Since any basis is as good a basis as any other, we can imagine expanding the $|k, j, m\rangle$ states in terms of the $|\vec{r}, s, m_s\rangle$ states. Since $\vec{J} = \vec{L} + \vec{S}$, the relationship between the eigenstates of $\vec{J}^2, J_z$ and the orbital angular momentum and spin angular momentum that <u>add</u> up to be the total angular momentum we desire is complicated. <u>We must first determine how to add angular momentum.</u>

Spin 0: Before doing this, however, we can consider the case of spin zero particles. In this situation the total angular momentum is just the orbital angular momentum: $\vec{J} = \vec{L}$. (i.e. $\vec{J} = \vec{L} + \vec{0}$)

$pq \sim \hbar$ $\quad [p][q] = \hbar$



When the particle has spin 0, the total ∡ momentum is just the orbital ∡ momentum. Now for orbital angular momentum

$\vec{J} = \vec{L} = \vec{R} \times \vec{P}$, we found in wavemechanics that $l$ and $m$ were integer valued only. This follows from the general properties of $L$, that is unlike spin $L = R \times P$.

(Aside: In particular $L_z = X P_y - Y P_x$

and $[X_i, P_j] = i\hbar\delta_{ij}$.

Now define 4 operators that are hermitian:

$$q_1 \equiv \frac{X + P_y \frac{a^2}{\hbar}}{\sqrt{2}}$$

$a$ = const. with dim of length

i.e. $a = \frac{4\pi\epsilon_0 \hbar^2}{me^2}$ Bohr radius

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$$q_2 \equiv \frac{X - P_y \frac{a^2}{\hbar}}{\sqrt{2}}$$

$$p_1 \equiv \left[\frac{\frac{a^2}{\hbar} P_x - Y}{\sqrt{2}}\right] \hbar/a^2$$

$$p_2 \equiv \left[\frac{\frac{a^2}{\hbar} P_x + Y}{\sqrt{2}}\right] \hbar/a^2$$

$$[q_1, p_2] = \frac{i\hbar}{\hbar 2a^2}[X, P_x] + \frac{\hbar a^2}{2\hbar}[P_y, Y]\frac{1}{a^2} = 0$$
$$[q_2, p_1] = \frac{a^2\hbar}{2\hbar a^2}[X, P_x] + \frac{\hbar a^2}{2\hbar}[P_y, Y]\frac{1}{a^2} = 0$$

Now for $i=j$

$$\boxed{[q_i, p_j] = \frac{\hbar a^2}{2\hbar a^2}[X, P_x] - \frac{\hbar a^2}{2\hbar a^2}[P_y, Y] = i\hbar\delta_{ij}} = i\hbar\delta_{ij}$$

Further $[q_1, q_2] = 0 = [p_1, p_2]$

$\Rightarrow$

$$\boxed{\begin{aligned}[q_i, q_j] &= 0 = [p_i, p_j] \\ [q_i, p_j] &= i\hbar\delta_{ij}\end{aligned}} \quad \text{CCR.}$$

Further

$$X = \frac{1}{\sqrt{2}}(q_1 + q_2)$$
$$Y = \frac{1}{\sqrt{2}}\frac{a^2}{\hbar}(p_2 - p_1)$$
$$P_x = \frac{1}{\sqrt{2}}(p_1 + p_2)$$
$$a^2 P_y = \frac{\hbar}{\sqrt{2}}(q_1 - q_2)$$

$$L_z = X P_y - Y P_x$$
$$= \frac{\hbar}{a^2 2}[q_1^2 - q_2^2]$$
$$- \frac{a^2}{2\hbar}[p_2^2 - p_1^2]$$

let $a=1$

$$\boxed{L_z = \left(\frac{a^2}{2\hbar}p_1^2 + \frac{1}{2}\frac{\hbar}{a^2}q_1^2\right) - \left(\frac{a^2}{2\hbar}p_2^2 + \frac{1}{2}\frac{\hbar}{a^2}q_2^2\right)}$$

This is just Hamiltonian for 2 independent 1-d SHO with $m = \hbar/a^2$, $\omega = 1$

$$a^2/\hbar = \frac{1}{m},$$

Hence their individual ev. are

$$L_z = h_1 - h_2 \;;\quad h_1 \to E_1 = \hbar(n_1 + \tfrac{1}{2})$$
$$h_2 \to E_2 = \hbar(n_2 + \tfrac{1}{2})$$
$$n_{1,2} = 0, 1, 2, \ldots$$

$\Rightarrow$

$$L_z \to m\hbar = E_1 - E_2$$
$$= \hbar(n_1 + \tfrac{1}{2}) - \hbar(n_2 + \tfrac{1}{2})$$
$$= \hbar(n_1 - n_2)$$

$\Rightarrow$

$m = n_1 - n_2 = 0, \pm 1, \pm 2, \ldots \Rightarrow$ <u>integer</u>

(<u>not</u> $\frac{1}{2}$ integer!)

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Indeed this is what we found explicitly in the $\{|\vec{r}\rangle\}$ basis

$$\langle \vec{r}|\vec{J}|\psi\rangle = \langle \vec{r}|\vec{L}|\psi\rangle = \langle \vec{r}|\vec{R}\times\vec{P}|\psi\rangle$$
$$= (\vec{r} \times \tfrac{\hbar}{i}\vec{\nabla}_{\vec{r}})\psi(\vec{r})$$

That is $\langle\vec{r}|\vec{J} = \langle\vec{r}|\vec{L} = (\vec{r}\times\frac{\hbar}{i}\vec{\nabla}_{\vec{r}})\langle\vec{r}|$ for the spin 0 subspace.

In spherical polar coordinates we have as found in wavemechanics

$$\langle\vec{r}|L_x = \frac{\hbar}{i}\left[-\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right]\langle r,\theta,\varphi|$$

$$\langle\vec{r}|L_y = \frac{\hbar}{i}\left[\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right]\langle r,\theta,\varphi|$$

$$\langle\vec{r}|L_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}\langle r,\theta,\varphi|$$

Thus

$$\langle\vec{r}|L_\pm = \hbar e^{\pm i\varphi}\left[\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right]\langle r,\theta,\varphi|$$

and

$$\langle\vec{r}|\vec{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\langle r,\theta,\varphi|$$

Since these operators are independent of the radius r, they act only on the spherical angle variables we can introduce a spherical angle basis $|\theta,\varphi\rangle$, The action of $\vec{L}$ in this basis is given above. ($\{|\theta,\varphi\rangle\}$ is not complete, we need also the radial dependence of the wavefunction, but since it is not relevant for the following discussion we surpress it, as the k in the $|k,j,m\rangle$ basis).

Now since $\vec{J}=\vec{L}$ we know that the eigen states $|l,m\rangle$ of $\vec{L}^2$ and $L_z$ are given by

$$\vec{L}^2|l,m\rangle = l(l+1)\hbar^2 |l,m\rangle$$

$$L_z|l,m\rangle = m\hbar |l,m\rangle$$

with $l=0,1,2,\ldots$ and $m=-l,-l+1,\ldots,l-1,l$.

Hence the $|l,m\rangle$ wavefunctions obey the angular differential equation

$$\langle\theta,\varphi|\vec{L}^2|l,m\rangle = l(l+1)\hbar^2\langle\theta,\varphi|l,m\rangle$$

$$= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\langle\theta,\varphi|l,m\rangle$$

and

$$\langle\theta,\varphi|L_z|l,m\rangle = m\hbar\langle\theta,\varphi|lm\rangle$$

$$= \frac{\hbar}{i}\frac{\partial}{\partial\varphi}\langle\theta,\varphi|l,m\rangle .$$

The solutions to these differential equations, as we found in wave mechanics, are just the spherical harmonics.

$$\langle\theta,\varphi|l,m\rangle = Y_l^m(\theta,\varphi)$$ with

$l = 0, 1, 2, \ldots$ and $m = -l, -l+1, \ldots, l-1, l.$

Since $\langle l,m|l,m\rangle = 1$ we have

$$\int_{\varphi=0}^{2\pi} d\varphi \int_{\theta=0}^{\pi} \sin\theta\, d\theta\, |Y_l^m(\theta,\varphi)|^2 = 1.$$

Hence we have transformed from one basis $|l,m\rangle$ to another $|\theta,\varphi\rangle$ by means of the spherical harmonics

$$Y_l^m(\theta,\varphi) = \langle\theta,\varphi|l,m\rangle.$$

Finally any wavefunction for a spin 0 particle has the expansion in terms of the <u>standard basis wavefunctions</u>

$$\psi_{k,l,m}(\vec{r}) = \langle\vec{r}|k,l,m\rangle$$
$$= R_{k,l,m}(r)\, Y_l^m(\theta,\varphi)$$

Since $\vec{L}^2$ and $L_z$ only act on the angular variables, $R_{klm}(r)$ acts as a constant factor in the $\langle\vec r|\vec L^2|k,l,m\rangle = l(l+1)\hbar^2/r^2 \langle\vec r|k,l,m\rangle$ and $\langle\vec r|L_z|k,l,m\rangle = m\hbar\langle\vec r|k,l,m\rangle$ differential equations.

Using $L_\pm$ we find $R_{k,l,m\pm1}(r) = R_{k,l,m}(r)$ hence $R_{k,l,m}(r) = R_{kl}(r)$ is independent of $m$. So

$$\psi_{k,l,m}(\vec r) = R_{kl}(r)\, Y_l^m(\theta,\varphi)$$

form a complete set of spin 0 wave functions which are orthonormal (by convention)

$$\int d^3r\, \psi^*_{k,l,m}(\vec r)\,\psi_{k',l',m'}(\vec r)$$
$$= \int_0^\infty dr\, r^2 R^*_{kl}(r)\, R_{k'l'}(r) \times$$
$$\times \int_{4\pi} d\Omega\, Y_l^{m*}(\theta,m)\, Y_{l'}^{m'}(\theta,\varphi)$$
$$= \langle k,l,m|k',l',m'\rangle = \delta_{kk'}\delta_{ll'}\delta_{mm'}$$

$\Rightarrow R_{kl}$ must be normalized to

$$\int_0^\infty dr\, r^2 R^*_{kl}(r)\, R_{k'l}(r) = \delta_{kk'}.$$

Indeed we applied precisely these techniques to the 3-d

to find explicitly

The {energy, (∡ mom.)$^2$, (∡ mom.)$_z$} eigenfunctions!

i.e. $\{k, \ell, m\} \to \{n, l, m\}$

$E_n = \hbar\omega(n + \frac{3}{2})$

Thus we have a basis for any central potential problem

Spin: Besides our well known spin 0 results — we now have new types of particles with the intrinsic property of spin.

In particular we have spin $\frac{1}{2}$ particles $e^-$, $\mu$, etc. also bound states p, n, atoms → which at lower energy behave as particles with intrinsic spin.