We would now like to find all possible matrices $\vec{S}_{ij}$ and hence $D^{(s)}(R)$. We explicitly constructed the matrices for spin $0, \frac{1}{2}, 1$. To determine the general matrix structure for $\vec{S}$ we can turn to consideration of the eigenvalue determination of a set of commuting observables made from the $\vec{J}$ and whatever other operators may commute with them.

Angular Momentum Commutation Relations and the "Standard Basis"

The angular momentum commutation relations are

$$[J_i, J_j] = i\hbar\epsilon_{ijk} J_k .$$

Hence $\vec{J}^2 = \vec{J}\cdot\vec{J}$ commutes with J_i since it is a scalar (the dot product of 2 vector operators) operator

$$[\vec{J}^2, J_i] = 0 \quad \text{for } i = 1,2,3 .$$

Hence $\vec{J}^2$ & one component of $\vec{J}$ have simultaneous eigenvectors. Choose $\{\vec{J}^2, J_z\}$. Usually there are other operators that commute with $\vec{J}^2$ & J_z. Let's include them also in our set of operators that have simultaneous eigenvectors. Let's denote all of the other commuting operators A and assume $\{A, \vec{J}^2, J_z\}$ is a CSCO. (ex. $\{H, \vec{L}^2, L_z\}$ in the H-atom case)

Let the simultaneous eigenvectors be denoted

$|k, \lambda, m\rangle$ with

$$A|k,\lambda,m\rangle = k|k,\lambda,m\rangle$$

$$\vec{J}^2|k,\lambda,m\rangle = \lambda\hbar^2|k,\lambda,m\rangle$$

$$J_z|k,\lambda,m\rangle = m\hbar|k,\lambda,m\rangle.$$

Since $A, \vec{J}^2, J_z$ are Hermitian $\Rightarrow$

$$\boxed{k, \lambda, m \text{ are real.}}$$

1) Orthonormal eigenvectors

$$\langle k', \lambda', m' | k, \lambda, m \rangle = \delta_{k'k}\, \delta_{\lambda'\lambda}\, \delta_{m'm}.$$

2) Completeness (CSCO)

$$\underline{\mathbb{1}} = \sum_{k,\lambda,m}\!\!\!\!\!\!\int\; |k,\lambda,m\rangle\langle k,\lambda,m| .$$

Now construct Ladder Operators:

$$\boxed{J_\pm \equiv J_x \pm i J_y \qquad (J_0 \equiv J_z)}$$

invert

$$J_x = \tfrac{1}{2}(J_- + J_+) \qquad J_z = J_0$$
$$J_y = \tfrac{i}{2}(J_- - J_+)$$

$$J_i^\dagger = J_i \iff J_+^\dagger = J_- \,,\; J_0^\dagger = J_0$$

Also using the ∡ momentum [(SU(2)) Algebra]
Commutation relations

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$$

$$\Longleftrightarrow \quad [J_0, J_+] = \hbar J_+ \qquad [J_+, J_-] = 2\hbar J_0$$

$$[J_0, J_-] = -\hbar J_-$$

We can write $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$

$$= \tfrac{1}{2}\left(J_+J_- + J_-J_+\right) + J_0^2$$

So since $[\vec{J}^2, J_i] = 0 \Longleftrightarrow [\vec{J}^2, J_{\substack{+\\-\\0}}] = 0$

we can only "diagonalize" $\vec{J}^2$ and one of the component operators $J_{\substack{+\\-\\0}}$, $\boxed{\text{Choose } J_0.}$

So we need to consider the products J_+J_- and J_-J_+ since they appear in $\vec{J}^2$ and $J_+^\dagger = J_-$ etc.

$$J_+J_- = (J_x + iJ_y)(J_x - iJ_y)$$

$$= J_x^2 + J_y^2 - i(J_xJ_y - J_yJ_x)$$

$$= J_x^2 + J_y^2 - i[J_x, J_y]$$

$$= J_x^2 + J_y^2 + \hbar J_0$$

Likewise

$$J_-J_+ = J_x^2 + J_y^2 - \hbar J_0$$

So
$$\boxed{\begin{aligned} J_+J_- &= \vec{J}^2 - J_0^2 + \hbar J_0 \\ J_-J_+ &= \vec{J}^2 - J_0^2 - \hbar J_0 \end{aligned}}$$

---

1) $\vec{J}^2$ is a non-negative operator, for any state

$$\langle\psi|\vec{J}^2|\psi\rangle = \langle\psi|J_x^2|\psi\rangle + \langle\psi|J_y^2|\psi\rangle + \langle\psi|J_z^2|\psi\rangle$$

$$= \langle\psi|J_x^\dagger J_x|\psi\rangle + \langle\psi|J_y^\dagger J_y|\psi\rangle + \langle\psi|J_z^\dagger J_z|\psi\rangle$$

$$= \|J_x|\psi\rangle\|^2 + \|J_y|\psi\rangle\|^2 + \|J_z|\psi\rangle\|^2$$

$$\geq 0.$$

So if $|\psi\rangle$ is an eigenvector of $\vec{J}^2$:

$$\vec{J}^2|\psi\rangle = \lambda\hbar^2|\psi\rangle \quad \text{then}$$

$$\langle\psi|\vec{J}^2|\psi\rangle = \lambda\hbar^2\,\||\psi\rangle\|^2 \geq 0$$

Since $\||\psi\rangle\|^2 > 0 \Rightarrow \boxed{\lambda \geq 0}$

(Like SHO $a^\dagger a$ was non-negative operator)

By convention we write $\lambda \equiv j(j+1)$ with $j \geq 0$.
Since both $\lambda, j \geq 0$ the eq.

$\lambda = j(j+1)$ has only one positive or zero root

$$j = -\frac{1}{2} + \frac{1}{2}\sqrt{1+4\lambda}.$$

Thus, given j we know λ; given λ we know j, uniquely.

Similarly $J_z = J_z^\dagger$ thus $m \in \mathbb{R}$.

So the ev equations are

$$\vec{J}^2 |k,j,m\rangle = j(j+1)\hbar^2 |k,j,m\rangle$$

$$J_z |k,j,m\rangle = m\hbar \, |k,j,m\rangle$$

with $j \geq 0$, $m \in \mathbb{R}$.

Lemma 1: Properties of $\vec{J}^2$ & J_z eigenvalues

With j, m defined above and associated with the same eigenket $|k,j,m\rangle$, then

$$-j \leq m \leq +j.$$

Proof: Since $J_+^\dagger = J_-$ we have

$$\| J_+ |k,j,m\rangle \|^2 = \langle k,j,m | J_- J_+ | k,j,m\rangle \geq 0$$

and

$$\| J_- |k,j,m\rangle \|^2 = \langle k,j,m | J_+ J_- | k,j,m\rangle \geq 0.$$

Since
$$J_+ J_- = \vec{J}^2 - J_z^2 + \hbar J_z$$
$$J_- J_+ = \vec{J}^2 - J_z^2 - \hbar J_z$$
we can evaluate the norms

1) $\langle k,j,m | J_- J_+ | k,j,m\rangle = \langle k,j,m | \vec{J}^2 - J_z^2 - \hbar J_z | k,j,m\rangle$
$$= [j(j+1) - m^2 - m]\hbar^2 \geq 0$$

2) $\langle k,j,m | J_+ J_- | k,j,m\rangle = \langle k,j,m | \vec{J}^2 - J_z^2 + \hbar J_z | k,j,m\rangle$
$$= [j(j+1) - m^2 + m]\hbar^2 \geq 0$$

Thus equation 1) yields

1') $j(j+1) - m(m+1) = (j-m)(j+m+1) \geq 0$

while equation 2) yields

2') $j(j+1) - m(m-1) = (j-m+1)(j+m) \geq 0.$

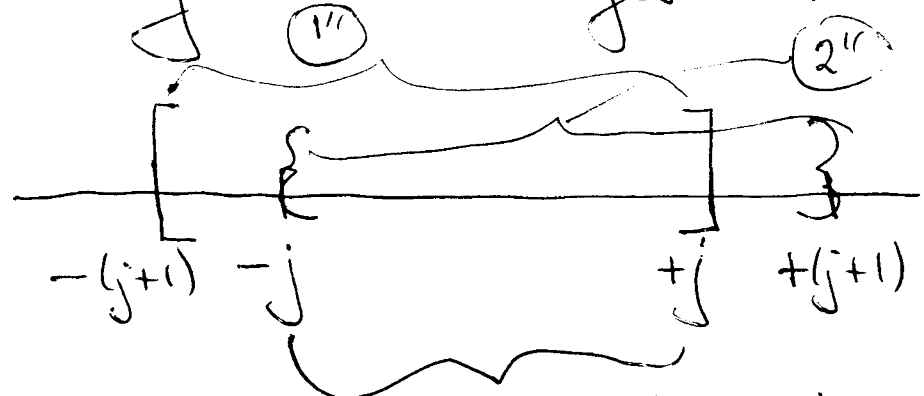
In order for $(j-m)(j+m+1) \geq 0$ we must have that

$$j \geq m \quad \text{and} \quad m \geq -(j+1)$$

That is 1'') $\boxed{-(j+1) \leq m \leq j}$. We also have that 2') implies that $m \leq j+1$ and $m \geq -j$

That is 2'') $\boxed{-j \leq m \leq j+1}$

Pictorially those ranges are



The overlap of the 1'') and 2'') regions imply

$$-j \leq m \leq +j \quad , \text{as required.}$$

Lemma 2: Properties of $J_-|k,j,m\rangle$.

Let $|k,j,m\rangle$ be an eigenvector of $\vec{J}^2$ and J_z with eigenvalues $j(j+1)\hbar^2$ and $m\hbar$, respectively.

a.) Iff $m=-j$, then $J_-|k,j,m\rangle = 0$

b.) If $m > -j$, then $J_-|k,j,m\rangle$ is

a non-zero eigenvector of $\vec{J}^2$ and J_z with the eigenvalues $j(j+1)\hbar^2$ and $(m-1)\hbar$, respectively.

Proof: a) From above

$$\|J_-|k,j,m\rangle\|^2 = [j(j+1) - m(m-1)]\hbar^2,$$

hence if $m=-j$, the RHS $= 0 \Rightarrow$

$$\|J_-|k,j,-j\rangle\|^2 = 0 \Rightarrow$$

$$J_-|k,j,-j\rangle = 0, \text{ the zero vector.}$$

Further, if $J_-|k,j,m\rangle = 0 \Rightarrow m=-j$ since we can ~~operate~~ operate with J_+ ~~to~~ find

$$J_+ J_- |k,j,m\rangle = 0$$
$$= \hbar^2 [j(j+1) - m^2 + m] |k,j,m\rangle$$
$$= \hbar^2 (j+m)(j-m+1) |k,j,m\rangle = 0$$

$\Rightarrow (j+m)(j-m+1) = 0$; But $-j \le m \le j$,

so there is only one solution $\Rightarrow \boxed{m = -j.}$

b) if $m > -j$, then from above

$$\| J_- |k,j,m\rangle \|^2 = [j(j+1) - m(m-1)]\hbar^2 \neq 0.$$

Thus $J_- |k,j,m\rangle$ is <u>not</u> the zero vector.

Using the angular momentum algebra we ~~have~~ that

$$[\vec{J}^2, J_-] = 0 \quad ; \text{ thus}$$

$$[\vec{J}^2, J_-] |k,j,m\rangle = 0$$

$$\Rightarrow \quad \vec{J}^2 (J_- |k,j,m\rangle) = J_- (\underbrace{\vec{J}^2 |k,j,m\rangle}_{= j(j+1)\hbar^2 |k,j,m\rangle})$$

$$= j(j+1)\hbar^2 (J_- |k,j,m\rangle)$$

Hence $J_-|k,j,m\rangle$ is an eigenvector of $\vec{J}^2$ with eigenvalue $j(j+1)\hbar^2$.

Now the angular momentum algebra also has

$$[J_z, J_-] = -\hbar J_- \quad ; \text{ so}$$

$$J_z\left(J_-|k,j,m\rangle\right) = J_-\underbrace{\left(J_z|k,j,m\rangle\right)}_{=m\hbar|k,j,m\rangle} - \hbar J_-|k,j,m\rangle$$

$$= (m-1)\hbar\left(J_-|k,j,m\rangle\right).$$

Thus $J_-|k,j,m\rangle$ is also an eigenvector of J_z with the eigenvalue $(m-1)\hbar$.

<u>Lemma 3</u>: Properties of $J_+|k,j,m\rangle$.

Let $|k,j,m\rangle$ be an eigenvector of $\vec{J}^2$ & J_z with eigenvalues $j(j+1)\hbar^2$ and $m\hbar$, respectively.

a.) Iff $m=j$, then $J_+|k,j,m\rangle = 0$.

b.) If $m<j$, then $J_+|k,j,m\rangle$ is a non-zero eigenvector of $\vec{J}^2$ and J_z with the eigenvalues $j(j+1)\hbar^2$ and $(m+1)\hbar$, respectively.

Proof: a) As in Lemma 2,

$$\| J_+ |k,j,m\rangle \|^2 = \hbar^2 \left(j(j+1) - m(m+1) \right),$$

so $m = +j \Rightarrow \| J_+ |k,j,j\rangle \|^2 = 0 \Rightarrow \boxed{J_+ |k,j,j\rangle = 0}$

Conversely if $J_+ |k,j,m\rangle = 0$, then act again with J_- to find

$$J_- J_+ |k,j,m\rangle = \hbar^2 [j(j+1) - m^2 - m] |k,j,m\rangle$$

$$= \hbar^2 [(j-m)(j+m+1)] |k,j,m\rangle = 0.$$

But $-j \le m \le j$, hence the only solution to $(j-m)(j+m+1) = 0 \Rightarrow m = j$.

b) If $m < j$ we have from above that

$$\| J_+ |k,j,m\rangle \|^2 = \hbar^2 [j(j+1) - m(m+1)] \neq 0,$$

hence $J_+ |k,j,m\rangle$ is <u>not</u> the zero vector.

As in Lemma 2, the angular momentum algebra implies $[\vec{J}^2, J_+] = 0$ so that

$$\vec{J}^2\left(J_+|k,j,m\rangle\right) = J_+\left(\vec{J}^2|k,j,m\rangle\right)$$
$$= j(j+1)\hbar^2\left(J_+|k,j,m\rangle\right)$$

and using the algebra again, $[J_z, J_+] = +\hbar J_+$ yields

$$J_z\left(J_+|k,j,m\rangle\right) = J_+\left(J_z|k,j,m\rangle\right) + \hbar\left(J_+|k,j,m\rangle\right)$$
$$= (m+1)\hbar\left(J_+|k,j,m\rangle\right).$$

Hence $J_+|k,j,m\rangle$ has $\vec{J}^2$, J_z eigenvalues $\hbar^2 j(j+1)$ and $(m+1)\hbar$, respectively.

We are now in a position to determine the spectrum of $\{\vec{J}^2, J_z\}$ eigenvalues. (The analysis is similar to that used in the SHO case.)

Suppose we have an (non-null) eigenvector $|k,j,m\rangle \neq 0$ with

$$\vec{J}^2|k,j,m\rangle = j(j+1)\hbar^2|k,j,m\rangle$$
$$J_z|k,j,m\rangle = m\hbar|k,j,m\rangle \text{ and}$$

by Lemma 1, $-j \le m \le +j$.

Consider the set of vectors

$$J_+|k,j,m\rangle, J_+^2|k,j,m\rangle, \ldots, J_+^p|k,j,m\rangle, \ldots.$$

Since $-j \le m \le j$, if $m = j$, $J_+|k,j,j\rangle = 0$ by Lemma 3. If $m < j$, $J_+|j,m\rangle$ is non-zero and has $\{\vec{J}^2, J_z\}$ eigenvalues $\{j(j+1)\hbar^2, (m+1)\hbar\}$, in particular $m+1 \le j$. If $m+1 = j$, then $J_+^2|k,j,m\rangle = 0$ by Lemma 3. If $m+1 < j$, $J_+^2|k,j,m\rangle$ is a non-zero eigenvector with $\{\vec{J}^2, J_z\}$ eigenvalues $\{j(j+1)\hbar^2, (m+2)\hbar\}$. This process can be continued. However, the series must terminate at some point or else we will make non-zero eigenvectors with J_z eigenvalue $> j$, in contradiction with Lemma 1. Therefore there exists an integer $p \ge 0$ such that $J_+^p|k,j,m\rangle$ is a non-zero eigenvector of $\{\vec{J}^2, J_z\}$ with eigenvalues $\{j(j+1)\hbar^2, (m+p)\hbar\}$ such that $J_+(J_+^p|k,j,m\rangle) = 0$

that is $J_+^{p+1}|k,j,m\rangle = 0$ is the zero vector.

Thus by Lemma 3, $m+p = j$. Hence

we have that $j-m=p$ is an integer ≥ 0. Further the p-vectors

$$J_+|k,j,m\rangle, J_+^2|k,j,m\rangle, \ldots, J_+^p|k,j,m\rangle$$

are non-zero eigenstates of $\vec{J}^2$ with all the same eigenvalue $j(j+1)\hbar^2$ of $\vec{J}^2$ and are also eigenstates of J_z with eigenvalues $(m+1)\hbar, (m+2)\hbar, \ldots, (m+p)\hbar = j\hbar$ of J_z, respectively.

Analogously we can consider the set of vectors

$$J_-|k,j,m\rangle, J_-^2|k,j,m\rangle, \ldots, J_-^q|k,j,m\rangle, \ldots$$

Since $-j \leq m \leq j$, if $m=-j$, $J_-|k,j,-j\rangle = 0$ by Lemma 2. If $m > -j$, $J_-|k,j,m\rangle$ is non-zero with $\{\vec{J}^2, J_z\}$ eigenvalues $\{j(j+1)\hbar^2, (m-1)\hbar\}$ with $(m-1) \geq j$. If $m-1=j$ then $J_-^2|k,j,m\rangle = 0$ by Lemma 2. If $m-1>j$, then $J_-^2|k,j,m\rangle \neq 0$ with $\{\vec{J}^2, J_z\}$ eigenvalues $\{j(j+1)\hbar^2, (m-2)\hbar\}$. And so on, as before this process must terminate or there will be non-zero eigenvectors with J_z eigenvalue $< -j$, in contradiction with Lemma 1. Therefore there exists an integer $q \geq 0$ such that $J_-^q|k,j,m\rangle$ is a non-zero eigenvector with $\{\vec{J}, J_z\}$ eigenvalue $\{j(j+1)\hbar^2, (m-q)\hbar\}$ and $J_-(J_-^q|k,j,m\rangle) = 0$.

By Lemma 2 $m-q=-j$. Thus $j+m=q\geq 0$ is a non-negative integer. Hence the vectors

$$J_-|k,j,m\rangle,\ J_-^2|k,j,m\rangle, \ldots, J_-^q|k,j,m\rangle,$$

all have $\vec{J}^2$ eigenvalue $j(j+1)\hbar^2$ and have J_z eigenvalues $(m-1)\hbar, (m-2)\hbar, \ldots, (m-q)\hbar = -j\hbar$ respectively.

Since p and q are non-negative integers, their sum is a non-negative integer also and

$$n = p+q = (j-m)+(j+m) = 2j$$

integer ≥ 0

Thus $\boxed{j = \frac{n}{2} \text{ with } n = 0, 1, 2, \ldots}$.

Hence the total angular momentum $\vec{J}^2$ eigenvalue, $j(j+1)\hbar^2$, is given by

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}, 3, \ldots$$

j can have integer as well as half-odd integer (half-integer), non-negative values. Furthermore, if $|k, j, m\rangle$ is a non-zero eigenvector of $\{\vec{J}^2, J_z\}$, then we can use $J_\pm$ to form the series of all non-zero eigenvectors of $\vec{J}^2$ with eigenvalue $j(j+1)\hbar^2$ and J_z with eigenvalues

$$-j\hbar, (-j+1)\hbar, (-j+2)\hbar, \ldots, (j-2)\hbar, (j-1)\hbar, j\hbar;$$

there are $(2j+1)$ eigenvectors of J_z with the same $\vec{J}^2$ eigenvalue $j(j+1)\hbar^2$, given by $m = -j, -j+1, \ldots, j-1, j$. Hence we ~~have~~ found the complete spectrum of $\{\vec{J}^2, J_z\}$.

<u>Theorem:</u> Let $\vec{J}$ be an operator (the angular momentum operator) obeying the commutation relations

$$[J_i, J_j] = i\hbar\epsilon_{ijk} J_k \quad .$$

If $j(j+1)\hbar^2$ and $m\hbar$ denote the eigenvalues of $\vec{J}^2$ and J_z, respectively, then

1) The only possible values for j are

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}, \ldots$$

2) For a given j, m can take on any of the $2j+1$ values only

$$-j, -j+1, -j+2, \ldots, j-2, j-1, +j.$$

If j is integral, m is integral; if j is half-integral, m is half-integral.

We next exploit the lowering, J_-, and raising, J_+, operators ~~to~~ construct the eigenvectors of $\{J^2, J_z\}$ and hence to form the "standard basis" for $\mathcal{H}$, since $\{A, \vec{J}^2, J_z\}$ are a CSCO, $\{|k, j, m\rangle\}$. In doing this we will just bring together the results we already have shown above.

Indeed we have that $J_\pm$ raises & lowers the J_z eV $\pm\hbar$. So if $J_\pm|k,j,m\rangle \neq 0$ we have

$$\| J_\pm |k,j,m\rangle \|^2 = \langle k,j,m| \underset{J_\pm^\dagger}{\underset{\parallel}{J_\mp}} J_\pm |k,j,m\rangle$$

$$= \langle k,j,m| \vec{J}^2 - J_z^2 \mp \hbar J_z |kjm\rangle$$

$$= \big(j(j+1) - m(m\pm 1)\big)\hbar^2 \langle k,j,m|k,jm\rangle$$

So for orthonormal states $\langle k_1,j_1,m_1|k_2,j_2,m_2\rangle \equiv \delta_{k_1k_2}\delta_{j_1j_2}\delta_{m_1m_2}$

we can define

$$\boxed{|k,j,m\pm 1\rangle \equiv \frac{1}{\hbar\sqrt{j(j+1)-m(m\pm 1)}} J_\pm |k,j,m\rangle}$$

And so we can build up the "standard" eigenbasis from one rung to another for each (k,j)

Further since we have

$$J_z|k,j,m\rangle = m\hbar|k,j,m\rangle$$
$$J_\pm|k,j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm 1)}\,|k,j,m\pm 1\rangle$$

we can determine the $\vec{J}$ operator matrix ele's in this bases:

$$\langle k, j, m | \vec{J} | k', j', m' \rangle \equiv \left(\vec{J}^{(j)}\right)_{mm'} \delta_{kk'} \delta_{jj'}$$

for each j it's a $(2j+1) \times (2j+1)$ matrix (all indep. of k). Further, the $\vec{J}$-commutation relations $\Rightarrow$ $\vec{J}^{(j)}$ obeys matrix commutation relations

$$\left([J_a^{(j)}, J_b^{(j)}]\right)_{mm'} = i\hbar\epsilon_{abc}\left(J_c^{(j)}\right)_{mm'}.$$

Thus we may find Matrix Representations of the Ang. momentum operator:

1) $j=0 \Rightarrow m=0 \Rightarrow \vec{J}^{(0)}$ is a 1×1 matrix i.e. a #
$|k,j,m\rangle \to |k\rangle$ & $=0$.
So $j=0$ will correspond to a spin 0 particle

2) $j=\frac{1}{2} \Rightarrow m = \pm\frac{1}{2}$ & we have 2×2 matrices

i.e. $\left(J_z^{(\frac{1}{2})}\right)_{\frac{1}{2}\frac{1}{2}} = \frac{1}{2}\hbar$ $\qquad \left(J_z^{\frac{1}{2}}\right)_{\pm\frac{1}{2}\mp\frac{1}{2}} = 0$

$\left(J_z^{(\frac{1}{2})}\right)_{-\frac{1}{2}-\frac{1}{2}} = -\frac{1}{2}\hbar$

$$\Rightarrow \left(J_z^{(\frac{1}{2})}\right) = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Likewise $$J_+^{(\frac{1}{2})} = \hbar\begin{pmatrix}0&1\\0&0\end{pmatrix} ; \quad J_-^{(\frac{1}{2})} = \hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

Recall $J_\pm \equiv J_x \pm i J_y$

$$\Rightarrow \quad J_x = \tfrac{1}{2}(J_- + J_+)$$
$$J_y = \tfrac{i}{2}(J_- - J_+)$$

$$\Rightarrow \quad J_x^{(\frac{1}{2})} = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} \quad J_y^{(\frac{1}{2})} = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

These are just the Pauli Matrices σ^i

$$\vec{J}^{(\frac{1}{2})} = \frac{\hbar}{2}\vec{\sigma}$$

$$\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix} \quad \sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix} \quad \sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

3) $j=1 \Rightarrow m = \pm 1, 0 \Rightarrow 3\times 3$ matrices

etc

Further The basis els. are $|k, \frac{1}{2}, +\frac{1}{2}\rangle$
$|k, \frac{1}{2}, -\frac{1}{2}\rangle$

Any vector in the $j=\frac{1}{2}$ subspace can be expanded

$$|\chi\rangle = \chi_\uparrow |k, \tfrac{1}{2}, \tfrac{1}{2}\rangle + \chi_\downarrow |k, \tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

normal. with $|\chi_\uparrow|^2 + |\chi_\downarrow|^2 = 1$

So in this basis, the basis vectors are just column vectors

$$|k,j,m\rangle = \sum_{m'=-j}^{+j} (e^m)_{m'}\, |k,j,m'\rangle$$

$\Rightarrow$

$$(e^m)_{m'} = \delta_{mm'}$$

i.e.

$$|k,j,m\rangle \longrightarrow e^m = \begin{array}{c} \\ \left(\begin{array}{c} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{array}\right) \end{array} \begin{array}{l} m'= \\ j \\ \vdots \\ m'=m \\ \vdots \\ m'=-j \end{array}$$

for $j=\frac{1}{2}$

$$|k,\tfrac{1}{2},+\tfrac{1}{2}\rangle \longrightarrow e^\uparrow = \begin{pmatrix}1\\0\end{pmatrix}$$

$$|k,\tfrac{1}{2},+\tfrac{1}{2}\rangle \longrightarrow e^\downarrow = \begin{pmatrix}0\\1\end{pmatrix}$$

&

$$|\chi\rangle \longrightarrow (\chi)_m = \begin{pmatrix}\chi_\uparrow\\ \chi_\downarrow\end{pmatrix}_m \left(= \langle kjm|\chi\rangle\right).$$

So any state column vector is a sum over basis col. vectors

$$\chi = \chi_\uparrow e^\uparrow + \chi_\downarrow e^\downarrow .$$

Now this basis is not quite the "physical" basis we are used to. We want the "spin" standard basis. ~~the~~ We know that the particle may have ordinary orbital ∡ momentum and — found experimentally — ~~any~~ intrinsic spin ∡ momentum

experimental discovery!

~~Let~~

$$\underbrace{\vec{J}}_{\text{Total ∡ mom.}} = \underbrace{\vec{L}}_{\text{orbital ∡ mom}} + \underbrace{\vec{S}}_{\text{spin ∡ mom.}}$$

Since $\vec{L} = \vec{R} \times \vec{P} \Rightarrow [L_i, L_j] = i\hbar\epsilon_{ijk}L_k$

& by def. $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$.

~~Since~~ $\vec{S}$ is an intrinsic property of particle so we will ~~assume~~ use $[S^i, X^j] = 0 = [S^i, P^j] \Rightarrow [L^i, S^j] = 0$.

i.e. $[L^i, X^j / P^j] = i\hbar\epsilon_{ijk} X^k / P^k$
$[L^i + S^i, X^j / P^j] = 0 \Rightarrow [S^i, X^j / P^j]$

Hence $\Rightarrow$ $\boxed{[S^i, S^j] = i\hbar\epsilon_{ijk}S^k}$!!

$$[L^i + S^i, L^j + S^j] = [L^i, L^j] + \cancel{[S^i, L^j]}^{0} + \cancel{[L^i, S^j]}^{0} + [S^i, S^j]$$

$$= i\hbar\epsilon_{ijk}(\cancel{L^k} + S^k)$$

$$\Rightarrow [S^i, S^j] = i\hbar\epsilon_{ijk}S^k.$$

So the spin ∡ momentum operator obeys the SU(2) Algebra. Hence we can construct $\vec{S}^2$, S_z eigenstates $|s, m_s\rangle$

$$\vec{S}\,|s, m_s\rangle = s(s+1)\hbar^2\,|s, m_s\rangle$$

$$S_z\,|s, m_s\rangle = m_s\hbar\,|s, m_s\rangle$$

with $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ & $m_s = -s, \dots, +s$.

Spin raising and lowering $S_\pm \equiv S_x \pm i S_y$ & the <u>Orthonormal standard spin basis</u>

$$S_+\,|s, m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s+1)}\,|s, m_s+1\rangle$$

$$S_-\,|s, m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s-1)}\,|s, m_s-1\rangle$$

$$S_z\,|s, m_s\rangle = \hbar\, m_s\,|s, m_s\rangle.$$

The spin operators are represented by spin matrices in the standard spin basis

$$\vec{S}\,|s, m_s'\rangle = \sum_{m_s=-s}^{+s} \left(\vec{S}^{(s)}\right)_{m_s m_s'} |s, m_s\rangle$$

<u>i.e.</u>

$$\langle s, m_s|\,\vec{S}\,|s, m_s'\rangle = \left(\vec{S}^{(s)}\right)_{m_s m_s'}$$

These are just the same matrices as the total $\measuredangle$ momentum matrices

$$\left(\vec{S}^{(s)}\right)_{m_s m_s'} = \left(\vec{J}^{(s)}\right)_{m_s m_s'} .$$

i.e.
$s = \frac{1}{2}$
$m_s = \pm\frac{1}{2}$

$$\vec{S}^{(\frac{1}{2})} = \frac{\hbar}{2}\vec{\sigma} .$$

As well the spin basis can be represented by column vectors

spin up $e^{\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

spin-down $e^{\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

~~Now if $s=0$ we just have $\vec{J}=\vec{L}$~~

Now the standard basis consisted of simultaneous eigenvectors of CSCO $\{A, \vec{J}^2, J_z\}$, the $\{|k, j, m\rangle\}$.

Since $\vec{R}$ & $\vec{P}$ commute with $\vec{S}$, we may also use the set $\{\vec{R}, \vec{S}^2, S_z\}$ or $\{\vec{P}, \vec{S}^2, S_z\}$ as a CSCO. Clike

Since $[S^i, X^j] = 0$ the eigenkets $|\vec{r}, s, m_s\rangle$ can be written as

$$|\vec{r}, s, m_s\rangle = |\vec{r}\rangle \otimes |s, m_s\rangle .$$

They are complete

$$|\psi\rangle = \int d^3r \sum_{m_s=-s}^{+s} \psi^{(s)}_{m_s}(\vec{r})\, |\vec{r}, s, m_s\rangle$$

for a particle with a particular s.

So $\quad \psi^{(s)}_{m_s}(\vec{r}) = \langle \vec{r}\, s\, m_s | \psi\rangle$

is a multi-component wavefunction.

Since $\vec{J} = \vec{L} + \vec{S}$

$$\langle \vec{r}, s, m_s | \vec{J} | \psi\rangle = \langle \vec{r}, s, m_s | \vec{L} | \psi\rangle + \langle \vec{r}, s, m_s | \vec{S} | \psi\rangle$$

$$= \left[\left(\vec{r} \times \frac{\hbar}{i} \vec{\nabla}_{\vec{r}}\right) \delta_{m_s m_s'} + \left(\vec{S}^{(s)}\right)_{m_s m_s'} \right] \psi^{(s)}_{m_s'}(\vec{r})$$

$$\psi^{(s)}_{m_s'}(\vec{r}) = \langle \vec{r}, s, m_s' | \psi\rangle .$$