6.4) Hyperfine Splitting:

Besides the fine structure splitting of the energy levels there is the hyperfine splitting. This is due to the proton having a spin-magnetic dipole moment

$$\vec{\mu}_p = g \frac{e}{2m_p} \vec{S}_p \qquad (g \simeq 5.59)$$

which produces a dipole magnetic field

$$\vec{B} = \frac{\mu_0}{4\pi r^3}\left[3(\vec{\mu}_p \cdot \hat{r})\hat{r} - \vec{\mu}_p\right] + \frac{2\mu_0}{3}\vec{\mu}_p \delta^3(\vec{r})$$

at the location of the electron. The electron interacts with this magnetic field via its spin magnetic dipole moment

$$\vec{\mu}_e = -g_e \frac{e}{2m_e} \vec{S}_e \qquad (g_e \simeq 2)$$

$$H_{hf} = -\vec{\mu}_e \cdot \vec{B}$$

$$= \frac{\mu_0 g e^2}{8\pi m_p m_e} \frac{\left[3(\vec{S}_p \cdot \hat{r})(\vec{S}_e \cdot \hat{r}) - \vec{S}_p \cdot \vec{S}_e\right]}{r^3} + \frac{\mu_0 g e^2}{3 m_p m_e} \vec{S}_p \cdot \vec{S}_e \, \delta^3(\vec{r})$$

Once again according to R-S, the 1^{st}-order correction to the energy levels is given by the matrix elements of H_{hf}

$$E^{(1)}_{hf} = \frac{\mu_0 g e^2}{8\pi m_p m_e} \left\langle \frac{3(\vec{S}_p\cdot\hat{r})(\vec{S}_e\cdot\hat{r}) - \vec{S}_p\cdot\vec{S}_e}{r^3} \right\rangle$$

$$+ \frac{\mu_0 g e^2}{3 m_p m_e} \langle \vec{S}_p\cdot\vec{S}_e \rangle |\psi_{nlm}(0)|^2$$

Consider the ground state (or any $l=0$ state) $n=1$, $l=0$ $(m=0)$ In that case the ψ_{100} wavefunction is spherically symmetric, hence

$$\int d^3r\, f(r) r_i r_j = a\,\delta_{ij}$$

$$\Rightarrow a\cdot 3 = \int d^3r\, f(r) r^2 \qquad \left(\begin{array}{c}\text{problem}\\ 6.25\end{array}\right)$$

$$\Rightarrow a = \frac{4\pi}{3}\int_0^\infty dr\, r^4 f(r) = \frac{1}{3}\int d^3r\, f(r) r^2 .$$

This implies the first average vanishes.

Hence for the groundstate $|\psi_{100}(0)|^2 = \frac{1}{\pi a_0^3}$

$$E^{(1)}_{hf} = \frac{\mu_0 g e^2}{3\pi m_p m_e a_0^3} \langle \vec{S}_p\cdot\vec{S}_e \rangle$$

This is called the <u>spin-spin coupling</u>.

We must now include the proton spin in the definition of the total spin

$$\vec{S} = \vec{S}_e + \vec{S}_p \quad \text{and}$$

$$\vec{S}_p \cdot \vec{S}_e = \frac{1}{2}\left[S^2 - S_e^2 - S_p^2\right]$$

Since e^- & p are both spin $\frac{1}{2}$, we have $S_e^2 = S_p^2 = \frac{3}{4}\hbar^2$. In the triplet state $\uparrow_e \uparrow_p$ etc. $\Rightarrow$ total spin $S = 1$ $\Rightarrow S^2 = 2\hbar^2$; in the singlet state $\uparrow_e\downarrow_p - \downarrow_e\uparrow_p$ and $S^2 = 0$ $\Rightarrow$

$$E^1_{hf} = \frac{4g\hbar^4}{3m_p m_e^2 c^2 a_0^4} \begin{cases} \frac{1}{4} & \text{triplet} \\ -\frac{3}{4} & \text{singlet} \end{cases}$$

So we see that the spin-spin coupling breaks the spin degeneracy of the ground state. The energy gap is

$$\Delta E_{1S_{1/2}} = \frac{4g\hbar^4}{3m_p m_e^2 c^2 a_0^4} \simeq 5.88 \times 10^{-6}\,\text{eV}.$$

(Note $E_{hf} \propto \frac{m_e}{m_p} mc^2 \alpha^4$)

This is a frequency $\nu = \frac{\Delta E_{1S_{1/2}}}{h} = 1420\ MHz$, and is a wavelength $c/\nu = 21\,cm$! in the microwave region.