Chapter 9

Time Dependent Perturbation Theory

Up until now we have been dealing with time independent Hamiltonians. Using stationary state techniques we were able to analyze the bound state spectrum (as well as elastic scattering) properties of such Hamiltonians. In such cases the system was in a fixed energy state. We would also like to analyze the case where a particle undergoes a transition from one energy state to another. For instance, a hydrogen atom may absorb or emit a photon to make a transition from one energy level to another. Such a case could be described by a Hamiltonian involving a time dependent external electromagnetic field. As well, the Hamiltonian can be time independent but the state of the system does not have to be an energy eigenstate. In inelastic scattering the target is in different states before and after the scattering process. All these examples will be described.
by means of time-dependent perturbation theory.

Dirac Perturbation Theory

In general we will consider time-dependent Hamiltonian of the form

\[ H(t) = H_0 + H'(t) \]

where \( H'(t) = 0 \) for \( t < t_0 \). \( H_0 \) is time independent and it is assumed we know its spectrum and hence its stationary states

\[ H_0 |\psi_n\rangle = E_n |\psi_n\rangle. \]

In general \( E_n \) will have discrete as well as continuous values. As usual \( \{ |\psi_n\rangle \} \) are a complete set of orthonormal states

\[ \langle \psi_n | \psi_m \rangle = \delta_{nm} \]

\[ \sum_n |\psi_n\rangle \langle \psi_n| = 1. \]

(here we have used discrete notation,
for eigenstates with $E_n$ in the continuum, one should integrate over its values with the appropriate density of states and Dirac delta function normalization. We will continue to use the discrete sum for convenience, note that Schiff uses the sum/integral notation "$S_n$" for this point.

Since $H'(t) = 0$ for $t < t_0$, the $|\psi_n\rangle$ are also eigenstates of $H(t)$ for $t < t_0$. Hence a system initially prepared in state $|\psi_i\rangle$, any eigenstate of $H_0$, is also in an eigenstate of $H(t)$. Since $H'(t) = 0$ for $t > t_0$, the system is no longer in an eigenstate of $H(t)$ for $t > t_0$, i.e. $|\psi_i\rangle$ is not an eigenstate of $H(t)$. $H'(t)$ will cause transitions between the various eigenstates $|\psi_n\rangle$ to occur. If the system is initially prepared ($t < t_0$) in the eigenstate $|\psi_i\rangle$ of $H_0 (= H_{\text{free}})$, the probability that the system has made a transition and is in the state $|\psi_f\rangle$ at time $t > t_0$, the transition is due to $H'(t) \neq 0$ for $t > t_0$, is

$$P_{fi} (t, t_0) = |<\psi_f | H'(t) | \psi_i >|^2$$
where $|\Psi(t)\rangle$ is the state of the system at time $t$. $|\Psi(t)\rangle$ obeys the Schrödinger equation

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = H(t) |\Psi(t)\rangle$$

with the initial condition

$$|\Psi(t_0)\rangle = e^{-\frac{iE}{\hbar} t_0} |\psi_0\rangle$$

Since $H(t<t_0) = H_0$ and $H(t) |\psi_0\rangle = E_0 |\psi_0\rangle$ is a stationary state. Since the Schrödinger equation is a first order differential equation in time, the initial condition is all that is needed to uniquely specify the state at time $t$.

Since $\{|\psi_n\rangle\}$ form a complete set, we can expand $|\Psi(t)\rangle$ at any instant of time in terms of these vectors

$$|\Psi(t)\rangle = \sum_n C_n(t) |\psi_n\rangle$$

with the time dependent coefficients

$$C_n(t) = \langle \psi_n |\Psi(t)\rangle$$
The initial condition,

\[ |\Psi(t_0)\rangle = e^{-\frac{i}{\hbar} E\cdot t_0} |\Psi_i\rangle \]

implies that

\[ C_n(t_0) = e^{-\frac{i}{\hbar} E\cdot t_0} \delta_{n i} \]

is the initial condition for the coefficients, \( C_n(t) \).

The time evolution of the \( C_n(t) \) is found from the Schrödinger equation

\[ i\hbar \frac{d}{dt} |\Psi(t)\rangle = \hbar \sum_n \frac{dC_n(t)}{dt} |\Psi_n\rangle \]

\[ = H(t) |\Psi(t)\rangle = \sum_n H(t) C_n(t) |\Psi_n\rangle \]

Now

\[ H(t) = H_0 + H'(t) \]

So

\[ i\hbar \sum_n \frac{dC_n(t)}{dt} |\Psi_n\rangle = \sum_n \left( E_n + H'(t) \right) C_n(t) |\Psi_n\rangle \]

Projecting onto the \( |\Psi_n\rangle \) direction gives...
\[ i \hbar \sum_n \frac{dC_n(t)}{dt} = \delta_{mn} \]

\[ = \sum_n \left[ E_n C_n(t) \langle \psi_m | \psi_n \rangle + \langle \psi_m | H'(t) | \psi_n \rangle C_n(t) \right] \]

\[ \Rightarrow \]

\[ i \hbar \frac{dC_m}{dt} = E_mC_m(t) + \sum_n \langle \psi_m | H'(t) | \psi_n \rangle C_n(t) \]

This is a system of coupled (by matrix elements of \( H'(t) \)) first order in time differential equations with the initial condition

\[ C_m(t=0) = e^{\frac{i}{\hbar} E_m t} \delta_{mi} . \]

This system of equations for \( C_n(t) \) is equivalent to the original Schrödinger equation. The transition probability

\[ P_{fi}(t, t_0) = \left| \langle \psi_f | \psi_i(t) \rangle \right|^2 \]

\[ = \left| \langle \psi_f | \sum_n C_n(t) | \psi_n \rangle \right|^2 \]

\[ = \left| C_f(t) \right|^2 . \]
Of course we, in general, cannot solve the exact problem for the $C_n(t)$ but must develop approximation techniques for them. For the special case of $H'(t) = 0$, the $C_n(t)$ equations decouple and are trivially solved to yield

$$i\hbar \frac{dC_n(t)}{dt} = E_n C_n(t)$$

$$\Rightarrow \quad C_n(t) = C_n(0) e^{-\frac{i}{\hbar} E_n t}.$$  

(From the initial condition $C_n(0) = \delta_{ni}$.)

We can separate this unperturbed time evolution from that of the exact or full $C_n(t)$ since we are interested in the effects of the time-dependent Hamiltonian. Thus let

$$C_n(t) \equiv b_n(t) e^{-\frac{i}{\hbar} E_n t}.$$  

This is not an approximation, it is equivalent to transforming to the interaction (or Dirac) picture

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar} H_0 t} |\psi(t)\rangle_{ip}.$$  


Expanding in terms of the $\lambda_k(\nu n)$'s

\[ \lambda(t) = \sum_n C_n(t) |\nu n\rangle + \cdots \]

\[ \lambda(t)_I = \sum_n b_n(t) |\nu n\rangle, \text{ this definition becomes} \]

\[ \lambda(t) = \sum_n C_n(t) |\nu n\rangle = \sum_n e^{-\frac{i}{\hbar} H_0 t} C_n(t) \sum_n b_n(t) |\nu n\rangle \]

\[ = \sum_n e^{-\frac{i}{\hbar} E_{\nu n} t} b_n(t) |\nu n\rangle, \]

projecting on $|\nu m\rangle$ implies

\[ C_m(t) = e^{-\frac{i}{\hbar} E_{\nu m} t} b_m(t), \]

as required. Since we have removed the unperturbed time dependence from $\lambda(t)_I$, we expect that it, and so $b_n(t)$, to evolve in time according to $H(t)$.

Directly from above we have

\[ \frac{i}{\hbar} \frac{dn(t)}{dt} = \frac{i}{\hbar} \frac{db_n(t)}{dt} e^{-\frac{i}{\hbar} E_{\nu n} t} + E_{\nu n} b_n(t) e^{-\frac{i}{\hbar} E_{\nu n} t} \]
\[ \begin{align*}
&= E_n c_n(t) + \sum_{l} \left\langle \psi_n | H'(t) | \psi_l \right\rangle c_l(t) \\
&= E_n b_n(t) e^{-\frac{i}{\hbar} E_n t} + \sum_{l} \left\langle \psi_n | H'(t) | \psi_l \right\rangle b_l(t) e^{-\frac{i}{\hbar} E_l t} \\
\Rightarrow \quad & \quad \frac{i\hbar}{\hbar} \frac{db_n(t)}{dt} = \sum_{l} \left\langle \psi_n | H'(t) | \psi_l \right\rangle e^{-\frac{i}{\hbar} (E_l - E_n) t} b_l(t) \\
&= \sum_{l} \left\langle \psi_n | e^{\frac{i}{\hbar} H_0 t} H'(t) e^{-\frac{i}{\hbar} H_0 t} | \psi_l \right\rangle b_l(t) \\
\end{align*} \]

We recognize the operator matrix element as just that of the interaction Hamiltonian in the interaction picture

\[ H'_{\text{IP}}(t) = e^{\frac{i}{\hbar} H_0 t} H'(t) e^{-\frac{i}{\hbar} H_0 t} \]

The above equation is just the projection of motion

\[ \frac{i\hbar}{\hbar} \frac{d}{dt} | 14(t) \rangle_{\text{IP}} = e^{\frac{i}{\hbar} H_0 t} \frac{i\hbar}{\hbar} \frac{d}{dt} | 14(t) \rangle - H_0 e^{\frac{i}{\hbar} H_0 t} | 14(t) \rangle \]
\[ e^{\frac{i}{\hbar} H_0 t} \mathcal{A}(t) = e^{\frac{i}{\hbar} H_0 t} \left( H_0 e^{\frac{i}{\hbar} H_0 t} \mathcal{A}(t) \right) = H_0 + H_0^\prime(t) \]

\[ = e^{\frac{i}{\hbar} H_0 t} H_0^\prime(t) e^{\frac{-i}{\hbar} H_0 t} + e^{\frac{i}{\hbar} H_0 t} \mathcal{A}(t) \]

\[ = H_0^\prime(t) + \mathcal{A}(t) \]

\[ = H_0^\prime IP(t) \mathcal{A}(t) \]

Thus \[ H_0^\prime IP(t) \mathcal{A}(t) \] 's act hence the \[ b_{n\hbar^1} \]'s time evolution is determined by \[ H_0^\prime(t) \] in the IP, i.e. \[ H_0^\prime IP(t) \] (See Further Review).

Returning to the coefficient equations recall the initial condition

\[ C_{n}(t_0) = e^{\frac{-i}{\hbar} E_n t_0} S_{n\hbar} \]

but \[ C_{n}(t_0) = e^{\frac{-i}{\hbar} E_n t_0} b_n(t_0) \]

\[ \Rightarrow b_n(t_0) = S_{n\hbar} \]
Thus the time evolution of the system in the Interaction Picture is given by the system of coupled differential equations

\[ i\hbar \frac{db_n(t)}{dt} = \frac{i}{\hbar} (E_n - E_t) b_n(t) \]

with the initial condition

\[ b_n(t_0) = \delta_{n0} \]

As stated before, these equations are exact: they contain the complete time evolution of the system as originally given by the Schrödinger equation and no approximation has yet been made. Of course we cannot solve these equations, hence we will develop a perturbation scheme for their solution. We will consider the case where the effects of \( H'(t) \) are small corrections to those of \( H_0 \), that is \( H'(t) \) has the form

\[ H'(t) = \lambda H'(t) \quad \text{with} \quad |\lambda| < 1. \]
and the solution to the differential equation can be written as a power series in $\lambda$

$$b_n(t) = \sum_{l=0}^{\infty} \lambda^l b_n^{(l)}(t)$$

$$= b_n^{(0)}(t) + \lambda b_n^{(1)}(t) + \ldots$$

First the initial condition implies

$$b_n(t_0) = \delta_{ni}$$

$$= b_n^{(0)}(t_0) + \lambda b_n^{(1)}(t_0) + \ldots$$

Equating equal powers of $\lambda \Rightarrow$

$$b_n^{(0)}(t_0) = \delta_{ni}$$

$$b_n^{(l)}(t_0) = 0 \text{ for } l \geq 1.$$ 

From the differential equation we have

$$i\hbar \sum_{l=0}^{\infty} \lambda^l \frac{db_n^{(l)}(t)}{dt} = \sum_{l=0}^{\infty} \lambda^l \sum_{m} e^{i\hbar} \langle \psi_m | H(t) | \psi_n \rangle \times b_m^{(l)}(t)$$
Again equating equal powers of \( t \) \( \Rightarrow \)

\[ l=0 : \quad \text{i} \hbar \frac{\partial b_n^{(0)}(t)}{\partial t} = 0 \]

\[ l \geq 1 : \quad \text{i} \hbar \frac{\partial b_n^{(l)}(t)}{\partial t} = \sum_m \frac{i(E_n - E_m)}{\hbar} \langle \psi_n | \hat{A}(t) | \psi_m \rangle b_m^{(l-1)}(t) \]

We solve these equations recursively. \( \text{first} \)

\[ l=0, \text{eq.} \Rightarrow b_n^{(0)}(t) = b_n^{(0)}(t_0) = \delta_{ni} \] by \( \text{i.c.} \)

Using this in the \( l=1 \) equation \( \Rightarrow \)

\[ \text{i} \hbar \frac{\partial b_n^{(1)}(t)}{\partial t} = \sum_m \frac{i(E_n - E_m)}{\hbar} \langle \psi_n | \hat{A}(t) | \psi_m \rangle b_m^{(0)}(t) \]

\[ = \delta_{ni} \]

\[ = e^{-i(E_n - E_i) t} \langle \psi_n | \hat{A}(t) | \psi_i \rangle . \]

Integrating this from \( t_0 \) to \( t \) and using the initial condition gives
\[ b_{n}^{(1)}(t) - b_{n}^{(1)}(t_0) = \frac{i}{\hbar} \int_{t_0}^{t} dt' e^{\frac{i}{\hbar}(E_n - E_i) t'} \langle \psi_n | H(t') | \psi_i \rangle. \]

Substituting in the expansion for \( b_{n}(t) \), we have \( b_{n}^{(1)}(t) \) to order \( \hbar \).

\[ b_{n}(t) = \delta_{n,i} + \frac{i}{\hbar} \int_{t_0}^{t} dt' e^{\frac{i}{\hbar}(E_n - E_i) t'} \langle \psi_n | H'(t') | \psi_i \rangle + O(\hbar^2). \]

The transition probability becomes

\[ P_{f,i}(t, t_0) = |C_f(t)|^2 = |b_f(t) e^{\frac{i}{\hbar}(E_f - E_i) t}|^2 
= |b_f(t)|^2 
= |\delta_{f,i} + \frac{i}{\hbar} \int_{t_0}^{t} dt' e^{\frac{i}{\hbar}(E_f - E_i) t'} \langle \psi_f | H'(t') | \psi_i \rangle + O(\hbar^2)|^2. \]

For final states different from the initial state \( \delta_{f,i} = 0 \), the transition probability is first order in the perturbation.
Example: Model for energy loss of a charged particle traveling through matter.

Suppose we consider a heavy charged particle with charge $Ze$ moving in a straight line with constant velocity through matter. Treat it as a particle classically and supply whatever energy is necessary to it so that it moves on a straight line with constant speed $v$. Now consider the particle of matter as the target of such a classically moving particle. Suppose this matter is modelled by a particle of mass $m$ and charge $q$, quantum mechanically bound in a 3 dimensional harmonic oscillator potential. Choose the origin of coordinates to be such that it is centred in this particle's equilibrium position.
The classical projectile of charge $Zq$ moves along the fixed impulse trajectory:

$$\vec{r}_P = b \hat{x} + n \hat{z}$$

while the target neutral particle moves according to the Hamiltonian

$$H_0 = \frac{1}{2m} \vec{p}^2 + \frac{1}{2} m_0 c^2 \vec{r}^2$$

(in coordinate basis $\vec{R}_1> = \vec{r}_1 >$).

As we know, the eigenstates of $H_0$ are given by the direct products of the eigenstates of the $(x,y,z)$ number operators

$$|n_x, n_y, n_z> = |n_x> |n_y> |n_z>$$
where

\[ H_0 |n_x, n_y, n_z\rangle = \frac{\hbar}{2} \left( n_x + n_y + n_z + \frac{3}{2} \right) \times |n_x, n_y, n_z\rangle \]

with \( n_x, n_y, n_z = 0, 1, 2, \ldots \).

Initially (for times \( t < t_0 \)) we assume the matter target is in its ground state

\[ |\psi\rangle = |0, 0, 0\rangle. \]

Since both particles are charged there is a Coulomb force between them with potential energy describing the interaction Hamiltonian

\[ H(t) = \frac{Zq^2}{4\pi\varepsilon_0 |R - F(t)|}. \]

\( H(t) \) is time dependent since the Coulomb energy depends inversely upon the separation distance of the two charges! For \( R = 0 \) we can get the potential

\[ \frac{Zq^2}{4\pi\varepsilon_0} \left( \frac{1}{b^2 + (at)^2} \right) \]
In addition we assume that the heavy particle of charge \( Z q \) is far away from the lighter target particle, i.e. \( |r_p| << |\mathbf{R}| \). Hence we can expand the denominator of \( H^{(\pm)} \) (dipole approximation)

\[
H^{(\pm)} = \frac{Z q^2}{4\hbar c} \left( \frac{1}{r_p} + \frac{\mathbf{R} \cdot \mathbf{R}}{r_p^3} + \cdots \right)
\]

We can ask to find the probability that the target is in the lowest excited state, say \( 11,0,0 \rangle \), at time \( t \). Accordingly we need the matrix element \( \langle \psi_f | H^{(\pm)} | \psi_i \rangle \)

\[
\langle \psi_f | H^{(\pm)} | \psi_i \rangle = \langle 1,0,0| H^{(\pm)} | 10,0,0 \rangle
\]

\[
= \frac{Z q^2}{4\hbar c} \left( \langle 1,0,0| \left[ \frac{1}{r_p} + \frac{\mathbf{R} \cdot \mathbf{R}}{r_p^3} \right] 3 |10,0,0 \rangle \right)
\]

\[
= \frac{Z q^2}{4\hbar c} \left( \langle 10,0| \mathbf{R} |0,0 \rangle + \frac{2q^2 \mathbf{R}}{4\hbar c} \langle 10,0| \mathbf{R} |0,0 \rangle \right)
\]

The dipole matrix element is simply

\[
\langle 10,0| \mathbf{R} |0,0 \rangle = \langle 1| \mathbf{R} |0 \rangle + \langle 10,0| \mathbf{R} |10,0 \rangle
\]
Recalling that \( \langle 11 \bar{x} 10 \rangle = \langle 01a_x \left( \frac{1}{\sqrt{2}} \sqrt{\frac{\hbar}{2m\omega}} (a_x^+ + a_x) \right)_x \rangle_{10} \)

\[
= \sqrt{\frac{\hbar}{2m\omega}} \langle 01a_x a_x^+ 10 \rangle \\
= 1
\]

So \( \langle 1,0,0| \vec{R}| 10,0,0 \rangle = \left( \frac{\hbar}{2m\omega} \right)^{1/2} \) and hence

\[
\langle \psi_f | H(t) | \psi_i \rangle = \frac{2g^2}{4\pi \hbar} \sqrt{\frac{\hbar}{2m\omega}} \frac{\vec{r}_0 \cdot \vec{b}}{r_p^3}
\]

\[
= \frac{2g^2}{4\pi \hbar} \sqrt{\frac{\hbar}{2m\omega}} \frac{b}{(b^2 + \alpha^2 + z^2)^{3/2}}
\]

The general formula for the transition probability was

\[
P_{fi}(t,t_0) = \left| \frac{1}{i\hbar} \int_{t_0}^{t} dt' \ e^{-i\hbar(E_f - E_i)t'} \langle \psi_f | H(t') | \psi_i \rangle \right|^2
\]

For our model \( E_f - E_i = \hbar \omega \) and choosing \( t \to +\infty \) and \( t_0 \to -\infty \) we find

\[
P_{fi}(+\infty,-\infty) = \left( \frac{2g^2}{\hbar^4 \omega} \right)^2 \left( \frac{\hbar}{2m\omega} \right) \left| \int_{-\infty}^{+\infty} dt \frac{be^{-i\omega t}}{(b^2 + \alpha^2 + z^2)^{3/2}} \right|^2
\]
In order to evaluate the integral we write it as

\[
I = \int_{-\infty}^{\infty} dt \frac{be^{i\omega t}}{(b^2 + s^2t^2)^{3/2}} = -\frac{d}{db} \int_{-\infty}^{\infty} dt \frac{e^{i\omega t}}{b(1 + \frac{s^2t^2}{b^2})^{1/2}}
\]

letting \( s = \frac{v}{b} t \) \( \Rightarrow \)

\[
= \frac{1}{v} \left[ -\frac{d}{db} \int_{-\infty}^{\infty} ds \frac{e^{i\omega b s}}{(1+s^2)^{1/2}} \right]
\]

Calling \( y = \frac{\omega b}{v} \) so that \( \frac{d}{db} = \frac{\omega}{v} \frac{d}{dy} \) we get

\[
= \frac{\omega}{v^2} \left[ -\frac{d}{dy} \int_{-\infty}^{\infty} ds \frac{e^{iys}}{(1+s^2)^{1/2}} \right]
\]

\[
= \int_{-\infty}^{\infty} ds \frac{\cos ys}{(1+s^2)^{1/2}}
\]

\[
= 2 \int_{0}^{\infty} ds \frac{\cos ys}{(1+s^2)^{1/2}}
\]

\[
= \frac{2\omega}{v^2} \left[ -\frac{d}{dy} \int_{0}^{\infty} ds \frac{\cos ys}{(1+s^2)^{1/2}} \right]
\]
Recall that the modified Bessel functions are
\[ K_n(z) = \frac{\pi}{2} i^{-n+1} H_n(iz) \]  
(Gradsteyn & Ryzhik, section 8.4)
\[ = \frac{\pi}{2} i^{-n+1} \left[ J_n(iz) + i N_n(iz) \right] \]

and
\[ K_0(x) = \int_0^\infty \frac{\cos xs}{(1 + s^2)^{1/2}} \]  
(G&R integral 8.432.5)

and \( K_n \) obeys the recursion relation
\[ K_{n+1}(x) = -\frac{d}{dx} K_n(x) \]  
(G&R eq. 8.473.6)

or 6.8.486.18

As well, the \( K_1(x) \) has the asymptotic properties
\[ K_1(x) \sim \frac{e^{-x}}{\sqrt{x}} \]
\[ K_1(x) \sim -\frac{1}{x} \]

So we find for the integral
\[ I = \frac{2\omega}{N^2} K_1(x) \]
\[ = \frac{2\omega}{N^2} K_1 \left( \frac{\omega k}{N} \right) \]
So the transition probability is
\[ P_{fi}(\pm \alpha) = \left( \frac{2q^2\alpha^2}{\hbar \nu} \right)^2 \left( \frac{\hbar \nu}{\frac{1}{2} m v^2} \right) J_1 \left( \frac{\omega_b}{\nu} \right) \left| \frac{\omega_b}{\nu} \right|^2. \]

For a flux of charge \( Ze \) particles we know that the number of transitions to the excited state of the target particle per unit time is given by the differential inelastic cross section \( d\sigma_{fi}(b) \)

\[ J_{in} d\sigma_{fi}(b) = \frac{\text{# of transitions}}{\text{unit time}}. \]

Since the transition probability is the probability for the target excitation to occur per incoming projectile at a distance \( b \) from the target we have that the number of projectiles per unit time with thin b impact parameter \( b \) is just

\[ J_{in} (2\pi b db) \]

(azimuthal symmetry: change physics for all azimuths with \( b \) fixed)

\[ \text{area} = \pi b^2 \]
So,
\[ J_{in} \delta f_c(b) = \frac{\text{# of transitions}}{\text{unit time}} = (2\pi b b) J_{in} \times \delta f_c(b) \times P_{f_c}(+\infty, -\infty) \]

\[ \Rightarrow \]
\[ \delta f_c(b) = 2\pi b b P_{f_c}(+\infty, -\infty) \]

The total inelastic cross section is given by
\[ \sigma_{f_c} = 2\pi \int_{b_{\text{min}}}^{\infty} db \ b \ b \ P_{f_c}(+\infty, -\infty) \]

where we cut-off the impact parameter \( b \geq b_{\text{min}} \) at a minimum separation due to the dipole approximation \( |R| \ll |\vec{r}_{\text{p}}| \)

\[ \Rightarrow \]
\[ \sigma_{f_c} = 2\pi \left( \frac{2g^2}{\hbar^2 \mu_{\text{c}}} \right)^2 \left( \frac{\hbar \omega}{\frac{1}{2} m v^2} \right) \int_{b_{\text{min}}}^{\infty} db \ b \left| K_i \left( \frac{\omega b}{\omega} \right) \right|^2 \]
The functional behavior of $K_1$ is given by

$$K_1(\frac{N}{\omega}b)$$

The function is exponentially damped for $b > \frac{N}{\omega}$, hence the main contribution to the $b$-integral comes from $b$ less than some $b_{\text{max}}$, maximum value. For $b < b_{\text{max}}$ we can approximate $K_1(\frac{N}{\omega}b) \approx \frac{N}{\omega}b$.

So the integral yields

$$\Omega F_c = 2\pi \left( \frac{2g^2/k_b^2}{\hbar \delta} \right)^2 \left( \frac{\hbar \omega}{\frac{1}{2} m v^2} \right) \frac{v^2}{\omega^2} \int_{b_{\text{min}}}^{b_{\text{max}}} b b_{\text{max}} \frac{1}{b^2}$$

$$= \ln \left( \frac{b_{\text{max}}}{b_{\text{min}}} \right)$$

Thus

$$\Omega F_c = 2\pi \left( \frac{2g^2/k_b^2}{\hbar \delta} \right)^2 \left( \frac{\hbar \omega}{\frac{1}{2} m v^2} \right) \ln \frac{b_{\text{max}}}{b_{\text{min}}}$$
Since $\frac{b_{max}}{b_{min}}$ is slowly varying function of their ratio, the $C_i$ is not very sensitive to their exact values. Thus we have a model for the energy loss of a charged particle while it travels through matter. The more transitions that are induced (i.e., larger $C_i$) the more energy is lost by the incident particle (the energy goes into exciting the matter target). The slower the $Z$ charged particle goes, the more rapid the energy loss due to the $\frac{1}{r^2}$ in $T_{z1}$. 