Physical Description of Spin

Under rotations, the system & measuring apparati have been rotated from $\vec{r}$ to $\vec{r}\,' = R\vec{r}$. Hence the QM position eigenstates $|\vec{r}\rangle$ and $|R\vec{r}\rangle$ are related by $U(R(\vec{\theta}))$

$$|\vec{r}\,'\rangle = U(R(\vec{\theta}))\,|\vec{r}\rangle$$

$$= |R(\vec{\theta})\,\vec{r}\rangle$$

where we are using the shorthand $\vec{r}\,' = R\vec{r}$ to denote the component rotation formula

$$x_i' = R_{ij}(\vec{\theta})\, x_j$$

So $X_i\,|\vec{r}\,'\rangle = x_i'\,|\vec{r}\,'\rangle$

$$= R_{ij}(\vec{\theta})\, x_j\,|\vec{r}\,'\rangle$$

Consequently we can determine the rotation transformation properties of the position operator $\vec{R}$, since

$$U^{-1}(R)|\vec{r}\rangle = |R^{-1}\vec{r}\rangle$$

we have

$$\underline{X}_i U^{-1}(R)|\vec{r}\rangle = \underline{X}_i |R^{-1}\vec{r}\rangle$$
$$= (R^{-1}_{ij} x_j)|R^{-1}\vec{r}\rangle$$
$$= (R^{-1}_{ij} x_j) U^{-1}(R)|\vec{r}\rangle .$$

$(R^{-1}\vec{r})$ is a c-number so $U^{-1}(R)$ does not operate on it and the RHS above is

$$= U^{-1}(R)(R^{-1}_{ij} x_j |\vec{r}\rangle)$$

Multiplying by $U(R)$ we have

$$U(R(\vec{\theta}))\underline{X}_i U^{-1}(R(\vec{\theta}))|\vec{r}\rangle$$
$$= R^{-1}_{ij}(\vec{\theta}) x_j |\vec{r}\rangle$$
$$= R^{-1}_{ij}(\vec{\theta}) \underline{X}_j |\vec{r}\rangle .$$

This implies the operator relation

$$X_i' = \boxed{U(R(\vec{\theta}))\, X_i\, U^{-1}(R(\vec{\theta})) = R_{ij}^{-1}(\vec{\theta})\, X_j},$$

hence rotating the system one way is equivalent to rotating the coordinate axes the inverse or opposite way.

Any operator which transforms like $\vec{R}$ under rotation transformations we call a vector operator i.e. if

$$V_i' = U(R(\vec{\theta}))\, V_i\, U^{-1}(R(\vec{\theta})) = R_{ij}^{-1}(\vec{\theta})\, V_j$$

then V_i is a vector operator.

we have that $\vec{J}$, $\vec{P}$, $\vec{X}$ are vector operators.

These rotation properties allow us to classify operators according to how they transform under rotations, this results in an operator tensor analysis.

1) A scalar operator S is invariant under rotations

$$S' = U(R(\vec{\theta}))\, S\, U^{-1}(R(\vec{\theta})) \equiv S$$

Such operators commute with $U(R(\vec{\theta}))$ as indicated by their definition $US = SU$ and hence commute with $\vec{J}$; $[S, \vec{J}] = 0$.

2) $\vec{V}$ is a vector operator if it transforms as $\vec{R}$ under rotations

$$V_i' = U(R(\vec{\theta}))\, V_i\, U^{-1}(R(\vec{\theta})) \equiv R^{-1}_{ij}(\vec{\theta})\, V_j$$

$$\Rightarrow \quad [J_i, V_j] = i\hbar \epsilon_{ijk} V_k .$$

3) Rank 2 tensor operators T_{ij}, transform as the product $X_i X_j$ transforms

$$T_{ij}' = U(R(\vec{\theta}))\, T_{ij}\, U^{-1}(R(\vec{\theta}))$$

$$\equiv R^{-1}_{ii'}(\vec{\theta})\, R^{-1}_{jj'}(\vec{\theta})\, T_{i'j'} ,$$

(since $R^T = R^{-1}$ this can be written

$$T' = R^{-1} T R .)$$

4) Hence in general a rank n tensor operator $T_{i_1 i_2 \ldots i_n}$ transforms under rotations as the product

$$X_{i_1} X_{i_2} \cdots X_{i_n} \; ;$$

$$T'_{i_1 i_2 \cdots i_n} = U(R(\vec\theta))\, T_{i_1 i_2 \cdots i_n}\, U^{-1}(R(\vec\theta))$$

$$\equiv R^{-1}_{i_1 j_1}(\vec\theta)\, R^{-1}_{i_2 j_2}(\vec\theta) \cdots R^{-1}_{i_n j_n}(\vec\theta)\, T_{j_1 \cdots j_n} \, .$$

Since any rotation can be built up by making successive infinitesimal rotations, we can equivalently define the tensor classification of operators according to the above formulae for $\vec\theta = \vec\omega$ = infinitesimal, that is by their commutation relations with $\vec J$. So we have

$$R_{ij}(\vec\omega) = \delta_{ij} + \omega_{ij}$$

and hence $R^{-1}_{ij}(\vec\omega) = \delta_{ij} - \omega_{ij}$.

Further

$$U(R(\vec\theta)) = e^{-\frac{i}{\hbar}\vec\theta\cdot\vec J}$$

So for infinitesimal angles $U(R(\vec\omega)) = 1 - \frac{i}{\hbar}\vec\omega\cdot\vec J$

and $U^{-1}(R(\vec\omega)) = 1 + \frac{i}{\hbar}\vec\omega\cdot\vec J$

Then
$$T' = U(R(\vec{\omega}))\, T\, U^{-1}(R(\vec{\omega}))$$
$$= \left(1 - \frac{i}{\hbar}\vec{\omega}\cdot\vec{J}\right) T \left(1 + \frac{i}{\hbar}\vec{\omega}\cdot\vec{J}\right)$$
which to first order in $\vec{\omega}$ yields
$$= 1 - \frac{i}{\hbar}[\vec{\omega}\cdot\vec{J}, T]$$

The above classification of operators are equivalent to

1) Scalar operators S obey
$$[J_i, S] = 0$$

2) Vector operators V_j obey
$$[J_i, V_j] = i\hbar\epsilon_{ijk} V_k$$

3) Rank 2 Tensor operators T_{ij} obey
$$[J_i, T_{mn}] = i\hbar\epsilon_{imm'} T_{m'n} + i\hbar\epsilon_{inn'} T_{mn'}$$

4) Rank n Tensor operators $T_{i_1 \cdots i_n}$ obey

$$[J_i, T_{i_1 \cdots i_n}] = i\hbar\,\epsilon_{i i_1 j_1} T_{j_1 i_2 \cdots i_n} + \cdots + i\hbar\,\epsilon_{i i_n j_n} T_{i_1 i_2 \cdots i_{n-1} j_n}$$

As we know from classical physics, expressing the laws of physics in terms of tensor quantities insures that the laws are covariant under a transformation of the coordinates, in this case by rotations.

Since the structure of the Hilbert space is determined by the eigenstates of a CSCO, it is reasonable that if the operators have a tensor classification so do the states. Indeed we have been discussing in wave mechanics particles whose wavefunctions are invariant under (active) rotation transformations,

$$\psi'(\vec{r}\,') = \psi(\vec{r}),$$

(passive: the function O' uses in his frame is the same as O uses in his frame.) Since $|\psi\rangle$ and $|\psi'\rangle$ are related by $U(R(\vec{\theta}))$

$$|\psi'\rangle = U(R(\vec{\theta}))\,|\psi\rangle$$

we have that

$$\psi'(\vec{r}) \equiv \langle \vec{r}|\psi'\rangle = \langle \vec{r}|\mathcal{U}(R(\vec{\theta}))|\psi\rangle$$

$$= \langle R^{-1}(\vec{\theta})\vec{r}|\psi\rangle$$

$$= \psi(R^{-1}(\vec{\theta})\vec{r}), \text{ as stated}$$

above. Such a <u>scalar</u> wavefunction is <u>said</u> to describe systems with <u>zero spin</u>. This becomes clearer if we consider infinitesimal rotations.

$$\psi'(\vec{r}) = \psi(R^{-1}(\vec{\omega})\vec{r}) = \psi(\vec{r} - \vec{\omega}\times\vec{r})$$

$$= \psi(\vec{r}) - (\vec{\omega}\times\vec{r})\cdot\vec{\nabla}_{\vec{r}}\psi(\vec{r})$$

$$= \psi(\vec{r}) - \epsilon_{ijk}\,\omega_j x_j \frac{\partial}{\partial x_i}\psi(\vec{r})$$

$$= \psi(\vec{r}) - \vec{\omega}\cdot(\vec{r}\times\vec{\nabla}_{\vec{r}})\psi(\vec{r})$$

$$= \psi(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\left(\vec{r}\times\frac{\hbar}{i}\vec{\nabla}_{\vec{r}}\right)\psi(\vec{r})$$

On the other hand

$$\psi'(\vec{r}) = \langle \vec{r}|\psi'\rangle = \langle \vec{r}|\mathcal{U}(R(\vec{\omega}))|\psi\rangle$$

$$= \langle \vec{r}|\mathbb{1} - \frac{i}{\hbar}\vec{\omega}\cdot\vec{J}|\psi\rangle$$

$$\psi'(\vec{r}) = \psi(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\langle\vec{r}|\vec{J}|\psi\rangle$$
$$= \psi(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\left(\vec{r}\times\frac{\hbar}{i}\vec{\nabla}_{\vec{r}}\right)\psi(\vec{r})$$

Hence on the space of spin zero wave-functions we have

$$\langle\vec{r}|\vec{J}|\psi\rangle = \left(\vec{r}\times\frac{\hbar}{i}\vec{\nabla}_{\vec{r}}\right)\langle\vec{r}|\psi\rangle$$
$$= \langle\vec{r}|\vec{R}\times\vec{P}|\psi\rangle$$
$$= \langle\vec{r}|\vec{L}|\psi\rangle$$

So the total angular momentum $\vec{J}=\vec{L}$ is just the orbital angular momentum when acting on the space of scalar states (that is spin zero states).

Hence if
$$\psi'(\vec{r}) = \psi(R^{-1}(\vec{\theta})\vec{r})$$

(which is equivalent to $\vec{J}=\vec{L}$ for these states) the states of the system have spin zero.

As with tensor operators, we can introduce multi-component wavefunctions that describe states with spin > 0. These functions are not invariant but mimic the transformation properties of products of position operators X_i. For instance a vector wavefunction has 3 components at each value of space $\psi_i(\vec{r})$. Under rotations not only do the position components rotate into each other, but also do the wave-function components.

Thus we define a vector, or spin 1 wavefunction to transform as

$$\psi_i'(\vec{r}) \equiv R_{ij}(\vec{\theta})\,\psi_j(R^{-1}(\vec{\theta})\vec{r}) \, .$$

Hence we have that the state vector consists of 3 ordinary state vectors in direct product with the basis vectors for the three dimensional spin space

$$|\psi\rangle = \sum_{i=1}^{3} |\psi_i\rangle \otimes |e_i\rangle$$

where
$$\Psi(\vec{r}) \equiv \langle \vec{r} | \psi \rangle$$
$$= \sum_{i=1}^{3} \langle \vec{r} | \psi_i \rangle \quad | e_i \rangle$$
$$= \sum_{i=1}^{3} \psi_i(\vec{r}) | e_i \rangle \quad .$$

that is the 3 wavefunctions $\psi_i(\vec{r})$ are the component wavefunctions of $|\psi\rangle$ in the $\{|\vec{r}\rangle \otimes |e_i\rangle\}$ basis. The $\{|e_i\rangle\}$ basis spans a 3 dimensional spin Hilbert space with inner product

$$\langle e_i | e_j \rangle = \delta_{ij} \quad \text{and}$$

completeness,
$$\sum_{i=1}^{3} |e_i\rangle\langle e_i| = 1 \quad ,$$

in the 3-dimensional spin space.

The rotation operators are now the direct product of operators acting on the $|\psi_i\rangle$ kets and the $|e_i\rangle$ kets

$$|\psi'\rangle = U^{\text{orbital}}(R(\vec{\theta}))|\psi_i\rangle \otimes U^{\text{spin}}(R(\vec{\theta}))|e_i\rangle$$

where $U^{\text{orbital}}(R(\vec{\theta}))$ only acts on the spatial degrees of freedom while $U^{spin}(R(\vec{\theta}))$ only acts to rotate the 3-dimensional spin space basis vectors. That is,

$$\langle \vec{r} | U^{orbital}(R(\vec{\theta})) | \psi_i \rangle = \langle R^{-1}(\vec{\theta})\vec{r} | \psi_i \rangle$$

while

$$U^{spin}(R(\vec{\theta}))|e_i\rangle = R^{-1}_{ij}(\vec{\theta})|e_j\rangle.$$

The $U(R(\vec{\theta})) = U^{orbital}(R(\vec{\theta})) \otimes U^{spin}(R(\vec{\theta}))$ are tensor operators on the space $\{|\vec{r}\rangle\} \otimes E^3 = \mathcal{H}_1 \otimes E^3$. (another use of the word tensor operator)

$$\Psi'(\vec{r}) = \langle \vec{r} | \psi' \rangle$$

$$= \langle \vec{r} | U^{orbital}(R(\vec{\theta})) | \psi_i \rangle \, U^{spin}(R(\vec{\theta}))|e_i\rangle$$

$$= \langle R^{-1}(\vec{\theta})\vec{r} | \psi_i \rangle \, R^{-1}_{ij}(\vec{\theta})|e_j\rangle$$

$$= R_{ji}(\vec{\theta})\,\psi_i(R^{-1}(\vec{\theta})\vec{r})\,|e_j\rangle$$

Recalling that we can expand $|\psi'\rangle$ in terms of the basis $\{|e_j\rangle\}$

$$|\psi'\rangle = |\psi'_j\rangle \otimes |e_j\rangle$$

we have that

$$\Psi'(\vec{r}) = \langle\vec{r}|\psi'\rangle = \langle\vec{r}|\psi'_j\rangle|e_j\rangle = \psi'_j(\vec{r})|e_j\rangle$$
$$= R_{ji}(\vec{\theta})\,\psi_i(R^{-1}(\vec{\theta})\vec{r})|e_j\rangle$$

$$\Rightarrow \quad \boxed{\psi'_i(\vec{r}) = R_{ij}(\vec{\theta})\,\psi_j(R^{-1}(\vec{\theta})\vec{r})}$$

as we stated earlier.

(Recall the single component scalar field; spin zero wavefunction, transformed as
$$\psi'(\vec{r}) = \psi(R^{-1}(\vec{\theta})\vec{r})\text{)}$$

The relation of these multi-component wavefunction transformation law to spin 1 is most easily seen by considering infinitesimal rotations $R(\vec{\omega})$

$$\psi'_i(\vec{r}) = R_{ij}(\vec{\omega})\,\psi_j(R^{-1}(\vec{\omega})\vec{r})$$
$$= (\delta_{ij} - \epsilon_{ijk}\omega_k)\,\psi_j(\vec{r} - \vec{\omega}\times\vec{r})$$

This becomes, to first order in $\vec{\omega}$

$$\psi_i'(\vec{r}) = \psi_i(\vec{r}-\vec{\omega}\times\vec{r}) - \epsilon_{ijk}\omega_k \psi_j(\vec{r})$$

now Taylor expanding the first term on the RHS

$$= \psi_i(\vec{r}) - (\vec{\omega}\times\vec{r})\cdot\vec{\nabla}_{\vec{r}}\,\psi_i(\vec{r})$$
$$- \epsilon_{ijk}\omega_k \psi_j(\vec{r})$$

$$= \psi_i(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\left[\left(\vec{r}\times\frac{\hbar}{i}\vec{\nabla}_{\vec{r}}\right)\delta_{ij} + \vec{S}_{ij}\right]\times$$
$$\times\psi_j(\vec{r})$$

where the "spin vector" $(\vec{S})_{ij}$ has components

$$(S^k)_{ij} = \frac{\hbar}{i}\epsilon_{kij} = -\frac{\hbar}{i}(I^k)_{ij}$$

that is $\vec{S}_{ij} = -\frac{\hbar}{i}\vec{I}_{ij}$ where the $\vec{I}_{ij}$ matrices are defined

$$\boxed{(I_k)_{ij} \equiv \epsilon_{ikj}}$$

$(\vec{I})_{ij}$ are 3 matrices

$$(\vec{I})_{ij} = (I^1)_{ij}\,\hat{i} + (I^2)_{ij}\,\hat{j} + (I^3)_{ij}\,\hat{k}.$$

So

$$I_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad (= I_x)$$

$$I_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad (= I_y)$$

$$I_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (= I_z)$$

Since ϵ_{ijk} obeys the Jacobi identity

$$0 = \underbrace{\epsilon_{ijk}\,\epsilon_{mnk}}_{\text{sum}} + \underbrace{\epsilon_{jnk}\,\epsilon_{mik}}_{\text{sum}} + \underbrace{\epsilon_{nik}\,\epsilon_{mjk}}_{\text{sum}}$$

fixed fixed fixed

cyclic permutation of (i,j,n)

This is

$$0 = -\epsilon_{ijk}(\epsilon_{mkn}) + (\epsilon_{mik})(\epsilon_{kjn}) - (\epsilon_{mjk})(\epsilon_{kin})$$

Which is

$$\boxed{([I_i, I_j])_{mn} = \epsilon_{ijk}(I_k)_{mn}}$$

So

$$[i\hbar I_i, i\hbar I_j] = i\hbar \epsilon_{ijk}(i\hbar I_k)$$

$$\Rightarrow \quad [S_i, S_j] = i\hbar \epsilon_{ijk} S_k$$

The angular momentum commutation relations are obeyed by the spin vector $(\vec{S})_{ij}$.

(Aside: Define matrix $\Theta_{ij} \equiv \hat{\theta} \cdot (\vec{I})_{ij}$

$$= \begin{bmatrix} 0 & -\hat{\theta}_3 & \hat{\theta}_2 \\ \hat{\theta}_3 & 0 & -\hat{\theta}_1 \\ -\hat{\theta}_2 & \hat{\theta}_1 & 0 \end{bmatrix}$$

One can show that

$$R_{ij}(\vec{\theta}) = \delta_{ij} + (\Theta^2)_{ij}(1-\cos\theta) + (\Theta)_{ij}\sin\theta$$

$$= (e^{\theta\Theta})_{ij}$$
)

Further we have $U(R(\vec{\theta})) = e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}}$
So

$$\psi_i'(\vec{r}) = \left(\langle\vec{r}|\otimes\langle e_i|\right)|\psi'\rangle$$
$$= \left(\langle\vec{r}|\otimes\langle e_i|\right) e^{-\frac{i}{\hbar}\vec{\omega}\cdot\vec{J}}|\psi\rangle$$
$$= \left(\langle\vec{r}|\otimes\langle e_i|\right)\left[\mathbb{1} - \frac{i}{\hbar}\vec{\omega}\cdot\vec{J}\right]|\psi\rangle$$
$$= \psi_i(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\left(\langle\vec{r}|\vec{J}^{\,orbital}|\psi_i\rangle + \langle\vec{r}|\psi_j\rangle\langle e_i|\vec{J}^{\,spin}|e_j\rangle\right)$$

Which equals

$$= \psi_i(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\left(\left(\vec{r}\times\frac{\hbar}{i}\vec{\nabla}_{\vec{r}}\right)\delta_{ij} + \vec{S}_{ij}\right)\psi_j(\vec{r})$$

$$= \psi_i(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\left[\vec{L}\,\delta_{ij} + \vec{S}_{ij}\right]\psi_j(\vec{r})$$

Thus on states with spin one the angular momentum operator $\vec{J}$ is represented by the sum of orbital angular momentum $\vec{L}$ and spin angular momentum $\vec{S}$,

$$\vec{J} = \vec{L} + \vec{S}$$

where the $\vec{L}$ acts only on the spatial variables, $\vec{L} = (\vec{r} \times \frac{\hbar}{i}\vec{\nabla}_r)$, as usual, while $\vec{S}$ acts only on the spin space variables

Thus
$$\psi_i'(\vec{r}) = \psi_i(\vec{r}) - \frac{i}{\hbar}\vec{\omega} \cdot \left(\vec{L}\delta_{ij} + \vec{S}_{ij}\right)\psi_j(\vec{r})$$

The $\vec{L}$ and $\vec{S}$ operators commute and, as we know, the

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$$

and we have shown that (i.e. $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$ & $[L_i, L_j] = \ldots L_k$ & $\vec{J} = \vec{L} + \vec{S}$, $[\vec{L}, \vec{S}] = 0$ $\Rightarrow [S_i, S_j] = i\hbar\epsilon_{ijk}S_k$)

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k .$$

The $\vec{L}$ and $\vec{S}$ operators obey the same algebra as the $\vec{J}$ operators, hence they generate the same group of operators.
We have that the eigenvalues of $\vec{L}^2$ are $\hbar^2 l(l+1)$; $l = 0, 1, 2, \ldots$ and L_3 are $\hbar m$; $m = -l, -l+1, \ldots, +l$

At the same time the commuting set of matrices $\vec{S}^2$ and S_3 which we can make out of S_i are given by

$(S^k)_{ij} = +i\hbar\,\epsilon_{ikj}$; thus

$(\vec{S}^2)_{ij} = (i\hbar)^2 \epsilon_{ik\ell}\epsilon_{\ell kj} = -\hbar^2(-2\delta_{ij})$

$= 2\hbar^2\delta_{ij} = 1(1+1)\hbar^2\delta_{ij}$

$(\vec{S}^2)_{ij} = \hbar^2 s(s+1)\delta_{ij}$ with $s=1$

the total spin eigenvalue is $s=1$

$(S^3)_{ij} = i\hbar\begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$ and this can be

diagonalized to $(\tilde{S}^3)_{ij} = \hbar\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}_{ij}$

or finding the eigenvectors of $(S^3)_{ij}$ we have (in components)

$e_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$; $e_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$; $e_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

So

$(S^3)e_+ = \hbar\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}\frac{1}{\sqrt{2}} = \hbar\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}\frac{1}{\sqrt{2}} = \hbar e_+$

$(S^3)e_- = \hbar\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}\frac{1}{\sqrt{2}} = \hbar\begin{pmatrix} -1 \\ +i \\ 0 \end{pmatrix}\frac{1}{\sqrt{2}} = -\hbar e_-$

$(S^3)e_0 = 0\hbar\, e_0$

Then the eigenvalues of $\frac{1}{\hbar}S^3$ are $+s, s-1, \ldots, -s$ for $s=1$ this is $+1, 0, -1$.

The projection of the spin onto the z-axis has discrete values, just like the orbital angular momentum, ranging from $-s\hbar, \ldots, +s\hbar$.

For finite transformations we can exponentiate the angular momentum operator to find

$$|\psi'\rangle = U(R(\vec{\theta}))|\psi\rangle = e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}}|\psi\rangle$$

$$= e^{-\frac{i}{\hbar}\vec{\theta}\cdot(\vec{L}+\vec{S})}|\psi_i\rangle \otimes |e_i\rangle$$

$$= e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{L}}|\psi_i\rangle \otimes e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}|e_i\rangle$$

Now in general $e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}|e_i\rangle = \left(e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}\right)_{ji}|e_j\rangle$

and this matrix is defined as

$$D^{(s)}_{ij}(R(\vec{\theta})) \equiv \left(e^{\mp\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}\right)_{ij}$$

For $s=1$ we have $\vec{S} = -i\hbar\vec{I}$ and

$$D^{(1)}_{ij}(R(\vec{\theta})) = R_{ij}(\vec{\theta})$$ the vector

representation matrix. We can ask if there are other matrices for different spin. Certainly we could put several spin 1 states together in a direct product to obtain higher integer spins like we did for higher rank tensor operators. So by this we obtain all integer spin states; have we missed any states? As we shall show shortly the eigenvalue spectrum of $\vec{J}^2$ and J_z are $\hbar^2 j(j+1)$ and $-j, \dots, +j$ respectively with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$ There is also the possibility of odd-half integer spin. This suggests for $s=\frac{1}{2}$ we can find a 2×2 spin matrix $\vec{S}$. That is we desire a 2 component wavefunction that transforms under rotations as

$$\psi'_i(\vec{r}) = D^{(s=\frac{1}{2})}_{ij}(R(\vec{\theta}))\,\psi_j(R^{-1}(\vec{\theta})\vec{r})$$

where $i,j = 1,2$ now.

That is in state space we have

$$|\psi'\rangle = U(R(\vec{\theta}))|\psi\rangle$$
$$= e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{L}}|\psi_i\rangle \otimes U^{spin}(R(\vec{\theta}))|e_i\rangle$$
$$= e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{L}}|\psi_i\rangle \otimes \underbrace{\left(e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}\right)_{ji}}_{=D^{(1/2)}_{ji}(R(\vec{\theta}))}|e_j\rangle$$

Since $U(R(\vec{\theta}))$ is unitary we always have

$$\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle$$

$$\Rightarrow \quad D^{(s)*}_{ji}(R(\vec{\theta}))\,D^{(s)}_{jk}(R(\vec{\theta})) = \delta_{ik}$$

$\Rightarrow$ $D^{(s)}_{ij}(R(\vec{\theta}))$ are unitary matrices

$$D^{*(s)}_{ji}(R) = D^{-1(s)}_{ij}(R) \, .$$

Further $\det D^{(s)}(R) = \pm 1$ ~~we~~ only include proper Rotations so $\det D^{(s)}(R) = +1$ we have ~~that~~ since

$$D^{(s)}(R) = e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}$$

$$\det D^{(s)}(R(\vec{\theta})) = e^{\mathrm{Tr} \ln D^{(s)}(R(\vec{\theta}))}$$
$$= e^{\mathrm{Tr}\left(-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}\right)}$$

Since $\det D^{(s)} = 1 \Rightarrow \boxed{\mathrm{Tr}\,\vec{S} = 0}$

Further $D^{(s)}$ is unitary, so $\boxed{\vec{S} \text{ is Hermitian}}$

All 2×2 Hermitian matrices that are traceless can be written as linear combinations of the Pauli-matrices $\vec{\sigma}$

$$\sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (= \sigma_x)$$

$$\sigma_2 \equiv \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \quad (= \sigma_y)$$

$$\sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (= \sigma_z) .$$

The Pauli matrices have the properties that

$$\det \sigma_i = -1$$

$$\mathrm{Tr}\, \sigma_i = 0$$

$$\left.\begin{array}{l} \sigma_i^2 = 1 \\ [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \end{array}\right\} \sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k$$

Hence the matrices $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ obey the angular momentum algebra of

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$$

so that $D^{(1/2)}$ and hence U^{spin} will obey the rotation group multiplication law for

$$D^{(1/2)}_{ij}(R(\vec{\theta})) = \left(e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}\right)_{ij} .$$

For infinitesimal transformations we find

$$\psi'_i(\vec{r}) = \left(1 - \tfrac{i}{\hbar}\vec{\omega}\cdot\vec{S}\right)_{ij}\psi_j(\vec{r}) + \psi_i(\vec{r} - \vec{\omega}\times\vec{r})$$

$$= \psi_i(\vec{r}) - \tfrac{i}{\hbar}\vec{\omega}\cdot\left[(\vec{r}\times\tfrac{\hbar}{i}\vec{\nabla}_{\vec{r}})\delta_{ij} + \vec{S}_{ij}\right]\psi_j(\vec{r})$$

As usual

$$\psi_i'(\vec{r}) = \psi_i(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\Big[\langle\vec{r}|\vec{J}^{\,\text{orbital}}|\psi_i\rangle + \langle\vec{r}|\psi_j\rangle\langle e_i|\vec{J}^{\,\text{spin}}|e_j\rangle\Big]$$

From above

$$= \psi_i(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot\big[\vec{L}\,\delta_{ij} + \vec{S}_{ij}\big]\psi_j(\vec{r})$$

Then $\vec{J} = \vec{L} + \vec{S}$ again where in the space of spin $\frac{1}{2}$ wavefunctions, called spinor wavefunctions,

$$(\vec{S})_{ij} = \frac{\hbar}{2}\vec{\sigma}_{ij} \quad ; \text{the Pauli-matrices } \vec{\sigma}.$$

As before

$$(\vec{S}^2)_{ij} = \left(\frac{\hbar}{2}\right)^2(\vec{\sigma}^2)_{ij} = \frac{1}{4}\hbar^2\delta_{ij}\cdot 3$$

$$= \hbar^2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}+1\right)\delta_{ij} = \hbar^2 s(s+1)\delta_{ij}$$

with $s = \frac{1}{2}$; the total spin eigenvalue is $s = \frac{1}{2}$. Since the $S_i = \frac{\hbar}{2}\sigma_i$ do not commute, we can only simultaneously diagonalize one of them with $\vec{S}^2$; we choose $S_3 = \frac{\hbar}{2}\sigma_3$

$(S_3) = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ it ~~has~~ eigenvectors (in components) $e_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $e_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so that

$$S_3 e_\uparrow = +\tfrac{1}{2}\hbar e_\uparrow \ ; \ S_3 e_\downarrow = -\tfrac{1}{2}\hbar e_\downarrow$$

$|e_\uparrow\rangle$ has spin $+\frac{1}{2}\hbar$ when projected onto the z-axis, it is said to have "spin up", while $|e_\downarrow\rangle$ has spin $-\frac{1}{2}\hbar$ when projected onto the z-axis, it is said to have "spin down".

The projected spin eigenvalues are $+s, \ldots, s$ in this case $+\frac{1}{2}$ and $-\frac{1}{2}$.

The finite transformation matrix is

$$D^{(\frac{1}{2})}(R(\vec{\theta})) = \left(e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}\right)$$

$$= \left(e^{-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}}\right)$$

$$= \cos\frac{\theta}{2}\,\mathbb{1} - i\hat{\theta}\cdot\vec{\sigma}\,\sin\frac{\theta}{2}$$

Note that
$$D^{(\frac{1}{2})}(R(2\pi)) = \cos\frac{2\pi}{2} = -1$$

The representation $D^{(1/2)}$ is said to be double-valued $D^{(1/2)}(R(0)) = -D^{(1/2)}(R(2\pi))$

Thus on the spin $\frac{1}{2}$ states, the identity rotation, $\theta = 0$ or 2π, is represented by $+1$ or -1. Since $\pm|e_i\rangle$ is in the same Ray, it corresponds to the same physical state.

Of course we can imagine combining multiple spin $\frac{1}{2}$ states to obtain all integer and odd-half integer spin states. The spin-$\frac{1}{2}$ states form the fundamental representation of the rotation group ($O(3)$) $SU(2)$. To be sure we have not missed any states, we next determine the eigenvalue spectrum of J^2, J_3 and construct the spin matrices $(S)_{ij}$ in their eigenvector basis.

Thus in general we have spin states s that transform as

$$\psi'_i(\vec{r}) \equiv D^{(s)}_{ij}(R(\vec{\theta}))\psi_j(R^{-1}(\vec{\theta})\vec{r})$$

which for infinitesimal rotations we write

$$\psi'_i(\vec{r}) = \psi_i(\vec{r}) - \frac{i}{\hbar}\vec{\omega}\cdot[\vec{L}\delta_{ij} + \vec{S}_{ij}]\psi_j(\vec{r})$$

where $$D^{(s)}(R(\vec{\theta})) = e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}}.$$

The group multiplication law implies for $R(\vec{\theta}_3) = R(\vec{\theta}_2)R(\vec{\theta}_1)$ that

$$D^{(s)}(R(\vec{\theta}_3)) = D^{(s)}(R(\vec{\theta}_2))D^{(s)}(R(\vec{\theta}_1)),$$

that is the $D^{(s)}(R)$ matrices form a <u>matrix</u> representation of the SU(2) rotation group. Equivalently the group multiplication law implies the $\vec{S}$ matrices obey the SU(2) Lie algebra of rotations

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k.$$

We would now like to find all possible matrices $\vec{S}_{ij}$ and hence $D^{(s)}(R)$. We explicitly constructed the matrices for spin $0, \frac{1}{2}, 1$. To determine the general matrix structure for S we can turn to consideration of the eigenvalue determination of a set of commuting observables made from the $\vec{J}$ and whatever other operators may commute with them.

Angular Momentum Commutation Relations and the "Standard Basis"

The angular momentum commutation relations are

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k .$$

Hence $\vec{J}^2 = \vec{J}\cdot\vec{J}$ commutes with J_i since it is a scalar (the dot product of 2 vector operators) operator

$$[\vec{J}^2, J_i] = 0 \quad \text{for } i = 1, 2, 3 .$$