Symmetry in QM

2-ways to view equivalent descriptions of a system (along with its preparation and measuring apparati)

Active View : The system is prepared in a particular state ($|\psi\rangle$) and the observables (given by operator A) are measured at position $\vec{r}$ (and time t). In the same coordinate system, the system, the preparation apparatus and the observation apparatus is moved from $(\vec{r}, t)$ to a new position $\vec{r}'$ (and t'). Corresponding to the state $|\psi\rangle$ prepared at $(\vec{r}, t)$, the same state is prepared at $(\vec{r}', t')$. Denote this state $|\psi'\rangle$. In the first case the eigenstates of observable A are $|\phi\rangle$ while in the second case denote them $|\phi'\rangle$. The probability of measuring "ϕ" when A is measured with the system in state $|\psi\rangle$ is

$$|\langle\phi|\psi\rangle|^2 .$$

Likewise when measuring the same observable at $\vec{r}'$ (and t') we find

$|\langle\phi'|\psi'\rangle|^2$. Since this is the same experiment on the same system in the same state, we have

$$|\langle\phi|\psi\rangle|^2 = |\langle\phi'|\psi'\rangle|^2$$

(~~Clearly~~ Assume $t=t'$, or more correctly, we have 2-identical experimental setups at the 2 locations $\vec{r}$ & $\vec{r}'$ and the experiment is done at t (simultaneously, non-relativistically).)

Passive View: The one-system and measurement apparatus is viewed by 2-observers O and O' using different inertial frames $(\vec{r}, t)$ and $(\vec{r}', t')$. Observer O measures

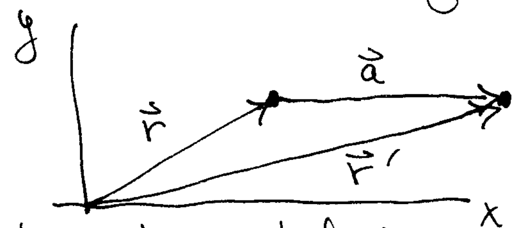
$|\langle\phi|\psi\rangle|^2$ while O' measures $|\langle\phi'|\psi'\rangle|^2$.

Since they are describing the same physical situation we require

$$|\langle\phi'|\psi'\rangle|^2 = |\langle\phi|\psi\rangle|^2 .$$

($|\psi'\rangle$ is a state in the Hilbert space of O that looks like $|\psi\rangle$ to O'.)

Ex. Active View of Translation:



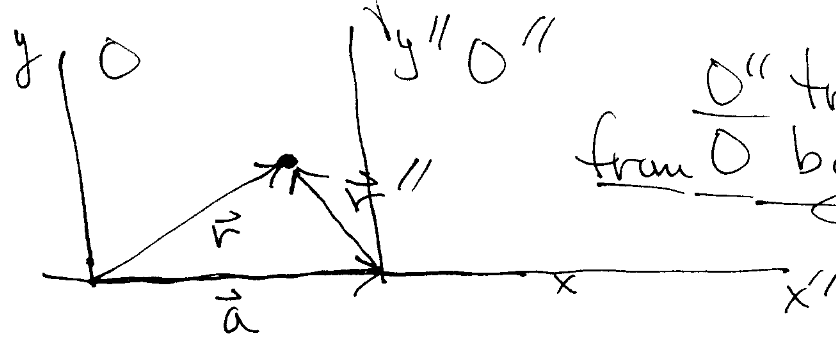
system translated from $\vec{r}$ to $\vec{r}+\vec{a}$

$$\vec{r}\,' = \vec{r} + \vec{a}$$

$$\psi'(\vec{r}\,') = \psi(\vec{r})$$

$$\Rightarrow \psi'(\vec{r}) = \psi(\vec{r}-\vec{a})$$

Passive View of Translation:



O'' translated from O by $\vec{a}$

$$\vec{r}\,'' = \vec{r} - \vec{a}$$

Still

$$\psi''(\vec{r}\,'') = \psi(\vec{r})$$

$$\Rightarrow \psi''(\vec{r}) = \psi(\vec{r}+\vec{a})$$

So moving system and equipment one direction (active: $\vec{r}\,' = \vec{r}+\vec{a}$, $\psi'(\vec{r}) = \psi(\vec{r}-\vec{a})$) is equivalent to moving the second observer the opposite direction (passive: $\vec{r}\,'' = \vec{r}-\vec{a}$, $\psi''(\vec{r}) = \psi(\vec{r}+\vec{a})$). Choose to work in one view: (active).

a) The Active view, moving the system & apparatus one way, is equivalent in the passive view to the observers being moved in the (inverse) opposite way. For example, if the apparatus is rotatated about the z-axis through angle φ; that is the same as observer O' using an inertial frame rotated about the z-axis of O (with common origin) by angle $-\varphi$.

b) We can generalize this discussion to include not only space-time transformations but other relationships between observers. For instance they could use 2 different conventions for the sign of electric charge, etc..

To be specific, let's consider a spatial translation as the transformation between cases.

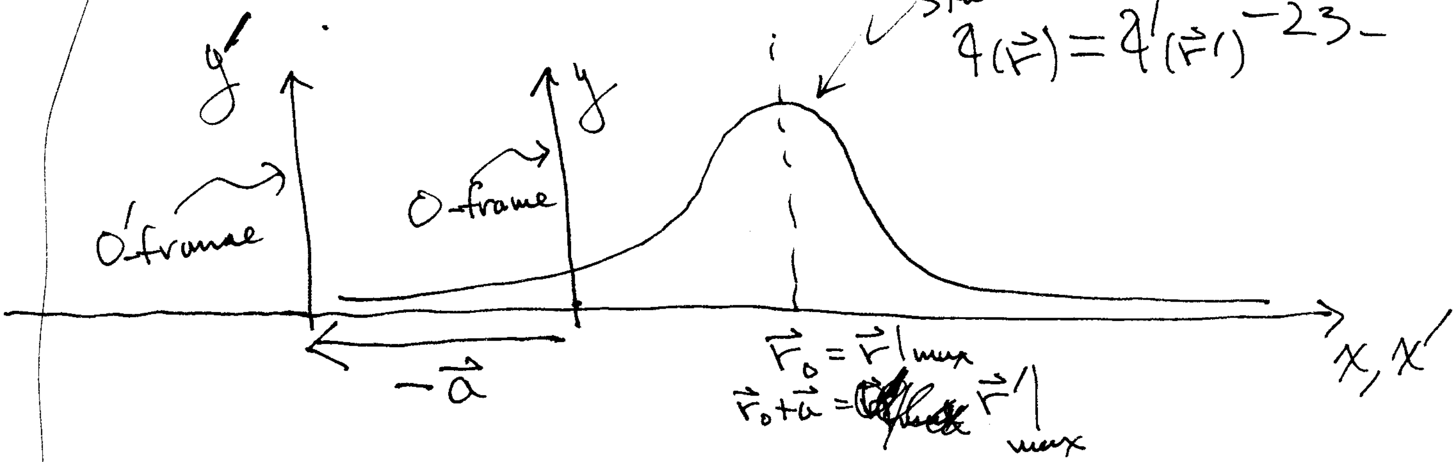
Active: prepare system in state $|\psi\rangle$ and measure A, all at $\vec{r}$. Then <u>translate</u> all apparatus & system to coordinate $\vec{r} + \vec{a} = \vec{r}\,'$.

The state is $|\psi'\rangle$. In <u>wavemechanics</u> the first case is described by wavefunction $\psi(\vec{r})$ while the second <u>identical</u> but translated system is described by $\psi'(\vec{r})$. Since the states of the system are prepared identically we have

$$\psi'(\vec{r}+a) = \psi(\vec{r}),$$

all we did was translate all the equipment etc from $\vec{r}$ to $\vec{r}+\vec{a}$.
So if $\psi(\vec{r})$ had a maximum at $\vec{r}_0$ then the same state prepared at the translated cite has a maximum at $\vec{r}_0+\vec{a}$.

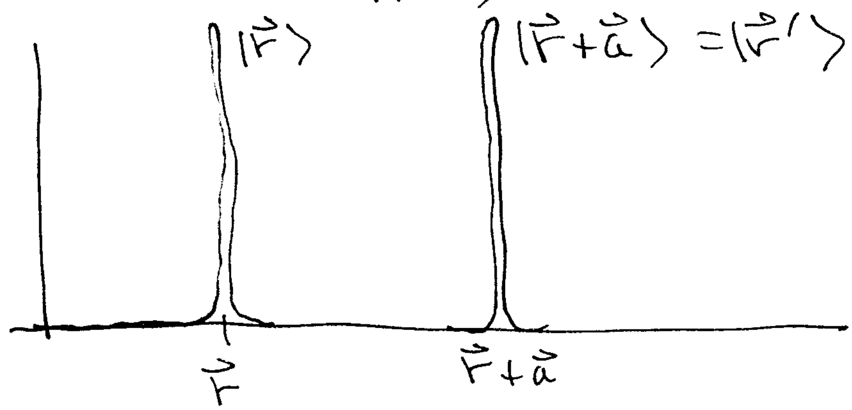
From the <u>passive</u> point of view the system is left stationary. The second observer has a coordinate frame displaced by $-\vec{a}$ w.r.t. the first observer



$$\vec r\,' = \vec r + \vec a$$

Suppose this was a very peaked state so that $|\psi\rangle \to |\vec r\rangle$
$|\psi'\rangle \to |\vec r\,'\rangle$

<u>Active view</u>



Now suppose we have a (unitary) (Show since $\vec p$ is Hermitian) operator $U(\vec a)$ that relates the states $|\psi\rangle$ and $|\psi'\rangle$ in our Hilbert space of states

$$|\psi'\rangle = U(\vec a)\,|\psi\rangle$$

In the coordinate representation, we have

$$\psi'(\vec{r}) = \langle \vec{r} | \psi' \rangle = \langle \vec{r} | U(\vec{a}) | \psi \rangle$$

But $\psi'(\vec{r}+\vec{a}) = \psi(\vec{r}) \Rightarrow \psi'(\vec{r}) = \psi(\vec{r}-\vec{a})$

we can Taylor expand $\psi(\vec{r}-\vec{a})$ about $\vec{r}$

$$\psi'(\vec{r}) = \psi(\vec{r}-\vec{a}) = \psi(\vec{r}) - \vec{a}\cdot\vec{\nabla}_{\vec{r}}\psi(\vec{r}) + \frac{1}{2!}(-\vec{a}\cdot\vec{\nabla}_{\vec{r}})^2\psi(\vec{r}) + \cdots$$

$$= e^{-\vec{a}\cdot\vec{\nabla}_{\vec{r}}}\psi(\vec{r}) = e^{-\vec{a}\cdot\vec{\nabla}_{\vec{r}}}\langle \vec{r} | \psi \rangle$$

In the coordinate basis:

$$\langle \vec{r} | \vec{\underline{P}} = \frac{\hbar}{i}\vec{\nabla}_{\vec{r}}\langle \vec{r} |$$

So

$$\psi'(\vec{r}) = \langle \vec{r} | \psi' \rangle = e^{-\vec{a}\cdot\vec{\nabla}_{\vec{r}}}\langle \vec{r} | \psi \rangle$$

$$= \langle \vec{r} | e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{\underline{P}}} | \psi \rangle$$

$$= \langle \vec{r} | U(\vec{a}) | \psi \rangle$$

Since $|\psi\rangle$ is arbitrary & $\langle \vec{r} |$ is a basis vector

$\Longrightarrow$

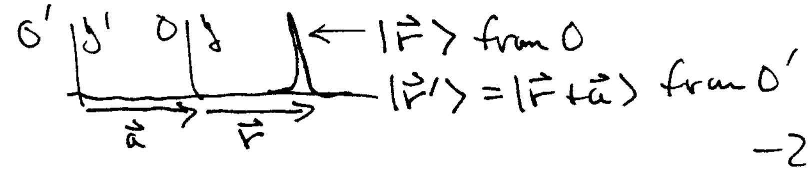
$$\boxed{\boxed{U(\vec{a}) = e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}}}}$$

The momentum operator is said to be the "generator of space translations" for states in the Hilbert space. Since $\vec{P} = \vec{P}^{\dagger}$, $\vec{P}$ is Hermitean $\Rightarrow$ $U(\vec{a})$ is unitary

$$U^{\dagger}(\vec{a}) = e^{+\frac{i}{\hbar}\vec{a}\cdot\vec{P}^{\dagger}} = e^{+\frac{i}{\hbar}\vec{a}\cdot\vec{P}}$$
$$= U^{-1}(\vec{a})$$

Note: $U^{\dagger}(\vec{a}) = U^{-1}(\vec{a}) = U(-\vec{a})$

Hence $\langle\phi'|\psi'\rangle = \langle\phi|U^{\dagger}(\vec{a})U(\vec{a})|\psi\rangle$
$$= \langle\phi|\psi\rangle \quad \text{and}$$
$$\Rightarrow \quad |\langle\phi'|\psi'\rangle|^2 = |\langle\phi|\psi\rangle|^2 \quad \text{as required.}$$

Passive View 

Mutatis Mutandis: p. -23- if $|\psi\rangle \to |\vec r\rangle$

Homogeneity of space ⇒ have there same ~~exact~~ results $|\psi'\rangle \to |\vec r'\rangle$

then $|\vec r'\rangle = U(\vec a)\,|\vec r\rangle = |\vec r + \vec a\rangle$

$\cancel{|\vec r + \vec a\rangle}$

$$1) \Rightarrow \quad \psi'(\vec r) = \langle \vec r|\psi'\rangle = \langle \vec r|\,U(\vec a)\,|\psi\rangle$$

$$= \langle \vec r|\,U^{\dagger}(-\vec a)\,|\psi\rangle = \langle \vec r - \vec a|\psi\rangle$$

$$= \psi(\vec r - \vec a)$$

$$2) \quad \vec R\, U^{-1}(\vec a)\,|\vec r\rangle = \vec R\,|\vec r - \vec a\rangle = (\vec r - \vec a)\,|\vec r - \vec a\rangle$$

$$= (\vec r - \vec a)\,U^{-1}(a)\,|\vec r\rangle$$

$$\Rightarrow$$

$$U(\vec a)\,\vec R\,U^{-1}(\vec a)\,|\vec r\rangle = (\vec r - \vec a)\,|\vec r\rangle$$

$$= (\vec R - \vec a)\,|\vec r\rangle$$

$$\Rightarrow$$

$$\boxed{\vec R' \equiv U(\vec a)\,\vec R\,U^{-1}(\vec a) = \vec R - \vec a}$$

Suppose $U(\vec{a}) = e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}}$ ← some operator $\vec{P}$.

$$U(\vec{\epsilon}) = 1 - \frac{i}{\hbar}\vec{\epsilon}\cdot\vec{P} \qquad \vec{\epsilon} = \text{infinitesimal}$$

Then

$$U(\vec{\epsilon})\vec{R}\,U^{-1}(\vec{\epsilon}) = \vec{R} - \vec{\epsilon}$$

$$\left(1 - \frac{i}{\hbar}\vec{\epsilon}\cdot\vec{P}\right) X_i \left(1 + \frac{i}{\hbar}\vec{\epsilon}\cdot\vec{P}\right) = X_i - \epsilon_i$$

$$\shortparallel$$

$$X_i - \frac{i}{\hbar}[\vec{\epsilon}\cdot\vec{P}, X_i] = X_i - \epsilon_i$$

$\Rightarrow$

$$[\vec{\epsilon}\cdot\vec{P}, X_i] = -i\hbar\,\epsilon_i$$

$$\epsilon_j [P_j, X_i] = -i\hbar\,\epsilon_j \delta_{ji}$$

$\Rightarrow$

$$\boxed{[P_j, X_i] = -i\hbar\,\delta_{ij}}$$

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3) Now make consecutive translations

$$\begin{tikzcd}
\overset{O}{\vec{r}} \arrow[r, "U(\vec{a})"'] \arrow[rr, bend right, "U(\vec{a}+\vec{b})"'] & \overset{O'}{\vec{r}+\vec{a}} \arrow[r, "U(\vec{b})"'] & \overset{O''}{(\vec{r}+\vec{a})+\vec{b}} = \vec{r}+(\vec{a}+\vec{b})
\end{tikzcd}$$

So $U(\vec{a}+\vec{b})|\vec{r}\rangle = |\vec{r}+\vec{a}+\vec{b}\rangle$

$$= U(\vec{a})U(\vec{b})|\vec{r}\rangle = |\vec{r}+\vec{a}+\vec{b}\rangle$$

$$= U(\vec{b})U(\vec{a})|\vec{r}\rangle = |\vec{r}+\vec{a}+\vec{b}\rangle$$

$$\Rightarrow \boxed{U(\vec{a}+\vec{b}) = U(\vec{a})U(\vec{b}) = U(\vec{b})U(\vec{a})}$$

Property of space trans. $\vec{a}+\vec{b} = \vec{b}+\vec{a}$

$\Rightarrow$

$$U(\vec{a})U(\vec{b}) = U(\vec{b})U(\vec{a})$$

$$e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}}\, e^{-\frac{i}{\hbar}\vec{b}\cdot\vec{P}} = e^{-\frac{i}{\hbar}\vec{b}\cdot\vec{P}}\, e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}}$$

infinitesimal

$$1-\frac{i}{\hbar}(\vec{a}+\vec{b})\cdot\vec{P} - \frac{i}{\hbar}\vec{a}\cdot\vec{P}\left(-\frac{i}{\hbar}\vec{b}\cdot\vec{P}\right)$$

$$= 1-\frac{i}{\hbar}(\vec{b}+\vec{a})\cdot\vec{P} - \frac{i}{\hbar}(\vec{b}\cdot\vec{P})\left(\frac{i}{\hbar}\vec{a}\cdot\vec{P}\right)$$

$\Rightarrow$

$$(\vec{a}\cdot\vec{P})(\vec{b}\cdot\vec{P}) = (\vec{b}\cdot\vec{P})(\vec{a}\cdot\vec{P})$$

$\vec{a}, \vec{b}$ arb.

$$P_i P_j = P_j P_i$$

$$\Rightarrow \boxed{[P_i, P_j] = 0}$$

4) As before $U(\vec{a})|\vec{r}\rangle = |\vec{r}+\vec{a}\rangle$

$$\Rightarrow \quad \psi'(\vec{r}) = \psi(\vec{r}-\vec{a})$$

$\vec{a}$ inf. $\Rightarrow$

$$\langle \vec{r}| e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}} |\psi\rangle = \psi(\vec{r}) - \vec{a}\cdot\vec{\nabla}_{\vec{r}}\psi(\vec{r})$$

$$\langle \vec{r}| 1 - \frac{i}{\hbar}\vec{a}\cdot\vec{P}|\psi\rangle = \langle \vec{r}|\psi\rangle - \vec{a}\cdot\vec{\nabla}_{\vec{r}}\langle\vec{r}|\psi\rangle$$

$\Rightarrow$

$$\langle \vec{r}| -\frac{i}{\hbar}\vec{a}\cdot\vec{P}|\psi\rangle = -\vec{a}\cdot\vec{\nabla}_r\langle\vec{r}|\psi\rangle$$

$$\Rightarrow \quad \boxed{\langle\vec{r}|\vec{P} = \frac{\hbar}{i}\vec{\nabla}_{\vec{r}}\langle\vec{r}|}$$

The translation of the system takes place "instantaneously". Making explicit the afore-unmentioned time variable we have $|\psi(t)\rangle$ describing the state of the system at time t in the frame of observer O who uses coordinates $\vec{r}$. On the other hand we have that

(passive)

$|\psi'(t)\rangle = U(\vec{a})\,|\psi(t)\rangle$ describes the same state of the system at time t but in the frame of observer O' who uses coordinates $\vec{r}\,' = \vec{r} - \vec{a}$.

We can then ask if $|\psi(t)\rangle$ evolves according to Hamiltonian H, ~~does the spatially translated state $|\psi'(t)\rangle$ correspond to possible motions of the system. That is~~ does the system as described by O' evolve in time into the transformation of the time evolution of the state $|\psi(t)\rangle$? Pictorially we have

$$\begin{tikzcd}
|\psi(t)\rangle \arrow[r, "H"] \arrow[d, "U(\vec{a})"] & |\psi(t+\epsilon)\rangle \arrow[d, "U(\vec{a})"'] \\
|\psi'(t)\rangle \arrow[r, "H", "/" marking] & |\psi'(t+\epsilon)\rangle
\end{tikzcd}$$

And clearly in general $|\psi'(t)\rangle$ does not evolve according to H into $|\psi'(t+\epsilon)\rangle$.

When does $|\psi'(t)\rangle \xrightarrow{H} |\psi'(t+\epsilon)\rangle$?

To answer this, consider the time derivative of $|\psi'(t)\rangle$

$$i\hbar \frac{d}{dt}|\psi'(t)\rangle = i\hbar \frac{d}{dt} U(\vec{a})|\psi(t)\rangle$$

$$= U(\vec{a}) \underbrace{i\hbar \frac{d}{dt}|\psi(t)\rangle}_{=H|\psi(t)\rangle}$$

$$= U(\vec{a}) H \underbrace{U^{\dagger}(\vec{a}) U(\vec{a})}_{=\mathbb{1}} |\psi(t)\rangle$$

$$= U(\vec{a}) H U^{\dagger}(\vec{a}) |\psi'(t)\rangle$$

$$i\hbar \frac{d}{dt}|\psi'(t)\rangle \equiv H'|\psi'(t)\rangle .$$

Thus $|\psi'(t)\rangle$ evolves according to

$$H' = U(\vec{a}) H U^{\dagger}(\vec{a})\ ;$$ only if $U(\vec{a}) H U^{\dagger}(\vec{a}) = H$ does $|\psi'(t)\rangle$ obey the same Schrödinger

equation as $|\Psi(t)\rangle$, then $|\Psi'(t)\rangle$ is a physically possible state of the system.

Multiplying $H' = H$ by $U(\vec{a})$ on the right yields

$$U(\vec{a})H = H\,U(\vec{a})$$

$$\Rightarrow \quad [U(\vec{a}), H] = 0.$$ So only if

$U(\vec{a})$ commutes with H is $|\Psi'(t)\rangle$ still a physically allowed state. For infinitesimal displacements $U(\vec{\epsilon}) = 1 - \frac{i}{\hbar}\vec{\epsilon}\cdot\vec{P}$ and

$$U(\vec{\epsilon})H\,U^{\dagger}(\vec{\epsilon}) = H$$

$$= \left(1 - \frac{i}{\hbar}\vec{\epsilon}\cdot\vec{P}\right)H\left(1 + \frac{i}{\hbar}\vec{\epsilon}\cdot\vec{P}\right)$$

to first order $\Rightarrow$

$$= H - \frac{i}{\hbar}[\vec{\epsilon}\cdot\vec{P}, H]$$

$$\Rightarrow \quad [\vec{P}, H] = 0.$$

$\Rightarrow H = H(\vec{R}, \vec{P}) = H(\vec{R} + \vec{a}, \vec{P})$

Ehrenfest thm.

$$\frac{d}{dt}\langle\vec{P}\rangle = \frac{i}{\hbar}\underbrace{\langle[H, \vec{P}]\rangle}_{=0} + \underbrace{\left\langle\frac{\partial\vec{P}}{\partial t}\right\rangle}_{=0} = 0$$

On the one hand this tells us that the momentum is a constant of the motion. Thus a physical system that can be displaced in space and still be a possible physical system is characterized by a constant momentum as well as energy. Since $\vec{R}$ and $\vec{P}$ do not commute, $[\vec{P}, H] = 0$ implies H is a function of $\vec{P}$ only. The simplest case is

$$H = \frac{1}{2m}\vec{P}^2 \; ; \quad \text{a free particle.}$$

Clearly, $V(\vec{R}) \neq 0$ corresponds to external forces acting on the system and translation of the system through the force field does not leave the system unaffected. So on the other hand the momentum operator is the generator of translations in space and $[\vec{P}, H] = 0$ implies that space-translations are a <u>symmetry</u> of the system. That is the ~~Hamiltonian~~ exhibits <u>translation invariance</u>. As in classical mechanics symmetries of the system imply conservation laws, $[H, \vec{P}] = 0$, and vice versa.

i.e. $U(\vec{a})\, H_{(\vec{R},\vec{P})}\, U^{\dagger}(\vec{a}) = \underbrace{H(\vec{R}+\vec{a}, \vec{P}) \equiv H(\vec{R}, \vec{P})}_{\text{Translation invariant}}$

Not only is the state preparation apparatus translated in space but also the measuring apparatus, that is the observables. We denote the observables at $\vec{r}$ by operators A and after translation to $\vec{r}+\vec{a}$ by operators A'. By definition, the states were prepared to have the same properties at the two locations. Thus the matrix elements of the observables are the same (the eigenvalue spectrum is unchanged),

$$\langle \phi' | A' | \psi' \rangle = \langle \phi | A | \psi \rangle .$$

But this gives

$$\langle \phi' | A' | \psi' \rangle = \langle \phi | U^\dagger(\vec{a}) A' U(\vec{a}) | \psi \rangle$$

$$= \langle \phi | A | \psi \rangle$$

hence, as we found with the Hamiltonian,

$$\boxed{A' = U(\vec{a}) A U^\dagger(\vec{a})} .$$

So an observable, A, at location $\vec{r}$ becomes the observable $A' = U(\vec{a})\, A\, U^{\dagger}(\vec{a})$ when translated to location $\vec{r}+\vec{a}$.
A is said to be <u>translationally invariant</u> if

$$A' = A = U(\vec{a})\, A\, U^{\dagger}(\vec{a}),$$

that is $[U(\vec{a}), A] = 0$ or equivalently if

$$[\vec{P}, A] = 0 .$$

Such observables are simultaneously measurable (diagonalizable) with the momentum $\vec{P}$.

For example the momentum operator is invariant

$$\vec{P}' = U(\vec{a})\, \vec{P}\, U^{\dagger}(\vec{a}) = \vec{P}$$

but the position operator is not

$$\vec{R}' = U(\vec{a})\, \vec{R}\, U^{\dagger}(\vec{a}) = \vec{R} - \frac{i}{\hbar}\underbrace{[\vec{a}\cdot\vec{P}, \vec{R}]}_{\substack{= a_i[P_i, \vec{R}] \\ = -i\hbar\vec{a}}}$$

$$\boxed{\vec{R}' = \vec{R} - \vec{a}} .$$

If we make two (active) translations

$$\vec{r} \longrightarrow \vec{r}+\vec{a} \longrightarrow (\vec{r}+\vec{a})+\vec{b}$$

(passive view: that is introduce a third observer O'' translated wrt O' by $\vec{b}$), this should be equivalent to making one overall translation

$$\vec{r} \longrightarrow \vec{r}+\vec{c}$$

where $\vec{c} = \vec{a}+\vec{b}$. Hence the unitary operators $U(\vec{a})$ relating the states in the translated systems obey a similar product rule. For the system at the position $\vec{r}$ the state is $|\psi\rangle$, at $\vec{r}+\vec{a}$ the state is denoted $|\psi'\rangle$ and at $\vec{r}+\vec{a}+\vec{b}$ it is $|\psi''\rangle$ where

$$|\psi''\rangle = U(\vec{b})|\psi'\rangle = U(\vec{b})U(\vec{a})|\psi\rangle$$

but on the other hand

$$|\psi''\rangle = U(\vec{c})|\psi\rangle = U(\vec{a}+\vec{b})|\psi\rangle.$$

Thus $U(\vec{b})U(\vec{a}) = U(\vec{a}+\vec{b})$. Since

$\vec{a}+\vec{b}=\vec{b}+\vec{a} \Rightarrow U(\vec{a}+\vec{b})=U(\vec{b}+a)=U(\vec{a})U(\vec{b})$, that is the U operators commute
$$U(\vec{b})U(\vec{a})=U(\vec{a})U(\vec{b}).$$

So we have a product law for the translation operators $U(\vec{a})$. They form a representation of the group of translations, that is they form a group under this multiplication law that is isomorphic to the translation group. The translation group T is the set of all displacement vectors $\vec{a}$ such that ($T=\{\vec{a} \mid \vec{a} = \text{displacement vector}\}$)

1) $\vec{a}+\vec{b}=\vec{c}\in T$ vector addition

2) $\vec{a}+\vec{0}=\vec{a}$ $\vec{0}=$ identity element $\in T$

3) $\vec{a}+(-\vec{a})=\vec{0}$ $-\vec{a}=$ inverse $\in T$

4) $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ associative law

The quantum mechanical translation group $\mathcal{T}$ is the set of all $U(\vec{a})$ for $\vec{a}\in T$ with the composition law given above such that

1) $\mathcal{U}(\vec{a})\,\mathcal{U}(\vec{b}) = \mathcal{U}(\vec{a}+\vec{b}) \in \mathcal{T}$

2) $\mathcal{U}(\vec{a})\,\mathcal{U}(\vec{0}) = \mathcal{U}(\vec{a})$; $\mathcal{U}(\vec{0}) = \mathbb{1}$ the identity operator $\in \mathcal{T}$

3) $\mathcal{U}(\vec{a})\,\mathcal{U}(-\vec{a}) = \mathcal{U}(\vec{0}) = \mathbb{1}$; $\mathcal{U}(-\vec{a}) = \mathcal{U}^{-1}(\vec{a}) \in \mathcal{T}$

4) $\left(\mathcal{U}(\vec{a})\,\mathcal{U}(\vec{b})\right)\mathcal{U}(\vec{c}) = \mathcal{U}(\vec{a})\left(\mathcal{U}(\vec{b})\,\mathcal{U}(\vec{c})\right)$ associative law.

Since the order of vectors under addition is irrelevant $\vec{a}+\vec{b} = \vec{b}+\vec{a}$ T is called an abelian group. Likewise since the operators $\mathcal{U}(\vec{a})$ and $\mathcal{U}(\vec{b})$ commute

$$\mathcal{U}(\vec{a})\,\mathcal{U}(\vec{b}) = \mathcal{U}(\vec{b})\,\mathcal{U}(\vec{a}),$$

$\mathcal{T}$ is also an abelian group; it is a representation of the group T on vectors in $\mathcal{H}$.

The group composition law implies algebraic properties for the generators of the transformations. In the abelian translation group case we have

$$U(\vec{a})\,U(\vec{b}) = U(\vec{a}+\vec{b}) \text{ where}$$

$$U(\vec{a}) = e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}}.$$

For infinitesimal transformations $U(\vec{a})$ and $U(\vec{b})$, $\vec{a}, \vec{b}$ infinitesimal vectors, we can expand the operators to first order in $\vec{a}$ or $\vec{b}$

$$U(\vec{a})\,U(\vec{b}) = \left(1-\frac{i}{\hbar}\vec{a}\cdot\vec{P}\right)\left(1-\frac{i}{\hbar}\vec{b}\cdot\vec{P}\right)$$

$$= 1 - \frac{i}{\hbar}\left(\vec{a}\cdot\vec{P}+\vec{b}\cdot\vec{P}\right)$$

$$+ \frac{i^2}{\hbar^2}(\vec{a}\cdot\vec{P})(\vec{b}\cdot\vec{P})$$

$$= 1 - \frac{i}{\hbar}(\vec{a}+\vec{b})\cdot\vec{P} - \frac{1}{\hbar^2}(\vec{a}\cdot\vec{P})(\vec{b}\cdot\vec{P})$$

On the other hand this equals

$$U(\vec{a}+\vec{b}) = 1 - \frac{i}{\hbar}(\vec{a}+\vec{b})\cdot\vec{P} + \frac{1}{2!}\left(\frac{-i}{\hbar}\right)^2\left((\vec{a}+\vec{b})\cdot\vec{P}\right)^2$$

$$= 1 - \frac{i}{\hbar}(\vec{a}+\vec{b})\cdot\vec{P}$$

$$- \frac{1}{2\hbar^2}\left((\vec{a}\cdot\vec{P})(\vec{b}\cdot\vec{P}) + (\vec{b}\cdot\vec{P})(\vec{a}\cdot\vec{P})\right)$$

This must $= U(\vec{a})\,U(\vec{b})$

The only way this can be true is if

$$-\frac{1}{\hbar^2}(\vec{a}\cdot\vec{P})(\vec{b}\cdot\vec{P}) = -\frac{1}{2\hbar^2}\left[(\vec{a}\cdot\vec{P})(\vec{b}\cdot\vec{P}) + (\vec{b}\cdot\vec{P})(\vec{a}\cdot\vec{P})\right]$$

$$\Rightarrow (\vec{a}\cdot\vec{P})(\vec{b}\cdot\vec{P}) = (\vec{b}\cdot\vec{P})(\vec{a}\cdot\vec{P})$$

$$\Rightarrow a_i b_j P_i P_j = a_i b_j P_j P_i$$

Since $\vec{a}, \vec{b}$ are arbitrary $\Rightarrow$

$$P_i P_j = P_j P_i \quad \text{or}$$

$$\boxed{[P_i, P_j] = 0}$$

Of course in this simple case we knew the result already for the P-P commutation relations. In general though the symmetry group multiplication law implies the commutation relations for the generators.

Alternatively, given the commutation relations for a set of symmetry generators, we can reconstruct the group multiplication law. For example knowing that

$$[P_i, P_j] = 0 \quad \text{implies that}$$

$$e^{\frac{i}{\hbar}\vec{a}\cdot\vec{P}}\, e^{\frac{i}{\hbar}\vec{b}\cdot\vec{P}} = e^{\frac{i}{\hbar}(\vec{a}+\vec{b})\cdot\vec{P}}$$

by the Baker-Campbell-Hausdorff formula. Hence

$$U(\vec{a})\,U(\vec{b}) = U(\vec{a}+\vec{b})\,.$$

If we do not introduce time translations into our group of transformations we have not asked for them to be symmetries of the Hamiltonian. That is if the multiplication law between the time translation operator and the other transformation operators has not been specified then neither has the commutator of the Hamiltonian H with the generators of the other transformations.

However in general for either relativistic or non-relativistic physics we find experimentally that time translations and space translations can be made independently of each other, thus we find the commutation relation $[H, \vec{P}] = 0$.

The forces we observe in nature are between bodies, there is no background force field in space preventing $[\vec{P}, H] = 0$. Hence, from a general point of view, we will imbed time translations in our group of transformations, in a way, of course, that is consistent with the precepts of Newtonian, that is, non-relativistic physics. When studying certain models we may go to a particular frame of reference, thus replacing the two-body potential for example with a background one body potential. We can then ask for an appropriate subset of the transformation generators to commute with the reduced Hamiltonian. Of course we can just study the "instantaneous" transformations by decoupling the time translations from set of operators.

Summary

So the homgeneity of space has led to the same experiment performed in two different places yielding the same result:

$$|\langle\phi|\psi\rangle|^2 = |\langle\phi'|\psi'\rangle|^2 .$$

The translation operator $U(\vec{a}) = e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{P}}$ is generated by the momentum: $\vec{P}$.

If the Hamiltonian is translation invariant
$H' = U(\vec{r})\, H\, U^{+}(\vec{r}) = H \Leftrightarrow [\vec{P}, H] = 0$
i.e. $H(\vec{R}+\vec{a}, \vec{P}) = H(\vec{R},\vec{P})$. (symmetry of H)
$\Leftrightarrow$ Then the momentum is conserved

$$\frac{d\langle\vec{P}\rangle}{dt} = 0 \quad \text{(Ehrenfest's Thm)}$$

Ehrenfest's Thm.:

$$\frac{d}{dt}\langle O\rangle = \frac{i}{\hbar}\langle[H, O]\rangle + \left\langle\frac{\partial O}{\partial t}\right\rangle$$

in this case $\vec{P}$ has no explicit time dependence so $\frac{\partial \vec{P}}{\partial t} = 0$ & $[H, \vec{P}] = 0 \Rightarrow$

$$\frac{d}{dt}\langle\vec{P}\rangle = 0 .$$