\varphi_m\rangle \Rightarrow$

$$c_m(t) = e^{-\frac{i}{\hbar}E_m t}\,b_m(t), \quad \text{as}$$

required. Since we have removed the unperturbed time dependence from $|\psi(t)\rangle_{IP}$, we expect that it, and so $b_n(t)$, to evolve in time according to $H'(t)$.

Directly from above we have

$$i\hbar\frac{dc_n(t)}{dt} = i\hbar\frac{db_n(t)}{dt}e^{-\frac{i}{\hbar}E_n t} + E_n b_n(t) e^{-\frac{i}{\hbar}E_n t}$$

$$= E_n c_n(t) + \sum_l{}' \langle \varphi_n | H'(t) | \varphi_l \rangle c_l(t)$$

$$= E_n b_n(t) e^{-\frac{i}{\hbar} E_n t} + \sum_l{}' \langle \varphi_n | H'(t) | \varphi_l \rangle b_l(t) e^{-\frac{i}{\hbar} E_l t}$$

$\Rightarrow$

$$\boxed{i\hbar \frac{d\, b_n(t)}{dt} = \sum_l{}' \langle \varphi_n | H'(t) | \varphi_l \rangle e^{-\frac{i}{\hbar}(E_l - E_n)t}\, b_l(t)}$$

$$= \sum_l{}' \langle \varphi_n | e^{+\frac{i}{\hbar} H_0 t} H'(t) e^{-\frac{i}{\hbar} H_0 t} | \varphi_l \rangle b_l(t)$$

We recognize the operator matrix element as just that of the interaction Hamiltonian in the interaction picture

$$H'_{IP}(t) \equiv e^{+\frac{i}{\hbar} H_0 t} H'(t) e^{-\frac{i}{\hbar} H_0 t}.$$

The above equation is just the projection onto $|\varphi_n\rangle$ of the I.P. state equation of motion

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle_{IP} = e^{+\frac{i}{\hbar} H_0 t}\, i\hbar \frac{d}{dt} |\Psi(t)\rangle$$
$$- H_0 e^{\frac{i}{\hbar} H_0 t} |\Psi(t)\rangle$$

$$= e^{\frac{i}{\hbar}H_0 t} \underbrace{H(t)}_{= H_0 + H'(t)} |\psi(t)\rangle - H_0 e^{\frac{i}{\hbar}H_0 t} |\psi(t)\rangle$$

$$= \underbrace{e^{\frac{i}{\hbar}H_0 t} H'(t) e^{-\frac{i}{\hbar}H_0 t}}_{\equiv H'_{IP}(t)} \underbrace{e^{+\frac{i}{\hbar}H_0 t} |\psi(t)\rangle}_{= |\psi(t)\rangle_{IP}}$$

$$= \boxed{H'_{IP}(t) |\psi(t)\rangle_{IP} = i\hbar \frac{d}{dt} |\psi(t)\rangle_{IP}}$$

Thus $|\psi(t)\rangle_{IP}$'s and hence the $b_n(t)$'s time evolution is determined by $H'(t)$ in the I.P., i.e. $H'_{IP}(t)$
(See Picture Review)

Returning to the coefficient equations recall the initial condition

$$C_n(t_0) = e^{-\frac{i}{\hbar}E_i t_0} \delta_{ni}$$

but $$C_n(t_0) \equiv e^{-\frac{i}{\hbar}E_n t_0} b_n(t_0)$$

$$\Rightarrow \quad \boxed{b_n(t_0) = \delta_{ni}}$$

Thus the time evolution of the system in the Interaction Picture is given by the system of coupled differential equations

$$i\hbar \frac{d b_n(t)}{dt} = \sum_l e^{\frac{i}{\hbar}(E_n - E_l)t} \langle \varphi_n | H'(t) | \varphi_l \rangle b_l(t)$$

with the initial condition

$$b_n(t_0) = \delta_{ni} \quad ,$$

As stated before, these equations are exact, they contain the complete time evolution of the system as originally given by the Schrödinger equation; no approximation has yet been made. Of course we cannot solve these equations, hence we will develop a perturbation scheme for their solution. We will consider the case where the effects of $H'(t)$ are small corrections to those of H_0, that is $H'(t)$ has the form

$$H'(t) = \lambda \hat{H}'(t) \quad \text{with } |\lambda| \ll 1,$$

and the solution to the differential equation can be written as a power series in λ

$$b_n(t) = \sum_{l=0}^{\infty} \lambda^l b_n^{(l)}(t)$$

$$= b_n^{(0)}(t) + \lambda b_n^{(1)}(t) + \cdots .$$

First the initial condition implies

$$b_n(t_0) = \delta_{ni}$$

$$= b_n^{(0)}(t_0) + \lambda b_n^{(1)}(t_0) + \cdots .$$

Equating equal powers of $\lambda \Rightarrow$

$$\boxed{\begin{aligned} b_n^{(0)}(t_0) &= \delta_{ni} \\ b_n^{(l)}(t_0) &= 0 \quad \text{for } l \geq 1. \end{aligned}}$$

From the differential equation we have

$$i\hbar \sum_{l=0}^{\infty} \lambda^l \frac{d b_n^{(l)}(t)}{dt} = \sum_{l=0}^{\infty} \lambda^{l+1} \sum_m e^{\frac{i}{\hbar}(E_n - E_m)t} \langle \varphi_n | \hat{H}'(t) | \varphi_m \rangle \times b_m^{(l)}(t)$$

Again equating equal powers of $\lambda \Rightarrow$

$l=0$: $\quad i\hbar \dfrac{db_n^{(0)}(t)}{dt} = 0$

$l \geq 1$: $\quad i\hbar \dfrac{db_n^{(l)}(t)}{dt} = \sum_m' e^{\frac{i}{\hbar}(E_n - E_m)t} \langle \varphi_n | \hat{H}'(t) | \varphi_m \rangle b_m^{(l-1)}(t)$

We solve these equations recursively, first

$l=0$ eq. $\Rightarrow$ $\boxed{b_n^{(0)}(t) = b_n^{(0)}(t_0) = \delta_{ni}}$ by the i.c.

Using this in the $l=1$ equation $\Rightarrow$

$$i\hbar \frac{db_n^{(1)}(t)}{dt} = \sum_m' e^{\frac{i}{\hbar}(E_n - E_m)t} \langle \varphi_n | \hat{H}'(t) | \varphi_m \rangle \underbrace{b_m^{(0)}(t)}_{= \delta_{mi}}$$

$$= e^{\frac{i}{\hbar}(E_n - E_i)t} \langle \varphi_n | \hat{H}'(t) | \varphi_i \rangle .$$

Integrating this from t_0 to t and using the initial condition gives

$$b_n^{(1)}(t) - \cancel{b_n^{(1)}(t_0)}^{\to 0} = \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\, e^{\frac{i}{\hbar}(E_n - E_i)t_1} \langle \varphi_n | \hat{H}'(t_1) | \varphi_i \rangle$$

Substituting in the expansion for $b_n(t)$, we have to order λ

$$\boxed{b_n(t) = \delta_{ni} + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\, e^{\frac{i}{\hbar}(E_n - E_i)t_1} \langle \varphi_n | H'(t_1) | \varphi_i \rangle + O(H'^2)}\,.$$

The transition probability becomes

$$P_{fi}(t, t_0) = |c_f(t)|^2 = |b_f(t)\, e^{-\frac{i}{\hbar}E_f t}|^2$$

$$= |b_f(t)|^2$$

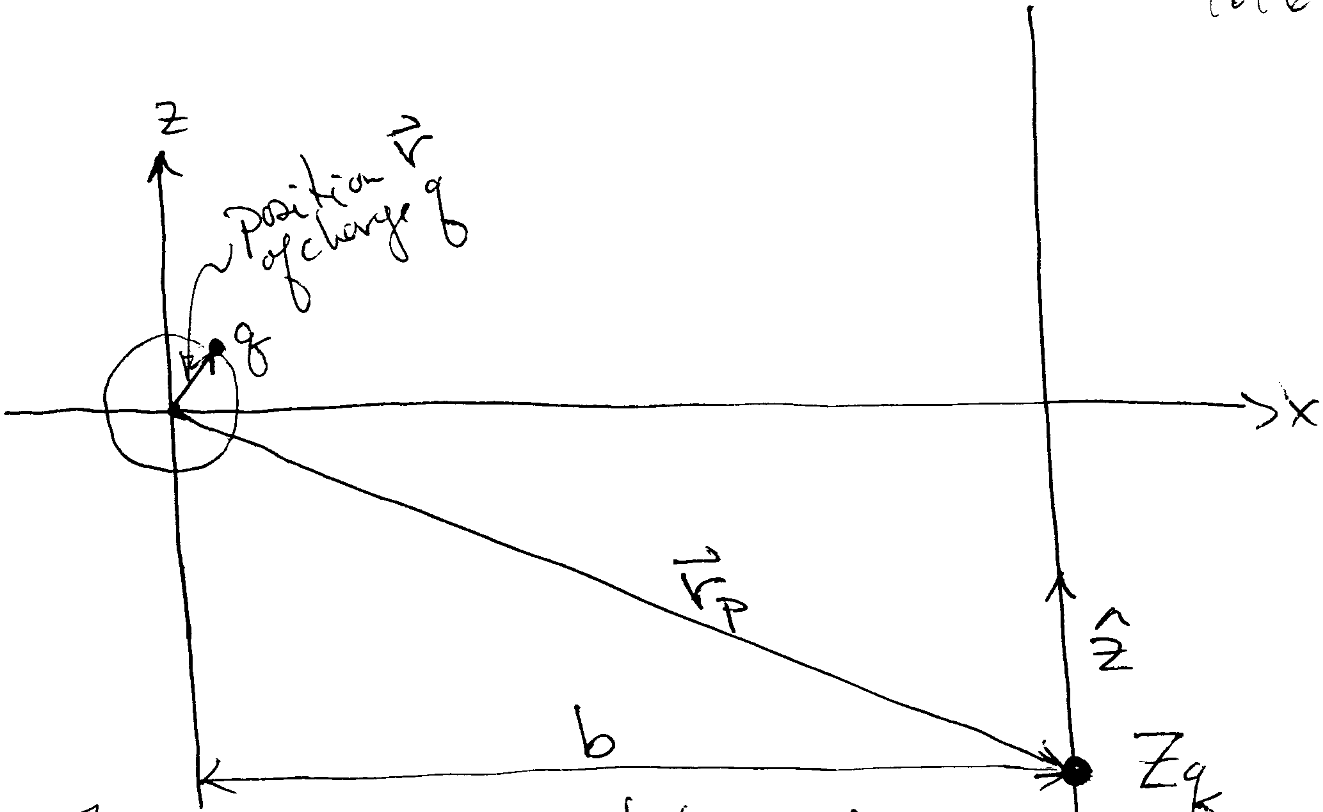
$$= \left| \delta_{fi} + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\, e^{\frac{i}{\hbar}(E_f - E_i)t_1} \langle \varphi_f | H'(t_1) | \varphi_i \rangle + O(H'^2) \right|^2.$$

For final states different from the initial state $\delta_{fi} = 0$, Thus the transition probability to first order in the perturbation is

$$\boxed{P_{fi}(t,t_0) = \left| \frac{1}{i\hbar} \int_{t_0}^{t} dt_1 e^{\frac{i}{\hbar}(E_f - E_i)t_1} \langle \varphi_f | H'(t_1) | \varphi_i \rangle \right|^2}$$

Example: Model for energy loss of a charged particle travelling through matter.

Suppose we consider a heavy charged particle with charge Zq moving in a straight line with constant velocity through matter. That is we treat this particle classically and supply whatever energy is necessary to it so that it moves on a straight line with constant speed v. Now consider the particle of matter as the target of such a classically moving particle. Suppose this matter is modelled by a particle of mass m and charge q, quantum mechanically bound in a 3 dimensional harmonic oscillator potential. Choose the origin of coordinates to be such that it is centered on this particle's equilibrium position



The classical projectile of charge Zq moves along the fixed (impulse approximation) trajectory

$$\vec{r}_p = b\hat{x} + vt\hat{z}$$

while the target matter particle moves according to the Hamiltonian

$$H_0 = \frac{1}{2m}\vec{P}^2 + \frac{1}{2}m\omega^2\vec{R}^2 .$$

(in coordinate basis $\vec{R}|\vec{r}\rangle = \vec{r}\,|\vec{r}\rangle$).

As we know the eigenstates of H_0 are given by the direct products of the eigenstates of the (x,y,z) number operators

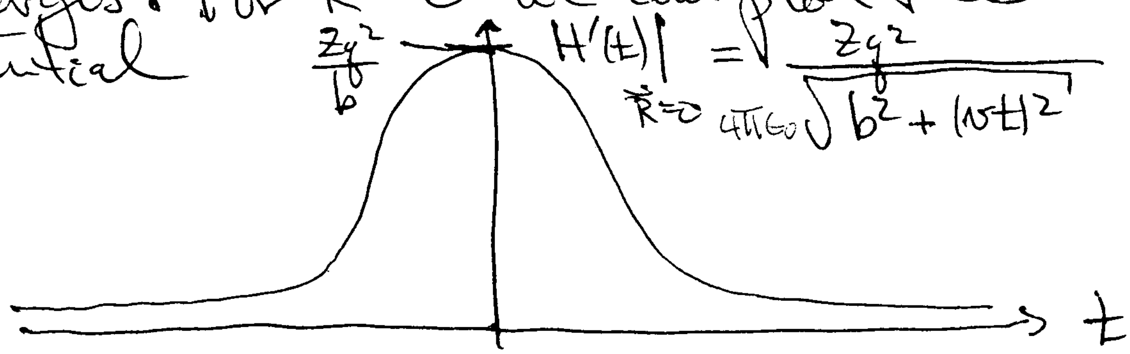
$$|n_x, n_y, n_z\rangle = |n_x\rangle|n_y\rangle|n_z\rangle$$

where $$H_0|n_x,n_y,n_z\rangle = \hbar\omega(n_x+n_y+n_z+\tfrac{3}{2})\times$$
$$\times|n_x,n_y,n_z\rangle$$
with $n_x,n_y,n_z = 0,1,2,\ldots$.

Initially (for times $t<t_0$) we assume the matter target is in its ground state
$$|\varphi_i\rangle = |0,0,0\rangle.$$

Since both particles are charged there is a Coulomb force between them with potential energy describing the interaction Hamiltonian $H'(t)$:
$$H'(t) = \frac{Zq^2}{4\pi\epsilon_0|\vec{R}-\vec{r}_p(t)|}.$$

$H'(t)$ is time dependent since the Coulomb energy depends inversely upon the separation distance of the two charges! For $\vec{R}=0$ we can plot the potential
$$H'(t)\Big|_{\vec{R}=0} = \frac{Zq^2}{4\pi\epsilon_0\sqrt{b^2+(vt)^2}}$$



In addition we assume that the heavy particle of charge Zq is far away from the matter target particle i.e. $|\vec{R}| << |\vec{r}_p|$
Hence we can expand the denominator of $H'(t)$ (dipole approximation)

$$H'(t) = \frac{Zq^2}{4\pi\epsilon_0}\left(\frac{1}{r_p} + \frac{\vec{r}_p \cdot \vec{R}}{r_p^3} + \cdots\right)$$

We can ask to find the probability that the target is in the lowest excited state, say $|1,0,0\rangle$, at time t. Accordingly we need the matrix element of $H'(t)$

$$\langle \psi_f | H'(t) | \psi_i \rangle = \langle 1,0,0 | H'(t) | 0,0,0 \rangle$$

$$= \frac{Zq^2}{4\pi\epsilon_0} \langle 1,0,0 | \left\{ \frac{1}{r_p} + \frac{\vec{r}_p \cdot \vec{R}}{r_p^3} \right\} | 0,0,0 \rangle$$

$$= \frac{Zq^2}{4\pi\epsilon_0 r_p} \cancelto{0}{\langle 1,0,0 | 0,0,0 \rangle} + \frac{Zq^2 \vec{r}_p \cdot}{4\pi\epsilon_0 r_p^3} \langle 1,0,0 | \vec{R} | 0,0,0 \rangle.$$

The dipole matrix element is simply

$$\langle 1,0,0 | \vec{R} | 0,0,0 \rangle = \langle 1 | X \hat{x} | 0 \rangle + \cancelto{0}{\langle 1,0,0 | Y\hat{y} + Z\hat{z} | 0,0,0 \rangle}$$

Recalling that $\langle 1|X|0\rangle = \langle 0|a_x\left(\frac{1}{\sqrt{2}}\sqrt{\frac{\hbar}{m\omega}}(a_x^\dagger + a_x)\right)\times$
$\times|0\rangle$

$$= \sqrt{\frac{\hbar}{2m\omega}}\,\underbrace{\langle 0|a_x a_x^\dagger|0\rangle}_{=1} \quad \left(\text{Recall [illegible] ...lator}\right).$$

So $\langle 1,0,0|\vec{R}|0,0,0\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\hat{x}$ and

hence

$$\langle \varphi_f|H'(t)|\varphi_i\rangle = \frac{Zq^2}{4\pi\epsilon_0}\sqrt{\frac{\hbar}{2m\omega}}\,\frac{\vec{r}_p\cdot\hat{x}}{r_p^3}$$

$$= \frac{Zq^2}{4\pi\epsilon_0}\sqrt{\frac{\hbar}{2m\omega}}\,\frac{b}{(b^2+v^2t^2)^{3/2}}.$$

The general formula for the transition probability was

$$P_{fi}(t,t_0) = \left|\frac{1}{i\hbar}\int_{t_0}^{t} dt_1\, e^{\frac{i}{\hbar}(E_f-E_i)t_1}\langle \varphi_f|H'(t_1)|\varphi_i\rangle\right|^2.$$

For our model $E_f - E_i = \hbar\omega$ and choosing $t \to +\infty$ and $t_0 \to -\infty$ we find

$$P_{fi}(+\infty,-\infty) = \left(\frac{Zq^2}{\hbar 4\pi\epsilon_0}\right)^2\left(\frac{\hbar}{2m\omega}\right)\left|\int_{-\infty}^{+\infty} dt\,\frac{b e^{i\omega t}}{(b^2+v^2t^2)^{3/2}}\right|^2.$$

In order to evaluate the integral we write it as

$$I = \int_{-\infty}^{+\infty} dt \frac{b e^{i\omega t}}{(b^2+v^2t^2)^{3/2}} = -\frac{d}{db}\int_{-\infty}^{+\infty} dt \frac{e^{i\omega t}}{(b^2+v^2t^2)^{1/2}}$$

$$= -\frac{d}{db}\int_{-\infty}^{+\infty} dt \frac{e^{i\omega t}}{b\left(1+\frac{v^2t^2}{b^2}\right)^{1/2}}$$

letting $s = \frac{v}{b} t \Rightarrow$

$$= \frac{1}{v}\left[-\frac{d}{db}\int_{-\infty}^{+\infty} ds \frac{e^{i\frac{\omega b}{v}s}}{(1+s^2)^{1/2}}\right]$$

Calling $\gamma = \frac{\omega b}{v}$ so that $\frac{d}{db} = \frac{\omega}{v}\frac{d}{d\gamma}$ we get

$$= \frac{\omega}{v^2}\left[-\frac{d}{d\gamma}\underbrace{\int_{-\infty}^{+\infty} ds \frac{e^{i\gamma s}}{(1+s^2)^{1/2}}}\right]$$

$$= \int_{-\infty}^{+\infty} ds \frac{\cos\gamma s}{(1+s^2)^{1/2}} \qquad \left[\begin{array}{l}\sin\gamma s \\ \text{is odd \&} \\ \text{so vanishes}\end{array}\right]$$

$$= 2\int_0^{\infty} ds \frac{\cos\gamma s}{(1+s^2)^{1/2}}$$

$$= \frac{2\omega}{v^2}\left[-\frac{d}{d\gamma}\int_0^{\infty} ds \frac{\cos\gamma s}{(1+s^2)^{1/2}}\right]$$

Recall that the modified Bessel functions are

$$K_n(z) = \frac{\pi}{2} i^{n+1} H_n(iz) \quad \left(\begin{array}{l}\text{Gradshteyn \& Ryzhik} \\ \text{section 8.4}\end{array}\right)$$

$$= \frac{\pi}{2} i^{n+1} \left[J_n(iz) + i N_n(iz) \right]$$

and

$$K_0(\gamma) = \int_0^\infty ds \frac{\cos \gamma s}{(1+s^2)^{1/2}} \quad \left(\begin{array}{l}\text{G\&R} \\ \text{integral 8.432.5}\end{array}\right)$$

and K_n obeys the recursion relation

$$K_1(\gamma) = -\frac{d}{d\gamma} K_0(\gamma) \quad \left(\begin{array}{l}\text{G\&R eq. 8.473.6} \\ \text{or 8.486.18}\end{array}\right).$$

As well the $K_1(\gamma)$ has the asymptotic properties

$$K_1(\gamma) \underset{\gamma\to\infty}{\sim} \left(\frac{\pi}{2\gamma}\right)^{1/2} e^{-\gamma}$$

$$K_1(\gamma) \underset{\gamma\to 0}{\sim} \frac{1}{\gamma} .$$

So we find for the integral

$$I = \frac{2\omega}{v^2} K_1(\gamma)$$

$$= \frac{2\omega}{v^2} K_1\left(\frac{\omega b}{v}\right)$$

So the transition probability is

$$P_{fi}(+\infty,-\infty) = \left(\frac{Zq^2/4\pi\epsilon_0}{\hbar v}\right)^2 \left(\frac{\hbar\omega}{\frac{1}{2}mv^2}\right) \left|K_1\left(\frac{\omega b}{v}\right)\right|^2 .$$

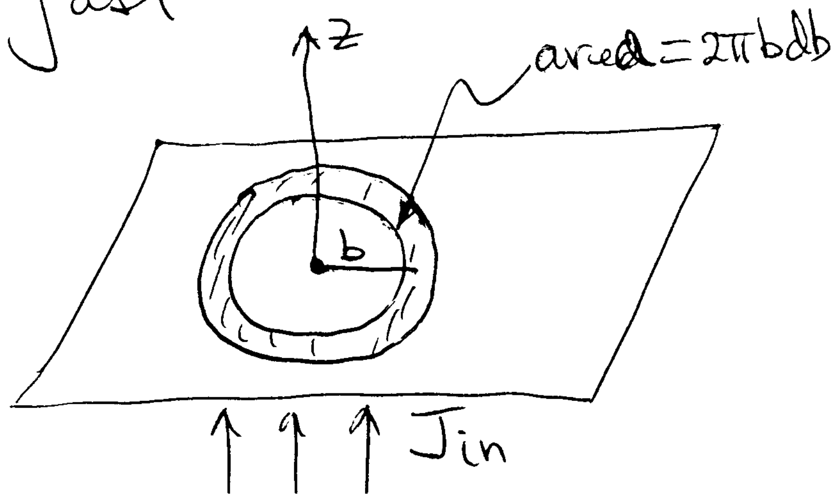
For a flux of charge Zq particles we have that the number of transitions to the excited state of the target particle per unit time is given by the differential inelastic cross-section $d\sigma_{fi}(b)$

$$J_{in}\, d\sigma_{fi}(b) \equiv \frac{\#\text{ of transitions}}{\text{unit time}}$$

Since the transition probability is the probability for the target excitation to occur per incoming projectile a distance b from the target we have that the number of projectiles per unit time with this impact parameter b is just

$$J_{in}\,(2\pi b\, db)$$

(azimuthal symmetry $\Rightarrow$ same physics for all azimuthal angles with b fixed)



So

$$J_{in}\, d\sigma_{fi}(b) = \frac{\#\text{ of transitions}}{\text{unit time}} = (2\pi b\,db) J_{in} \times P_{fi}(+\infty, -\infty)$$

$\Rightarrow$

$$d\sigma_{fi}(b) = 2\pi b\,db\, P_{fi}(+\infty, -\infty)$$

The total inelastic cross section is given by

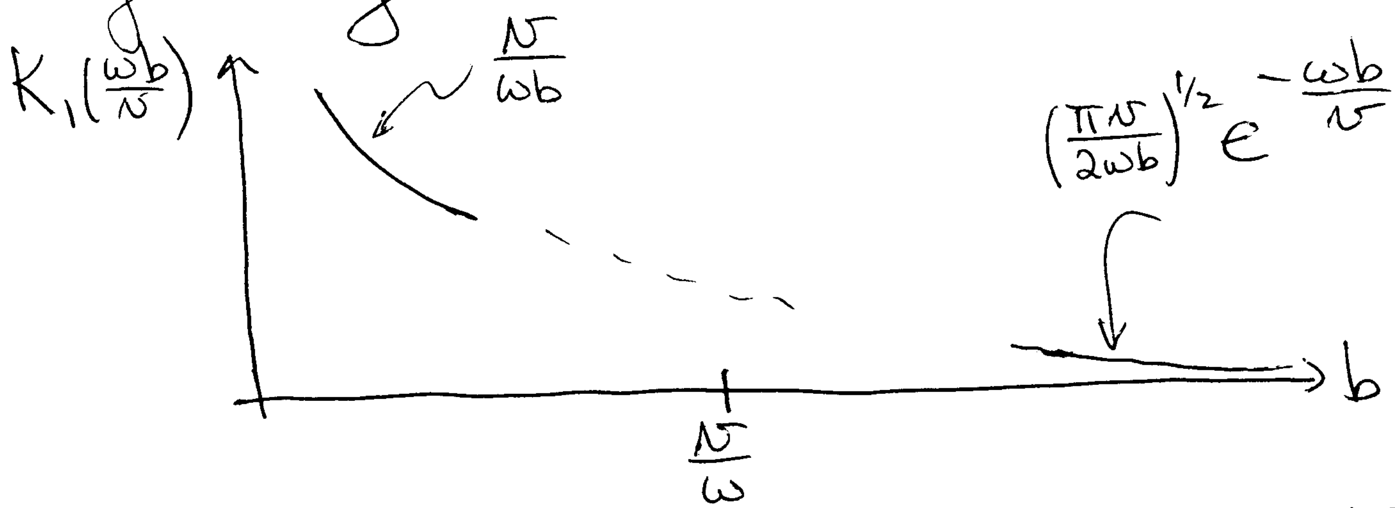
$$\sigma_{fi} = 2\pi \int_{b_{min}}^{\infty} db\, b\, P_{fi}(+\infty, -\infty)$$

where we cut-off the impact parameter $b \geq b_{min}$ at a minimum separation due to the dipole approximation $|\vec{R}| << |\vec{r}_p|$

$\Rightarrow$

$$\boxed{\sigma_{fi} = 2\pi \left(\frac{Zq^2/4\pi\epsilon_0}{\hbar v}\right)^2 \left(\frac{\hbar\omega}{\frac{1}{2}mv^2}\right) \int_{b_{min}}^{\infty} db\, b \left|K_1\left(\frac{\omega b}{v}\right)\right|^2}$$

The functional behavior of K_1 is given by



$K_1\left(\frac{\omega}{v} b\right)$ is exponentially damped for $b > \frac{v}{\omega}$, hence the main contribution to the b integral comes from b less than some, b_{max}, maximum value. For

$b < b_{max}$ we can approximate $K_1\left(\frac{\omega}{v} b\right) \simeq \frac{v}{\omega b}$,

So the integral yields

$$\sigma_{fi} = 2\pi \left(\frac{Z q^2/4\pi\epsilon_0}{\hbar v}\right)^2 \left(\frac{\hbar\omega}{\frac{1}{2} m v^2}\right) \left(\frac{v^2}{\omega^2}\right) \underbrace{\int_{b_{min}}^{b_{max}} db\, b \frac{1}{b^2}}_{= \ln \frac{b_{max}}{b_{min}}}$$

$\Rightarrow$

$$\boxed{\sigma_{fi} = 2\pi b^2 \left(\frac{Z q^2 \cancel{/b}}{\hbar\, \cancel{v}\, \omega}\right)^2 \left(\frac{\hbar\omega}{\frac{1}{2} m v^2}\right) \ln \frac{b_{max}}{b_{min}}}$$

$4\pi\epsilon_0$

Since $\ln \frac{b_{max}}{b_{min}}$ is slowly varying function of the ratio, the σ_{fi} is not very sensitive to their exact values. Thus we have a model for the energy loss of a charged particle while it travels through matter. The more transitions/sec that are induced (i.e. larger σ_{fi}) the more energy is lost by the incident particle (the energy goes into exciting the matter target). The slower the Zq charged particle goes the more rapid the energy loss due to the $\frac{1}{v^2}$ in σ_{fi}.
