5.) Spatial Rotations: Geometry of rotation and the rotation group.

Isotropy of space $\Leftrightarrow$ equivalence of experiments performed in frames rotated relative to each other

(Active view):

Rotate system & measuring apparatus from $\vec{r}$ to $\vec{r}\,'$. The 2-laboratories are related by a rotation of their components of $\vec{r}$ & $\vec{r}\,'$:

$$x'_i = R_{ij}\, x_j .$$

Since rotations about a common origin leave lengths of vectors unchanged

$$\vec{r}\,' \cdot \vec{r}\,' = \vec{r} \cdot \vec{r} = x_j x_j = x_j \delta_{jk} x_k$$

$$\vec{r}\,' \cdot \vec{r}\,' = R_{ij} x_j R_{ik} x_k = x_j (R^T_{ji} R_{ik}) x_k$$

$$\Rightarrow \quad R^T R = 1 \Rightarrow \boxed{R^T = R^{-1}}$$

R is a real, 3x3 orthogonal matrix.

Also $\det(RR^T) = \det 1 = 1$

$$\underset{(\det R)^2}{\overset{\shortparallel}{}} \implies \boxed{\det R = \pm 1}$$

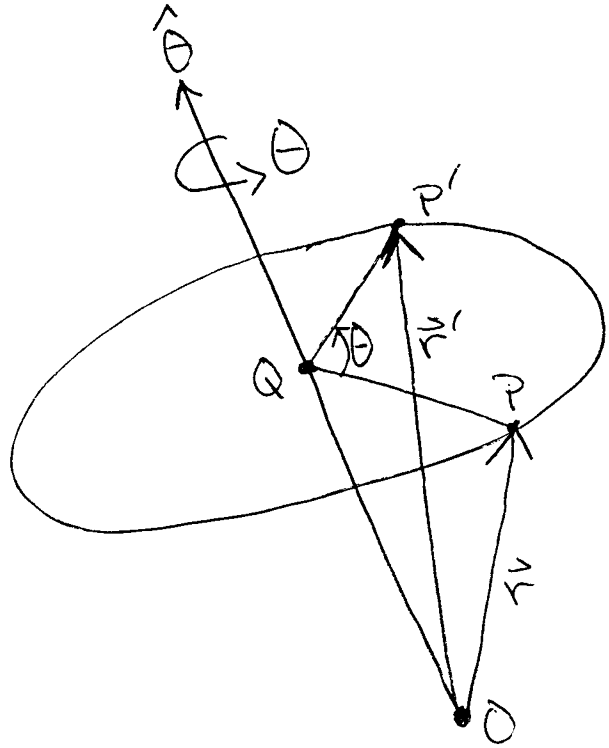
We ~~—~~ are interest in rotations connected to the identity so restrict ourselves to $\underline{\det R = +1}$: <u>proper rotations</u>

($\det R = -1$ Rotations are made up of $P_{arity} R_{proper}$)

R_{ij} has 9 real matrix elements. $R^T = R^{-1}$ is 6 conditions $\Rightarrow$ only 3 independent parameters needed to specify each matrix R.

Euler's Theorem: specify arbitrary rotation in terms of a rotation through angle θ about a fixed axis whose orientation is given by the unit vector $\hat{\theta}$. Hence the vector $\vec{\theta} \equiv \theta \hat{\theta}$ specifies the rotation $R_{ij} = R_{ij}(\vec{\theta})$ between $\vec{r}$ & $\vec{r}\,'$.

Given $\vec{\theta}$ we can find $R_{ij}(\vec{\theta})$ by considering the geometry of the rotation

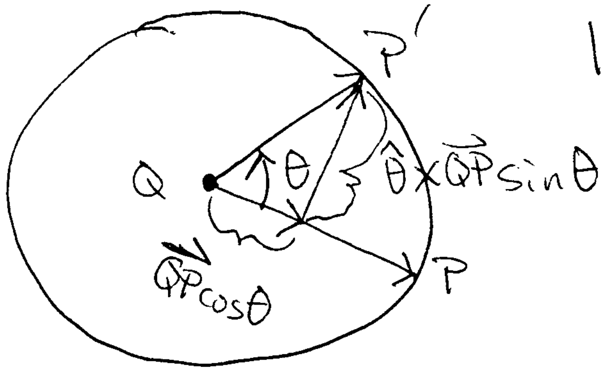


So

$$\vec{r}\,' = \overrightarrow{OQ} + \overrightarrow{QP'}$$

Now to find $\overrightarrow{QP'}$ look down the $\hat{\theta}$-axis from above

$$|\overrightarrow{QP}| = |\overrightarrow{QP'}|$$



$$\vec{r}\,' = \overrightarrow{OQ} + \overrightarrow{QP}\cos\theta + \hat{\theta}\times\overrightarrow{QP}\sin\theta$$

But vector $\overrightarrow{OQ} = (\vec{r}\cdot\hat{\theta})\hat{\theta}$ and

vector $\overrightarrow{QP} = \vec{r} - \overrightarrow{OQ} = \vec{r} - (\vec{r}\cdot\hat{\theta})\hat{\theta}$

So
$$\vec{r}\,' = (\vec{r}\cdot\hat{\theta})\hat{\theta} + \left(\vec{r}\cos\theta - (\vec{r}\cdot\hat{\theta})\hat{\theta}\cos\theta\right)$$
$$+\left(\hat{\theta}\times\vec{r}\sin\theta - (\vec{r}\cdot\hat{\theta})\underbrace{\hat{\theta}\times\hat{\theta}}_{0}\sin\theta\right)$$

$$\boxed{\vec{r}\,' = \vec{r}\cos\theta + (\vec{r}\cdot\hat{\theta})\hat{\theta}(1-\cos\theta) + \hat{\theta}\times\vec{r}\sin\theta}$$

In terms of components this yields

$$x_i' = \left[\delta_{ij}\cos\theta + \hat{\theta}_i\hat{\theta}_j(1-\cos\theta) + \epsilon_{ikj}\hat{\theta}_k\sin\theta\right]x_j$$
$$= R_{ij}(\vec{\theta})\,x_j$$

Hence given $\vec{\theta}$ we have

$$\boxed{R_{ij}(\vec{\theta}) = \delta_{ij}\cos\theta + \hat{\theta}_i\hat{\theta}_j(1-\cos\theta) + \epsilon_{ikj}\hat{\theta}_k\sin\theta}$$

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1) A rotation through θ can be built up by successive rotations about $\hat{\theta}$ since

$$R_{ij}(\theta\hat{\theta})\,R_{jk}(\theta'\hat{\theta}) = R_{ik}((\theta+\theta')\hat{\theta})$$

(note: $\displaystyle\lim_{N\to\infty}\left[R\left(\tfrac{\theta}{N}\hat{\theta}\right)\right]^N = R\Big(\underbrace{\left(\tfrac{\theta}{N}+\tfrac{\theta}{N}+\cdots+\tfrac{\theta}{N}\right)}_{N\text{-terms}}\hat{\theta}\Big)$

$= R(\theta\hat{\theta})$)

2) Suppose $\vec{\theta}$ is infinitesimal $\vec{\theta} \equiv \omega\hat{\theta} \equiv \vec{\omega}$

Then $R_{ij}(\vec{\omega}) = \delta_{ij} + \epsilon_{ikj}\,\underbrace{\theta\hat{\theta}_k}_{\equiv\,\omega\hat{\theta}_k = \omega_k}$

2) $R_{ij}(\vec{\omega}) = \delta_{ij} + \epsilon_{ikj}\omega_k$

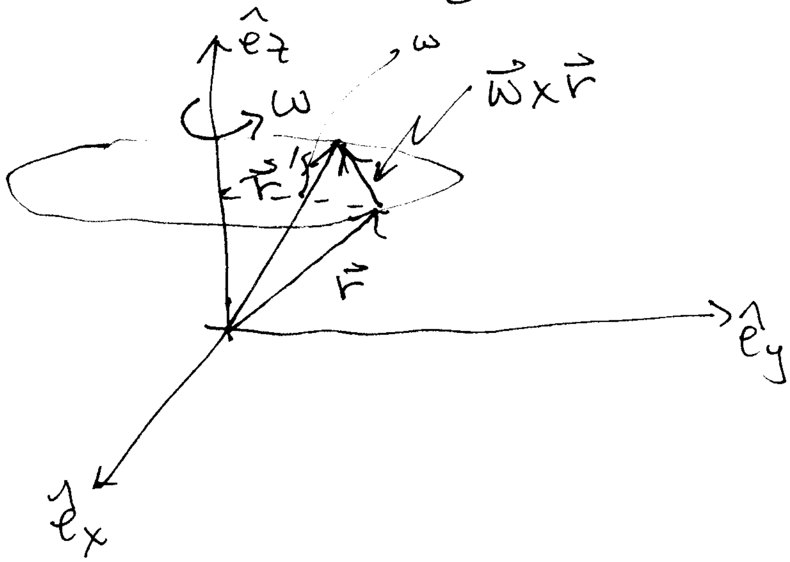
That is $x_i' = R_{ij}(\vec{\omega})\,x_j = x_i + \epsilon_{ikj}\omega_k x_j$

$$\Rightarrow \vec{r}\,' = \vec{r} + \vec{\omega}\times\vec{r}$$

($\vec{\omega}$ is ∡ not ∡'ar velocity here)

<u>ex.</u> Rotation about the z-axis

$$\vec{\omega} = \omega\,\hat{e}_z$$



$\vec{r}\,' = \vec{r} + \vec{\omega}\times\vec{r}$

$x' = x - \omega y$

$y' = y + \omega x$

$z' = z$

<u>Aside:</u> Useful to define $\omega_{ij} \equiv -\epsilon_{ijk}\omega_k$

So $x_i' = x_i + \omega_{ij}x_j$

Then invert $\omega_i = -\frac{1}{2}\epsilon_{ijk}\omega_{jk}$

So $\vec{\omega} = \omega \hat{e}_z \Rightarrow \quad \omega_{12} = -\omega = -\omega_{21}$

and a rotation about z-axis corresponds to a rotation in the $x_1 - x_2$ plane.

In general $\omega_{ij} \leftrightarrow$ rotation in $x_i - x_j$ plane.

Finally we consider sequential rotations

$$\begin{tikzcd} \vec{r} \arrow[r, "R(\vec{\theta}_1)"] \arrow[rr, bend right, "R(\vec{\theta}_3)"'] & \vec{r}\,' \arrow[r, "R(\vec{\theta}_2)"] & \vec{r}\,'' \end{tikzcd}$$

Then if

$$x_i'' = R_{ij}(\vec{\theta}_2)\, x_j'$$

$$x_j' = R_{jk}(\vec{\theta}_1)\, x_k$$

$$\Rightarrow \quad x_i'' = R_{ij}(\vec{\theta}_2) R_{jk}(\theta_1)\, x_k$$

$$= R_{ik}(\vec{\theta}_3)\, x_k$$

$$\Rightarrow R_{ij}(\vec{\theta}_2) R_{jk}(\vec{\theta}_1) = R_{ik}(\vec{\theta}_3)$$

So the composition law for rotations is matrix multiplication. Since $R(\vec{\theta}_1)$ and $R(\vec{\theta}_2)$ are orthogonal with determinant $= 1$ so is the product

$$\det\left(R(\vec{\theta}_2) R(\vec{\theta}_1)\right) = \underbrace{\det R(\vec{\theta}_2) \det R(\vec{\theta}_1)}_{=1}$$

$$\left(R(\vec{\theta}_2) R(\vec{\theta}_1)\right)^T = R(\vec{\theta}_1)^T R(\vec{\theta}_2)^T$$

$$= R^{-1}(\vec{\theta}_1) R^{-1}(\vec{\theta}_2) = \left(R(\vec{\theta}_2) R(\vec{\theta}_1)\right)^{-1}$$

Hence $R(\vec{\theta}_3) = R(\vec{\theta}_2)R(\vec{\theta}_1)$ is also a determinant $=1$ orthogonal matrix, and hence a rotation. Since $R(\vec{0}) = \mathbb{1}$ and $R^{-1}(\vec{\theta}) = R(-\vec{\theta})$ and matrix multiplication is associative we see that the set of all rotations forms a group - the group of 3×3 orthogonal matrices with determinant one. This group is denoted $SO(3)$, the Special ($\det = 1$), Orthogonal ($R^{-1} = R^T$) group of 3×3 matrices. It is a group whose elements $R(\vec{\theta})$ depend continuously on 3 parameters given by $\vec{\theta}$.

Finally to complete the specification of the group multiplication law, we would like to specify $\vec{\theta}_3$ in terms of $\vec{\theta}_1$ and $\vec{\theta}_2$. Clearly this is somewhat messy. Since we can build up finite rotations from successive infinitesimal ones, it will suffice to consider the group product for the transformations

$$
\begin{tikzcd}
\vec{r} \arrow[r, "R^{-1}(\vec{\theta})"] \arrow[d, "(1+\omega')"'] & \vec{r}\,' \arrow[r, "(1+\omega)"] & \vec{r}\,'' \arrow[dll, bend left, "R(\vec{\theta})"] \\
\vec{r}\,''' & &
\end{tikzcd}
$$

Thus
$$x_i''' = R_{ij}(\vec{\theta}) x_j''$$
$$x_j'' = (\delta_{jk} + \omega_{jk}) x_k'$$
$$x_k' = R^{-1}_{kl}(\vec{\theta})\; x_l$$

$\Rightarrow$
$$x_i''' = R_{ij}(\vec{\theta})(\delta_{jk} + \omega_{jk}) R^{-1}_{kl}(\vec{\theta})\; x_l$$

equivalently $\quad x_i''' = (\delta_{il} + \omega'_{il})\, x_l$

$\Rightarrow$
$$\delta_{il} + \omega'_{il} = R_{ij}(\vec{\theta})(\delta_{jk} + \omega_{jk}) R^{-1}_{kl}(\vec{\theta})$$
$$= \delta_{il} + \left(R(\vec{\theta})\,\omega\, R^{-1}(\vec{\theta})\right)_{il}$$

Hence given $\vec{\theta}$ and $\vec{\omega}$ we have the composite rotation resulting from the above sequence of transformations is

$$\boxed{\omega'_{ij} = \left(R(\vec{\theta})\,\omega\, R^{-1}(\vec{\theta})\right)_{ij}}$$

i.e. $R_{ij}(\vec{\omega}') = \delta_{ij} + \omega'_{ij}$ so
$$R(\vec{\omega}')\, R(\vec{\theta}) = R(\vec{\theta})\, R(\vec{\omega})$$

$\left(R(\vec{\omega}') = R(\vec{\theta})\, R(\vec{\omega}) \times R^{-1}(\vec{\theta})\right)$

<u>Next</u>: consider the <u>quantum mechanical</u> implementation of a rotation of the system & measuring apparatus.

<u>Rotations in QM</u>:

If the state of the system is given by $|\psi\rangle$ before the system and the measuring apparatus are rotated, it is given by $|\psi'\rangle$ after it is rotated by $R(\vec{\theta})$ where

$$|\psi'\rangle = U(R(\vec{\theta}))\,|\psi\rangle .$$

Since any proper rotation can be written as a square $R(\vec{\theta}) = R^2(\tfrac{1}{2}\vec{\theta})$ we have

$$U(R(\vec{\theta})) = U^2(R(\tfrac{1}{2}\vec{\theta})),$$

hence $U(R)$ is unitary because the product of two unitary or two anti-unitary operators is unitary. So

$$U^\dagger(R(\vec{\theta})) = U^{-1}(R(\vec{\theta})) .$$

Since equivalence of physical states only requires the vectors to be equal up to a phase (if they are normalized the same) we have that the unitary operators $U(R)$ obey the rotation group multiplication law only up to a phase

$$\begin{tikzcd} \vec{r} \arrow[r, "R(\vec{\theta}_1)"] \arrow[rr, bend right] & \vec{r}\,' \arrow[r, "R(\vec{\theta}_2)"] & \vec{r}\,'' \end{tikzcd}$$

$$R(\vec{\theta}_3) = R(\vec{\theta}_2) R(\vec{\theta}_1)$$

$$\begin{tikzcd} |\psi\rangle \arrow[r, "U(R(\vec{\theta}_1))"] \arrow[rr, bend right] & |\psi'\rangle \arrow[r, "U(R(\vec{\theta}_2))"] & e^{+i\alpha(\vec{\theta}_2, \vec{\theta}_1)} |\psi''\rangle \\ & & |\psi''\rangle \end{tikzcd}$$

$$U(R(\vec{\theta}_3)) = e^{-i\alpha(\vec{\theta}_2, \vec{\theta}_1)} U(R(\vec{\theta}_2)) U(R(\vec{\theta}_1))$$

So

$$\boxed{U(R(\vec{\theta}_3)) = e^{-i\alpha(\vec{\theta}_2, \vec{\theta}_1)} U(R(\vec{\theta}_2)) U(R(\vec{\theta}_1))}$$

Now we can keep track of the phases and show at the end that it is equivalent to having no phase at all by re-defining the generators for rotations to include the phase in a consistent manner.

So we might as well set the phase to zero to begin with just to keep the details less cumbersome.

So $$\boxed{U(R(\vec{\theta}_3)) = U(R(\vec{\theta}_2))\,U(R(\vec{\theta}_1))}$$

Now since $R_{ij}(\vec{\theta} \to 0) \to \delta_{ij}$ we assume

$U(R(\vec{\theta}))$ depends continuously on $\vec{\theta}$ ($\delta_{ij} + \omega_{ij}$)

so that $\quad U(\delta_{ij} + \omega_{ij}) \to \underset{\text{identity in Hilbert space}}{\mathbb{1}}$ as $\omega_{ij} \to 0$

So as $\omega \to 0$

$$\mathbb{1} = U^{\dagger} U = [1 + (U-1)]^{\dagger}\,[1 + (U-1)]$$
$$= \mathbb{1} + (U - \mathbb{1}) + (U - \mathbb{1})^{\dagger} \quad \text{to}$$

first order in ω

$$\Rightarrow \boxed{U - \mathbb{1} = i(\text{Hermitian operator})}$$

Define $$U(1+\omega) \equiv \mathbb{1} + \frac{i}{2}\,\omega_{ij}\,\frac{J_{ij}}{\hbar}$$

first order

Hermitean operator.

Now recall that $\omega_i = -\frac{1}{2}\epsilon_{ijk}\omega_{jk}$
and inverse

$$\omega_{ij} = \epsilon_{ijk}\omega_k$$

$\Rightarrow$ $\boxed{\omega_{ij} = -\omega_{ji}}$ anti-symmetric 3x3 real matrix $\Leftrightarrow$ vector in 3 space

Hence we have $J_{ij} = -J_{ji}$

and J_{ij} is Hermitian $J_{ij} = J_{ij}^{\dagger}$
as an operator in Hilbert space.

The J_{ij} are 3-independent Hermitian operators.

They are the generators of <u>spatial</u> <u>rotations</u>
in Hilbert space.

<u>Note:</u> $J_{ij} = -J_{ji}$ $\Rightarrow$ $J_i \equiv \frac{1}{2}\epsilon_{ijk}J_{jk}$

	Rotation about
$J_1 \equiv J_{23}$	x
$J_2 \equiv J_{31}$	y
$J_3 \equiv J_{12}$	z

invert:

$$J_{ij} = \epsilon_{ijk}J_k.$$

Since $R(\vec{\theta}_3) = R(\vec{\theta}_2) R(\vec{\theta}_1)$ according to the rotation group multiplication law we have that the unitary operators represent this group up to a phase. As we have seen $\vec{\theta}_3$ is determined by $\vec{\theta}_1$ and $\vec{\theta}_2$, as well $U(R(\vec{\theta}_3))$ is determined by $U(R(\vec{\theta}_1))$ and $U(R(\vec{\theta}_2))$ (up to a phase).
This group multiplication law will imply algebraic relations (commutation relations) for the generators J_{ij}. Consider again the group multiplication law as embodied in the successive transformations

$$\begin{array}{ccccc} \vec{r} & \xrightarrow{R^{-1}(\vec{\theta})} & \vec{r}\,' & \xrightarrow{(1+\omega)} & \vec{r}\,'' \\ \big\downarrow {\scriptstyle (1+\omega')} & & & & \\ \vec{r}\,''' & & \xleftarrow{R(\vec{\theta})} & & \end{array}$$

The group multiplication law

$$R(\vec{\omega}') = R(\vec{\theta}) R(\vec{\omega}) R^{-1}(\vec{\theta})$$

$$\Rightarrow \quad \omega'_{ij} = \left(R(\vec{\theta})\, \omega\, R^{-1}(\vec{\theta})\right)_{ij}$$

Quantum mechanically we have that

$$\begin{array}{ccccc} |\psi\rangle & \xrightarrow{U(R^{-1})} & |\psi'\rangle & \xrightarrow{U((1+\omega))} & |\psi''\rangle \\ \big\downarrow {\scriptstyle U(1+\omega')} & & & & \\ |\psi'''\rangle = e^{+i\alpha(\omega,\vec{\theta})}|\psi'''\rangle & & \xleftarrow{U(R)} & & \end{array}$$

$\boxed{\text{Again, let } \alpha = 0}$

Thus
$$\boxed{\mathcal{U}(\vec{R}(|\vec{\theta}|))\,\mathcal{U}(1+\omega)\,\mathcal{U}(R^{-1}(|\vec{\theta}|)) = \mathcal{U}(1+\omega')}$$

Recall $\mathcal{U}(R^{-1}) = \mathcal{U}^{-1}(R)$ i.e. $\mathcal{U}(R)\mathcal{U}(R^{-1}) = \mathcal{U}(RR^{-1}) = \mathcal{U}(\mathbb{1}) = \mathbb{1} \Rightarrow \mathcal{U}^{-1}(R) = \mathcal{U}(R^{-1})$

and $\omega' = R(\vec{\theta})\,\omega\,R^{-1}(\vec{\theta})$

i.e. $\omega'_{ij} = R_{ik}(\vec{\theta})\,\omega_{kl}\,R^{-1}_{lj}(\vec{\theta})$

So
$$\mathcal{U}(1+\omega') = \mathbb{1} + \frac{i}{2}\frac{1}{\hbar}\left(R(\vec{\theta})\omega R^{-1}(\vec{\theta})\right)_{kl} J_{kl}$$

$$\mathcal{U}(1+\omega) = \mathbb{1} + \frac{i}{2}\frac{1}{\hbar}\,\omega_{ij} J_{ij}$$

Now use the above group (SO(3)) multiplication law:

$$U(R(\vec{\theta}))\left[\mathbb{1} + \frac{i}{2}\frac{\omega_{ij}}{\hbar} J_{ij}\right] U^{-1}(R(\vec{\theta}))$$
$$= \mathbb{1} + \frac{i}{2}\frac{1}{\hbar}\left(R(\vec{\theta})\,\omega\, R^{-1}(\vec{\theta})\right)_{k\ell} J_{k\ell}$$

$$\Longrightarrow$$

$$\frac{i}{2}\frac{\omega_{ij}}{\hbar}\; U(R(\vec{\theta}))\, J_{ij}\, U^{-1}(R(\theta))$$
$$= \frac{i}{2}\frac{\omega_{ij}}{\hbar}\left[R_{ki}(\vec{\theta})\, R^{-1}_{j\ell}(\vec{\theta})\, J_{k\ell}\right]$$

Since ω_{ij} is arbitrary $\Rightarrow$

$$\boxed{U(R(\vec{\theta}))\, J_{ij}\, U^{-1}(R(\vec{\theta})) = R_{ki}(\vec{\theta})\, R^{-1}_{j\ell}(\vec{\theta})\, J_{k\ell}}$$

Now $\vec{\theta}$ is completely arbitrary, So choose it to be infinitesimal ~~also~~ let

$$\vec{\theta} = \vec{\alpha}\,; \qquad \alpha_i = -\tfrac{1}{2}\epsilon_{ijk}\,\alpha_{jk}$$
$$\alpha_{ij} = -\epsilon_{ijk}\,\alpha_k\,; \quad \alpha_{ij} = -\alpha_{ji}$$

$$\Rightarrow \boxed{R_{ij}(\vec{\alpha}) = \delta_{ij} + \alpha_{ij}\,;\; R^{-1}_{ij}(\vec{\alpha}) = \delta_{ij} - \alpha_{ij}}$$

$$U(1+\alpha) = \mathbb{1} + \frac{i}{2}\frac{\alpha_{ij}}{\hbar} J_{ij} \quad ; \quad J_{ij} = -J_{ji}$$

So we find

$$\left(\mathbb{1} + \frac{i}{2}\frac{\alpha_{mn}}{\hbar} J_{mn}\right) J_{ij} \left(\mathbb{1} - \frac{i}{2}\frac{\alpha_{mn}}{2} J_{mn}\right)$$

$$= J_{kl}(\delta_{ki} + \alpha_{ki})(\delta_{jl} - \alpha_{jl})$$

factor out a common α_{mn} (keep linear in α_{mn})

$$\Rightarrow \frac{i}{2}\frac{\alpha_{mn}}{\hbar}[J_{mn}, J_{ij}] = \alpha_{mn}[\delta_{ni} J_{mj} - \delta_{mj} J_{in}]$$

Since $\alpha_{mn} = -\alpha_{nm}$, we must anti-symmetrize the RHS before we can set the "J"-terms equal

$$\frac{i}{2}\frac{\alpha_{mn}}{\hbar}[J_{mn}, J_{ij}] = \frac{\alpha_{mn}}{2}\left[\delta_{ni} J_{mj} - \delta_{mj} J_{in} - \delta_{mi} J_{nj} + \delta_{nj} J_{im}\right]$$

Now since α_{mn} is arbitrary $\Rightarrow$

$$\boxed{[J_{mn}, J_{ij}] = -i\hbar\left(\delta_{in} J_{mj} - \delta_{im} J_{nj} + \delta_{jn} J_{im} - \delta_{jm} J_{in}\right)}$$

The angular momentum (SO(3)) commutation relations.

To make this look more familiar recall that $J_i = \frac{1}{2}\epsilon_{ijk} J_{jk}$; So

$$\frac{1}{2}\epsilon_{kmn}\frac{1}{2}\epsilon_{lij}[J_{mn}, J_{ij}] = [J_k, J_l]$$

$$= -\frac{i\hbar}{4}\Big[\epsilon_{kmn}\epsilon_{lij}\delta_{in} J_{mj} - \epsilon_{kmn}\epsilon_{lij}\delta_{im} J_{nj} + \epsilon_{kmn}\epsilon_{lij}\delta_{jn} J_{im} - \epsilon_{kmn}\epsilon_{lij}\delta_{jm} J_{in}\Big]$$

$$= -\frac{i\hbar}{4}\Big(\epsilon_{kmi}\epsilon_{lij} J_{mj} - \underbrace{\epsilon_{kin}\epsilon_{lij} J_{nj}}_{\text{relabel } n\to m} + \epsilon_{kmj}\epsilon_{lij} J_{im} - \underbrace{\epsilon_{kjn}\epsilon_{lij} J_{in}}_{\text{relabel } n\to m}\Big)$$

$$[J_k, J_l] = -\frac{i\hbar}{4}\epsilon_{lij}\Big(\epsilon_{kmi}J_{mj} - \overset{-\epsilon_{kmi}}{\overset{\parallel}{\epsilon_{kim}}}J_{mj}$$

$$+\epsilon_{kmj}J_{im} - \underset{-\epsilon_{kmj}}{\underset{\parallel}{\epsilon_{kjm}}}J_{im}\Big)$$

$$= -\frac{i\hbar}{2}\Big(\epsilon_{lij}\epsilon_{kmi}J_{mj}$$

$$+\underbrace{\epsilon_{lij}\epsilon_{kmj}J_{im}}_{i\leftrightarrow j}\Big)$$

$$= \epsilon_{lji}\epsilon_{kmi}J_{jm}$$

$$= -\epsilon_{lij}\epsilon_{kmi}(-J_{mj})$$

$$= +\epsilon_{lij}\epsilon_{kmi}J_{mj}$$

$$= -i\hbar\,\epsilon_{lij}\epsilon_{kmi}J_{mj}$$

$$= +i\hbar\,\underbrace{\epsilon_{lji}\epsilon_{kmi}}_{}J_{mj}$$

$$= (\delta_{lk}\delta_{jm} - \delta_{lm}\delta_{jk})$$

$$= i\hbar\,\delta_{lk}\cancelto{0}{J_{mm}} - i\hbar J_{lk} = +i\hbar J_{kl}$$

$$= \boxed{i\hbar\,\epsilon_{klm}J_m = [J_k, J_l]}$$

So we obtain the familiar form of the (SO(3)) angular momentum commutation relations
$\overset{\parallel}{SU(2)}$

$$\boxed{\begin{array}{l} [J_i, J_j] = i\hbar\, \epsilon_{ijk} J_k \\ \text{i.e.} \quad [J_x, J_y] = i\hbar J_z \quad , \text{ plus cyclic permutations.} \end{array}}$$

So for infinitesimal rotations, $R_{ij} = \delta_{ij} + \omega_{ij}$ we have found that

$$U(1+\omega) = \mathbb{1} + \frac{i}{2}\frac{\omega_{ij}}{\hbar} J_{ij}$$

but using $J_{ij} = \epsilon_{ijk} J_k$; $\omega_k = -\frac{1}{2}\epsilon_{kij}\omega_{ij}$

$$\frac{1}{2}\omega_{ij} J_{ij} = \frac{1}{2}\omega_{ij}\epsilon_{ijk} J_k = -\omega_k J_k$$
$$= -\vec{\omega}\cdot\vec{J}$$

$$U(1+\omega) = \mathbb{1} - \frac{i}{\hbar}\vec{\omega}\cdot\vec{J}$$

with $[J_i, J_j] = i\hbar\epsilon_{ijk} J_k$ obeying the SO(3) algebra.

Since $[\vec{\theta}\cdot\vec{J}, \vec{\theta}\cdot\vec{J}]=0$, we can build up any finite rotation transformation corresponding to the rotation $R(\vec{\theta})$ by successive infinitesimal rotations about the same axis $\hat{\theta}$ through angle $\frac{\theta}{N}$:

For finite rotations $R(\vec{\theta})$ we have

$$U(R(\vec{\theta})) = \lim_{N\to\infty}\left[U\left(R\left(\frac{\vec{\theta}}{N}\right)\right)\right]^N$$

$$= \lim_{N\to\infty}\left[\mathbb{1} - \frac{i}{\hbar}\frac{1}{N}\vec{\theta}\cdot\vec{J}\right]^N$$

$$\boxed{U(R(\vec{\theta})) = e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}}}$$

Note: Since $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$

$$e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}} = e^{-\frac{i}{\hbar}(\theta_x J_x + \theta_y J_y + \theta_z J_z)}$$

$$\neq e^{-\frac{i}{\hbar}\theta_x J_x}\, e^{-\frac{i}{\hbar}\theta_y J_y}\, e^{-\frac{i}{\hbar}\theta_z J_z}$$

Note: If H is Rotationally invariant:

$$H' = U(R(\vec{\theta}))\, H\, U^\dagger(R(\vec{\theta})) = H$$

$$\Leftrightarrow \quad [\vec{J}, H] = 0$$

Then Ehrenfest's Thm. $\Rightarrow$

$$\boxed{\frac{d}{dt}\langle \vec{J} \rangle = 0} \qquad \frac{d}{dt}\langle \vec{J} \rangle = \frac{i}{\hbar}\underbrace{\langle [H, \vec{J}] \rangle}_{=0} + \underbrace{\left\langle \cancel{\frac{\partial \vec{J}}{\partial t}} \right\rangle}_{0}$$

$\langle \vec{J} \rangle$ is constant in time hence

$\langle$ total $\measuredangle$ momentum $\rangle$ is conserved.

We have

Homogeneity of Time + Time translation Invariance (H time indep.) $\Longleftrightarrow$ $\langle$ Energy $\rangle$ is conserved (H = Energy)

Homogeneity of Space + Space Translation Invariance ($[\vec{P}, H] = 0$) $\Longleftrightarrow$ $\langle$ Total Linear Momentum $\rangle$ is conserved ($\vec{P}$ = Total Linear Momentum)

Isotropy of Space + Space Rotation Invariance ($[\vec{J}, H] = 0$) $\Longleftrightarrow$ $\langle$ Total $\measuredangle$ momentum $\rangle$ is conserved ($\vec{J}$ = Total $\measuredangle$ momentum)
