Indeed we applied precisely these techniques to the 3-d

to find explicitly

The {energy, ($\measuredangle$ mom.)^2, ($\measuredangle$ mom.)$_z$} eigenfunctions!

i.e. $\{k, \ell, m\} \to \{n, \ell, m\}$

$E_n = \hbar\omega(n+\frac{3}{2})$

Thus we have a basis for any central potential problem

Spin: Besides our well known spin 0 results — we now have new types of particles with intrinsic property of spin.

In particular we have spin $\frac{1}{2}$ particles e^-, μ etc. also boundstates p, n, atoms, → which at lower energy behave as particles with intrinsic spin.

perhaps spin matrix valued

Now consider the state of a system (particle) moving in potential $U(\vec{r})$ with spin s. Its Hilbert space of states is the direct product of its orbital (spatial) degrees of freedom $\mathcal{H}_{\vec{r}}$ and its spin degrees of freedom $\mathcal{H}_s$

$$\mathcal{H} = \mathcal{H}_{\vec{r}} \otimes \mathcal{H}_s$$

That is a basis for the dynamics of all the degrees of freedom of the particle is

$$\{ |\vec{r}, s, m_s\rangle = |\vec{r}\rangle \otimes |s, m_s\rangle \}$$

Hence the state of the system at time t, $|\psi(t)\rangle$, can be expanded into a multi-component wavefunction

$$|\psi(t)\rangle = \int d^3r \sum_{m_s=-s}^{+s} \psi^{(s)}_{m_s}(\vec{r},t)\, |\vec{r}, s, m_s\rangle$$

Schrödinger's Eq. is now a spin matrix valued differential equation since the Hamiltonian besides orbital variables i.e. $\vec{R}, \vec{P}$, may also involve the spin operator $\vec{S}$:

$$H = H(\vec{R}, \vec{P}, \vec{S})$$

and Schrödinger's Eq is

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = H |\Psi(t)\rangle$$

Projecting onto $|\vec{r}, s, m_s\rangle \Rightarrow$

$$i\hbar \frac{\partial}{\partial t} \Psi^{(s)}_{m_s}(\vec{r}, t) = \int d^3r' \sum_{m_s'} \overbrace{\langle \vec{r}, s, m_s | H | \vec{r}', s, m_s' \rangle}^{= (H)_{m_s m_s'} \delta^3(\vec{r} - \vec{r}')} \langle \vec{r}', s, m_s' | \Psi(t) \rangle$$

$$= \sum_{m_s'} (H^{(s)})_{m_s m_s'} \Psi^{(s)}_{m_s'}(\vec{r}, t)$$

Assume at first

Now if the orbital degrees of freedom evolve independently from the spin degrees of freedom we may separate the variables, that is the state or wavefunction into a product of a spatial wavefunction and a spinor wavefunction.

More specifically if

Assume at first

$$H = H_o + H_s$$

H_o ↑ orbital, H_s ↖ spin

$$H_s = H_s(\vec{S})$$

$$H_o = H_o(\vec{R}, \vec{P})$$

Then $|\Psi(t)\rangle = |\Psi_o(t)\rangle \otimes |\chi_s(t)\rangle$

where $|\Psi_o(t)\rangle \in \mathcal{H}_{\vec{r}}$ & $|\chi_s(t)\rangle \in \mathcal{H}_s$.

Further

$$i\hbar \frac{\partial}{\partial t}|\Psi(t)\rangle = \left(i\hbar \frac{\partial}{\partial t}|\Psi_o(t)\rangle\right) \otimes |\chi_s(t)\rangle + |\Psi_o(t)\rangle \otimes \left(i\hbar \frac{\partial}{\partial t}|\chi_s(t)\rangle\right)$$

$$= H\,|\Psi(t)\rangle$$

$$= \left(H_o|\Psi_o(t)\rangle\right) \otimes |\chi_s(t)\rangle + |\Psi_o(t)\rangle \otimes \left(H_s|\chi_s(t)\rangle\right)$$

Hence

$$i\hbar \frac{\partial}{\partial t}|\Psi_o(t)\rangle = H_o|\Psi_o(t)\rangle$$

$$i\hbar \frac{\partial}{\partial t}|\chi_s(t)\rangle = H_s|\chi_s(t)\rangle,$$

separately.

From a wavefunction point of view we have

$$\psi^{(s)}_{m_s}(\vec{r},t) = \langle \vec{r}, s, m_s | \psi(t) \rangle$$
$$= \langle \vec{r} | \psi_0(t) \rangle \langle s, m_s | \chi_s(t) \rangle$$
$$= \psi_0(\vec{r},t) \big(\chi_s(t)\big)_{m_s}$$

The wavefunction separates into the product of an orbital or space-time dependent wavefunct and a spin wavefunction.

Schrödinger's eq. also separates

$$i\hbar \frac{\partial}{\partial t} \psi_0(\vec{r},t) = H_0(\vec{R},\vec{P})\, \psi_0(\vec{r},t)$$
$$i\hbar \frac{\partial}{\partial t} \big(\chi_s(t)\big)_{m_s} = \big(H_s\big)_{m_s m_s'} \big(\chi_s(t)\big)_{m_s'}$$

Of course <u>not</u> <u>every</u> situation ~~has~~ has orbital & spin variables as separate — there may be a spin-orbit coupling eg. $\vec{L}\cdot\vec{S}$ then the evolution of these degrees of freedom are coupled.