## 2) Time Translation



Now, in similar manner, ~~Pumer~~ experimentally we also observe the homogeneity of time i.e the zero of time is inconsequential. The same experiment performed at 2 different times yields the same result.

2 Observers $t' = t + t_0 \Rightarrow$

Both start with same initial state $|\phi\rangle$

$$\begin{cases} |\psi(t')\rangle = |\phi\rangle \\ |\psi(t)\rangle = |\phi\rangle \end{cases}$$

Let the states evolve time $\epsilon \Rightarrow$ the states should be the same ~~state~~

$$|\psi(t'+\epsilon)\rangle = |\psi(t+\epsilon)\rangle$$

Now by Postulate 4 we know the time evolution generator — Hamiltonian

$$|\psi(t+\epsilon)\rangle = |\psi(t)\rangle + \epsilon \frac{\partial}{\partial t}|\psi(t)\rangle$$

$$= |\psi(t)\rangle - \frac{i\epsilon}{\hbar} \underbrace{i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle}_{= H(t)|\psi(t)\rangle}$$

$$= \left[1 - \frac{i\epsilon}{\hbar}H(t)\right]|\phi\rangle$$

Likewise

$$|\psi(t'+\epsilon)\rangle = [1 - \frac{i\epsilon}{\hbar} H(t')]|\phi\rangle$$

Now homogeneity of time $\Rightarrow$

$$|\psi(t'+\epsilon)\rangle = |\psi(t+\epsilon)\rangle$$

$$\Rightarrow \quad -\frac{i\epsilon}{\hbar} H(t')|\phi\rangle = -\frac{i\epsilon}{\hbar} H(t)|\phi\rangle$$

Since $|\phi\rangle$ is arbitrary $\Rightarrow$

$$H(t') = H(t)$$

or $$\boxed{\frac{dH}{dt} = 0}$$

So time translation invariance requires

$H(t+t_0) = H(t)$ to be time independent.

Now then Ehrenfest's thm $\Rightarrow$

$$\frac{d}{dt}\langle H\rangle = \frac{i}{\hbar}\langle\underbrace{[H,H]}_{=0}\rangle + \langle \cancel{\frac{\partial H}{\partial t}}^{\,0}\rangle$$

$$\Rightarrow \quad \boxed{\frac{d}{dt}\langle H\rangle = 0 \quad \text{Energy is conserved}}$$

If $H$ is time independent then we can solve Schrödinger's Eq. "easily"

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle$$

← time independent.

$$\Rightarrow$$

$$U(t,t_0)|\psi(t_0)\rangle = |\psi(t)\rangle = e^{-\frac{i}{\hbar}(t-t_0)H} |\psi(t_0)\rangle$$

$$(\text{let } t_0 = 0) \quad = e^{-\frac{i}{\hbar}tH} |\psi(0)\rangle$$

$$\Leftrightarrow \boxed{U(t) = e^{-\frac{i}{\hbar}tH}}$$

Unitary time evolution operator.
(or time translation)

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Time translation is abelian group

$$U(t_1)U(t_2) = U(t_1+t_2)$$
$$= U(t_2)U(t_1)$$
$$\Leftrightarrow [H, H] = 0.$$

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Experimentally all known ~~forces~~: interactions: e-m, weak, strong, grav. are space & time translation invariant.

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