4.) Time Inversion

Besides inverting the space coordinates, we can imagine a transformation that reverses the direction of time. More accurately, we can imagine that the motion of a system is reversed, momentum flowing in the opposite direction, direction of angular momentum opposite its original direction etc. This motion reversal, as we shall see, is equivalent to letting $t \to -t$ in our states.

A time reversal transformation is defined by $\vec{r}\,' = \vec{r}$ but

$t' = -t$. Thus we

define the operator that relates the states in these two frames as $U_T \equiv T$

$|\psi'\rangle = T|\psi\rangle$. The transformation

is defined to leave the coordinates unchanged so we define i.e. $T|\vec{r}\rangle = |\vec{r}\rangle$, $T|\vec{p}\rangle = |{-\vec{p}}\rangle$

$$T\vec{R}T^{-1} = \vec{R}.$$

But $t \to -t$ the velocities and hence momentum should be reversed (motion reversal), so we define

$$T\vec{P}T^{-1} = -\vec{P}.$$

Since $\vec{L} = \vec{R} \times \vec{P} \Rightarrow T\vec{L}T^{-1} = -\vec{L}$.

Now if the commutation relations are to be the same in each frame we must have

$$T[X^i, P^j]T^{-1} = T i\hbar\delta^{ij} T^{-1}$$

$$= TX^iT^{-1}TP^jT^{-1} - TP^jT^{-1}TX^iT^{-1}$$

$$= -X^iP^j + P^jX^i$$

$$= -[X^i, P^j]$$

$$= -i\delta^{ij}\hbar \quad = \hbar\delta^{ij}\, TiT^{-1}$$

$$\Rightarrow \boxed{TiT^{-1} = -i \Rightarrow Ti = -iT}$$

T must be <u>anti-linear</u>, hence (by Wigner's Theorem) it is <u>anti-unitary</u>

Since two time reversal operations result in the original system we have that

$$T^2|\psi\rangle = e^{i\varphi}|\psi\rangle \quad ; \varphi \in \mathbb{R}.$$

Using the <u>associative property</u> of <u>operator multiplication</u> we find

$$T^3|\psi\rangle = T^2(T|\psi\rangle) = T(T^2|\psi\rangle)$$

$$= T(e^{i\varphi}|\psi\rangle)$$

but T is anti-linear so

$$= e^{-i\varphi}(T|\psi\rangle).$$

Now for the sum of 2 states $(|\psi\rangle + T|\psi\rangle)$ we also have

$$T^2(|\psi\rangle + T|\psi\rangle) = e^{i\varphi'}(|\psi\rangle + T|\psi\rangle)$$

$$\parallel$$

$$T^2|\psi\rangle + T^3|\psi\rangle = e^{i\varphi}|\psi\rangle + e^{-i\varphi}T|\psi\rangle$$

$$\Rightarrow e^{i\varphi}|\psi\rangle + e^{-i\varphi}\,T|\psi\rangle$$
$$= e^{i\varphi'}\left(|\psi\rangle + T|\psi\rangle\right)$$
$$\Rightarrow e^{i\varphi} = e^{i\varphi'} = e^{-i\varphi} \Rightarrow \boxed{\varphi = 0 \text{ or } \pi \text{ for all phases}}$$

Thus $\boxed{T^2|\psi\rangle = \pm|\psi\rangle}$, 2 successive time reversal transformations need not be the identity.

Now consider the Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

Operate with T $\Rightarrow$

$$-i\hbar \frac{d}{dt}\left(T|\psi(t)\rangle\right) = THT^{-1}\left(T|\psi(t)\rangle\right)$$

T is anti-linear

Now let $t \to -t$

$\Rightarrow$

$$+i\hbar \frac{d}{dt}\left(T|\psi(-t)\rangle\right) = \underbrace{\left(T H T^{-1}\right)}_{H'}\left(T|\psi(-t)\rangle\right)$$

Now if $H' = H$ so that the Hamiltonian is time reversal invariant, then

$$\Rightarrow \; i\hbar \frac{d}{dt}\left(T|\psi(-t)\rangle\right) = H\left(T|\psi(-t)\rangle\right)$$

$\Rightarrow$ if $|\psi(t)\rangle$ is a solution to Sch.-eq.

So is $\left(T|\psi(-t)\rangle\right)$ a " " " "

(Note: for linear (unitary) symmetry operator the invariance of $H \Rightarrow$ conserved quantity — in the case of anti-linear (anti-unitary) the invariance of H does not imply a conserved quantity but instead solutions of Sch. Eq. come in pairs $|\psi(t)\rangle$ and $\left(T|\psi(-t)\rangle\right)$ (i.e. for stationary states $\xrightarrow{\text{sometimes}}$ degeneracy))

Suppose $H = \frac{\vec{P}^2}{2m} + V(\vec{R})$; m = real > 0.

$$H' = THT^{-1} = \frac{\vec{P}^2}{2m} + V^*(\vec{R})$$

if $H' = H \Rightarrow V^*(\vec{R}) = V(\vec{R})$

The potential is real if H is T-invariant.