Finally, let's consider a system of two spin $\frac{1}{2}$ particles (ignoring the orbital degrees of freedom ~~at first~~).

The total spin angular momentum of the system is

$$\vec{S} = \vec{S}^{(1)} + \vec{S}^{(2)}$$

while the state space of the system is spanned by the direct product space

$$|\overset{\frac{1}{2}}{s_1}, m_{s_1}\rangle \otimes |\overset{\frac{1}{2}}{s_2}, m_{s_2}\rangle = |s_1 m_{s_1}, s_2 m_{s_2}\rangle$$

These are just eigenstates of the individual $\vec{S}^{(1)2}_{(2)}$ & $S_z^{(1)}_{(2)}$ operators

$$\vec{S}^{(i)^2} |s_1 m_{s_1}, s_2 m_{s_2}\rangle = \hbar^2 \underbrace{s_i(s_i+1)}_{= \frac{1}{2}(\frac{1}{2}+1) = \frac{3}{4}} |s_1 m_{s_1}, s_2 m_{s_2}\rangle$$

$i = 1, 2$

$$S_z^{(i)} |\;"\;\rangle = \hbar m_i |\;"\;\rangle$$

Now since each particle is spin $\frac{1}{2} = s_i$ and $m_{s_i} = \pm\frac{1}{2}$ let's use the compact notation for the 4 possible product states that span the space i.e. 4-dimensional $\mathcal{H}$

$$|\uparrow\uparrow\rangle = |s_1=\tfrac{1}{2}\ m_{s_1}=+\tfrac{1}{2},\ s_2=\tfrac{1}{2}\ m_{s_2}=\tfrac{1}{2}\rangle$$
$$|\uparrow\downarrow\rangle = |\ \text{''}\quad \text{''},\quad \text{''}\quad m_{s_2}=-\tfrac{1}{2}\rangle$$
$$|\downarrow\uparrow\rangle = |s_1=\tfrac{1}{2},\ m_{s_1}=-\tfrac{1}{2},\ s_2=\tfrac{1}{2},\ m_{s_2}=+\tfrac{1}{2}\rangle$$
$$|\downarrow\downarrow\rangle = |\ \text{''}\quad \text{''}\quad \text{''}\quad m_{s_2}=-\tfrac{1}{2}\rangle$$

Since $S_z = S_z^{(1)} + S_z^{(2)}$ acts additively on these states, they are also eigenstates of S_z

$$S_z|s_1, m_{s_1}, s_2 m_{s_2}\rangle = \left(S_z^{(1)} + S_z^{(2)}\right)|\cdots\rangle$$
$$= \Big(\underbrace{S_z^{(1)}|s_1 m_{s_1}\rangle}_{=m_{s_1}\hbar|s_1 m_{s_1}\rangle}\Big)\otimes|s_2 m_{s_2}\rangle + |s_1 m_{s_1}\rangle\otimes\Big(\underbrace{S_z^{(2)}|s_2 m_{s_2}\rangle}_{=m_{s_2}\hbar|s_2, m_{s_2}\rangle}\Big)$$
$$= (m_{s_1}+m_{s_2})\hbar\,|s_1 m_{s_1} s_2 m_{s_2}\rangle$$

Now since each $\vec{S}^{(i)}$ obeys the $\angle$ mom. algebra so does $\vec{S}$

So we expect $\vec{S}^2$ to have ev $s(s+1)\hbar^2$; $s=0,\tfrac{1}{2},1,\ldots$
S_z " " $m\hbar$; $m=-s,\ldots,+s$

So we find
$$m = m_{s_1} + m_{s_2}$$

has 4 values

$|\uparrow\uparrow\rangle \quad m = \frac{1}{2} + \frac{1}{2} = 1$

$|\uparrow\downarrow\rangle \quad m = \frac{1}{2} - \frac{1}{2} = 0$

$|\downarrow\uparrow\rangle \quad m = -\frac{1}{2} + \frac{1}{2} = 0$

$|\downarrow\downarrow\rangle \quad m = -\frac{1}{2} - \frac{1}{2} = -1$

Since $m = -1, 0, 0, +1 \Rightarrow$

$s = 0, 1$. Note: Two $m=0$ states!

In particular the $m=+1$ state has $s=+1$, we can apply the lowering operator to it in order to obtain which combo of $|\uparrow\downarrow\rangle$ $|\downarrow\uparrow\rangle$ belongs to the spin 1 multiplet

$S_- = S_-^{(1)} + S_-^{(2)}$ So recall $S_-^{(i)}|\uparrow\rangle = \hbar\sqrt{s(s+1) - m(m-1)}\,|\downarrow\rangle = \hbar\sqrt{\frac{3}{4}+\frac{1}{4}}\,|\downarrow\rangle = \hbar|\downarrow\rangle$

$S_-|\uparrow\uparrow\rangle = (S_-^{(1)}|\uparrow\rangle) \otimes |\uparrow\rangle$

$+ |\uparrow\rangle \otimes (S_-^{(2)}|\uparrow\rangle)$

$= \hbar\left[|\downarrow\rangle\otimes|\uparrow\rangle + |\uparrow\rangle\otimes|\downarrow\rangle\right]$ also $S_+^{(i)}|\downarrow\rangle = \hbar|\uparrow\rangle$

* i.e. $\langle 0,0|1,0\rangle = \frac{1}{2}\left[\langle\uparrow\downarrow - \downarrow\uparrow|\uparrow\downarrow + \downarrow\uparrow\rangle\right]$

$= \frac{1}{2}\left[\langle\uparrow\downarrow|\uparrow\downarrow\rangle - \langle\downarrow\uparrow|\downarrow\uparrow\rangle\right] = 0$

So

$$S_-|\uparrow\uparrow\rangle = \hbar\left[|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right]$$

Normalizing this state to one, we have

$s=1$ <u>triplet</u>

$$\begin{cases} |s=1, m=1\rangle = |\uparrow\uparrow\rangle \\ |s=1, m=0\rangle = \frac{1}{\sqrt{2}}\left[|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right] \\ |s=1, m=-1\rangle = |\downarrow\downarrow\rangle \end{cases}$$

Note $S_-|s=1, m=0\rangle = \frac{\hbar}{\sqrt{2}}\left[|\downarrow\downarrow\rangle + |\downarrow\downarrow\rangle\right]$

$= \hbar\sqrt{2}\,|\downarrow\downarrow\rangle$

Normalize to one $\Rightarrow$

$|s=1, m=-1\rangle = |\downarrow\downarrow\rangle$. ✓

The state orthogonal to these triplet states is the $s=0$, $m=0$ <u>singlet</u>

$s=0$ <u>singlet</u> $\quad |s=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

* see above — Note that $\langle s=0, m=0|s=1, m\rangle = 0$

Now it was deduced that $s=0,1$ since $m=-s,\ldots,+s$ and $S_z = S_z^{(1)} + S_z^{(2)}$

This can be proven directly.

So we desire $\vec{S}^2$

$$\vec{S}^2 = (\vec{S}^{(1)} + \vec{S}^{(2)}) \cdot (\vec{S}^{(1)} + \vec{S}^{(2)})$$

$$= \vec{S}^{(1)2} + \vec{S}^{(2)2} + 2\,\vec{S}^{(1)} \cdot \vec{S}^{(2)}$$

Now $|s,m\rangle$ are eigenstates of $\vec{S}^{(i)2}$

i.e.
$$\vec{S}^{(i)2} |s,m\rangle = \hbar^2 s_i(s_i+1) |s,m\rangle$$

$$= \tfrac{3}{4}\hbar^2 |s,m\rangle$$

since both particles are spin $\frac{1}{2}$.

But $\vec{S}^{(1)} \cdot \vec{S}^{(2)}$ must be calculated explicitly!!

Now recall $S_\pm^{(i)} = S_x^{(i)} \pm i\, S_y^{(i)}$

$$\Rightarrow \quad S_x^{(i)} = \tfrac{1}{2}\left(S_+^{(i)} + S_-^{(i)}\right)$$

$$S_y^{(i)} = \tfrac{i}{2}\left(S_-^{(i)} - S_+^{(i)}\right)$$

So
$$\vec{S}^{(1)} \cdot \vec{S}^{(2)} = S_x^{(1)} S_x^{(2)} + S_y^{(1)} S_y^{(2)} + S_z^{(1)} S_z^{(2)}$$

$$= \tfrac{1}{4}\left[\cancel{S_+^{(1)} S_+^{(2)}} + S_+^{(1)} S_-^{(2)} + S_-^{(1)} S_+^{(2)} + \cancel{S_-^{(1)} S_-^{(2)}}\right]$$

$$+ S_z^{(1)} S_z^{(2)} \quad - \tfrac{1}{4}\left[\cancel{+S_-^{(1)} S_-^{(2)}} - S_-^{(1)} S_+^{(2)} - S_+^{(1)} S_-^{(2)} + \cancel{S_+^{(1)} S_+^{(2)}}\right]$$

$$\boxed{\vec{S}^{(1)}\cdot\vec{S}^{(2)} = S_z^{(1)}S_z^{(2)} + \tfrac{1}{2}\left[S_+^{(1)}S_-^{(2)} + S_-^{(1)}S_+^{(2)}\right]}$$

So we can determine the action of these operators on every product state

$$\vec{S}^{(1)}\cdot\vec{S}^{(2)}\,|\uparrow\uparrow\rangle = \hbar^2\Big[\underbrace{\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)}_{|\uparrow\uparrow\rangle} + \cancel{\tfrac{1}{2}(0+0)}\Big]$$

$$= \tfrac{1}{4}\hbar^2\,|\uparrow\uparrow\rangle$$

$$\vec{S}^{(1)}\cdot\vec{S}^{(2)}\,|\downarrow\downarrow\rangle = \tfrac{1}{4}\hbar^2\,|\downarrow\downarrow\rangle$$

$$\vec{S}^{(1)}\cdot\vec{S}^{(2)}\,|\uparrow\downarrow\rangle = -\tfrac{1}{4}\hbar^2\,|\uparrow\downarrow\rangle + \tfrac{1}{2}\hbar^2\,|\downarrow\uparrow\rangle$$

$$\vec{S}^{(1)}\cdot\vec{S}^{(2)}\,|\downarrow\uparrow\rangle = -\tfrac{1}{4}\hbar^2\,|\downarrow\uparrow\rangle + \tfrac{1}{2}\hbar^2\,|\uparrow\downarrow\rangle$$

Now we must bring all this together

$$\vec{S}^2 = \vec{S}^{(1)2} + \vec{S}^{(2)2} + 2\,\vec{S}^{(1)}\cdot\vec{S}^{(2)}$$
$$= \vec{S}^{(1)2} + \vec{S}^{(2)2} + 2 S_z^{(1)} S_z^{(2)} + \left[S_+^{(1)} S_-^{(2)} + S_-^{(1)} S_+^{(2)}\right]$$

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1) $\vec{S}^2 |\overset{S}{1}, \overset{M}{1}\rangle = \vec{S}^2 |\uparrow\uparrow\rangle$

$$= \left[\tfrac{3}{4}\hbar^2 + \tfrac{3}{4}\hbar^2 + 2\tfrac{1}{4}\hbar^2\right] |\uparrow\uparrow\rangle$$

$$= 2\hbar^2 |1,1\rangle = \hbar^2 (1+1)1\, |1,1\rangle$$
$$= \hbar^2 (s+1)s\, |1,1\rangle$$

$\Rightarrow \boxed{s = 1}$ ✓

2) $\vec{S}^2 |s=1, m=-1\rangle = \vec{S}^2 |\downarrow\downarrow\rangle$

$$= \left[\tfrac{3}{2}\hbar^2 + 2\tfrac{1}{4}\hbar^2\right] |\downarrow\downarrow\rangle$$

$$= 2\hbar^2 |1,-1\rangle = \hbar^2\, 1(1+1)\, |1,-1\rangle$$
$$= \hbar^2\, s(s+1)\, |1,-1\rangle$$

$\Rightarrow \boxed{s=1}$

3) $\vec{S}^2 |s=1, m=0\rangle = \frac{1}{\sqrt{2}}\left[\vec{S}^2 |\downarrow\uparrow\rangle + \vec{S}^2 |\uparrow\downarrow\rangle\right]$

$$= \frac{1}{\sqrt{2}}\Big\{\tfrac{3}{2}\hbar^2 |\downarrow\uparrow\rangle + 2\left(-\tfrac{1}{4}\hbar^2 |\downarrow\uparrow\rangle + \tfrac{1}{2}\hbar^2 |\uparrow\downarrow\rangle\right)$$
$$+ \tfrac{3}{2}\hbar^2 |\uparrow\downarrow\rangle + 2\left(-\tfrac{1}{4}\hbar^2 |\uparrow\downarrow\rangle + \tfrac{1}{2}\hbar^2 |\downarrow\uparrow\rangle\right)\Big\}$$

$$= \frac{1}{\sqrt{2}}\left\{\hbar^2\left(\tfrac{3}{2} - \tfrac{1}{2} + 1\right)|\downarrow\uparrow\rangle + \hbar^2\left(\tfrac{3}{2} - \tfrac{1}{2} + 1\right)|\uparrow\downarrow\rangle\right\}$$

$$= 2\hbar^2 \frac{1}{\sqrt{2}}\left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right)$$

$$= 2\hbar^2 |s=1, m=0\rangle \Rightarrow \boxed{s=1}\ !$$

So the triplet states indeed have $\underline{s=1}$

4) Finally for the singlet state

$$\vec{S}^2 |s=0, m=0\rangle = \frac{1}{\sqrt{2}} \left[\vec{S}^2 |\uparrow\downarrow\rangle - \vec{S}^2 |\downarrow\uparrow\rangle \right]$$

$$= \frac{1}{\sqrt{2}} \hbar^2 \left\{ \frac{3}{2} |\uparrow\downarrow\rangle + 2\left(-\frac{1}{4} |\uparrow\downarrow\rangle + \frac{1}{2} |\downarrow\uparrow\rangle\right) \right.$$
$$\left. - \frac{3}{2} |\downarrow\uparrow\rangle - 2\left(-\frac{1}{4} |\downarrow\uparrow\rangle + \frac{1}{2} |\uparrow\downarrow\rangle\right) \right\}$$

$$= \frac{1}{\sqrt{2}} \hbar^2 \left\{ \left(\cancel{\frac{3}{2} - \frac{1}{2}} - 1\right) |\uparrow\downarrow\rangle - \left(\cancel{\frac{3}{2} - 1} - \frac{1}{2}\right) |\downarrow\uparrow\rangle \right\}$$

$$= 0 = \hbar^2 s(s+1) |s=0, m=0\rangle$$

$$\Rightarrow \boxed{s=0} \;.\;! \text{ as expected.}$$

So we have added the spin ang. mom. of 2 particles and obtained the sum of two total spin multiplets the triplet & singlet. We have changed the basis from the product of individual spin states which diagonalizes CSCO $\{S^{(1)2}, S_z^{(1)}, S^{(2)2}, S_z^{(2)}\}$ to.

The total spin basis which diagonalizes the CSCO $\{\vec{S}^2, S_z, \vec{S}^{(1)2}, \vec{S}^{(2)2}\}$

decompose product of irred. reps of rotation group
into sum of " " " " ".

Symbolically we have the product of 2 spin $\frac{1}{2}$ Hilbert spaces equals the sum of a spin 1 Hilbert space and a spin 0 Hilbert space

$$\tfrac{1}{2} \otimes \tfrac{1}{2} = 1 \oplus 0 .$$

i.e. direct product of 2 spin $\frac{1}{2}$ Hilbert spaces
= direct sum of spin 1 & spin 0 Hil. space

Dimensions $\quad \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$

$$\underbrace{(2s_1+1) \cdot (2s_2+1)}_{= 2 \cdot 2 = 4} \qquad \underbrace{\underset{1}{\underset{(s=0)}{2s+1}} + \underset{3}{\underset{(s=1)}{2s+1}}}_{= 4} \checkmark$$

The general theorem for combining spin s_1 with spin s_2 (or j_1 & j_2) to get total spins S is

$$S = (s_1+s_2),\ (s_1+s_2-1),\ \dots,\ |s_1-s_2|$$

i.e. spins align $\qquad\qquad$ spins anti-parallel

ex. Hydrogen e^- spin with ℓ num $\to$ $j = \begin{matrix} \ell+\frac{1}{2} \\ \ell-\frac{1}{2} \end{matrix}$

if also include proton spin $j = \begin{cases} l+1 \\ l \leftarrow \text{2 states} \\ l-1 \end{cases}$

Thus $$s_1 \otimes s_2 = \bigoplus_{s=|s_1-s_2|}^{s_1+s_2} s$$

i.e. $$\tfrac{3}{2} \otimes 2 = \bigoplus_{s=\frac{1}{2}}^{7/2} s = \tfrac{1}{2} \oplus \tfrac{3}{2} \oplus \tfrac{5}{2} \oplus \tfrac{7}{2}$$

dimensions $4 \times 5 = 2+4+6+8 = 20$ ✓

From the basis vector pt. of view, the change of basis is simply

$$|s, m\rangle = \sum_{m=m_1+m_2} C^{s_1 s_2 s}_{m_1 m_2 m} \, |s_1 m_1\rangle |s_2 m_2\rangle$$

$C^{s_1 s_2 s}_{m_1 m_2 m} = \langle s, m | s_1 s_2 m_1 m_2 \rangle$

Since $S_z = S_z^{(1)} + S_z^{(2)}$ we only sum over m_1 & $m_2 \ni m = m_1 + m_2$ for given s, s_1, s_2.

The $C^{s_1 s_2 s}_{m_1 m_2 m}$ are called

Clebsch-Gordon coefficients.

There are tables of them.

As with any basis change — we can invert the transformation to find

$$|s_1 m_1\rangle |s_2 m_2\rangle = \sum_{s \atop (m_1+m_2=m)} C^{s_1 s_2 s}_{m_1 m_2 m} |s, m\rangle .$$

The same C-G coefficients.