1.3.9. Feynman's Path Integral Formulation of Quantum Mechanics

As we have seen for time independent potentials, the stationary states of the Schrödinger equation form a complete set of functions. The time independent Schrödinger equation is the energy eigenvalue equation

$$H\psi_n(\vec{r}) = E_n\psi_n(\vec{r})$$

with $\{\psi_n\}$ orthonormal: $\int d^3r\,\psi_m^*(\vec{r})\psi_n(\vec{r}) = \delta_{mn}$ and complete $\sum_n \psi_n^*(\vec{r}\,')\psi_n(\vec{r}) = \delta^3(\vec{r}-\vec{r}\,')$.

Hence any solution to the time dependent Schrödinger equation at time t $\psi(\vec{r},t)$, may be expanded in terms of $\psi_n(\vec{r})$

$$\psi(\vec{r},t) = \sum_n C_n(t)\psi_n(\vec{r})$$

where

$$C_n(t) = \int d^3r\,\psi_n^*(\vec{r})\psi(\vec{r},t)$$

from orthonormality of $\{\psi_n\}$. Since $\psi(\vec{r},t)$ obeys the time dependent Schrödinger equation, we have

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = H\psi(\vec{r},t)$$

$\Rightarrow$

$$\sum_n i\hbar\frac{d}{dt}c_n(t)\psi_n(\vec{r}) = \sum_n c_n(t)H\psi_n(\vec{r})$$
$$= \sum_n E_n c_n(t)\psi_n(\vec{r}).$$

Applying orthonormality yields $i\hbar\frac{d}{dt}c_n(t) = E_n c_n(t)$.
This has the solution

$$c_n(t) = e^{-i\frac{E_n(t-t_0)}{\hbar}}c_n(t_0),$$

where t_0 is some initial condition time, previously we chose $t_0=0$, but in general it is arbitrary. Thus

$$\psi(\vec{r},t) = \sum_n c_n(t_0)e^{-i\frac{E_n(t-t_0)}{\hbar}}\psi_n(\vec{r}).$$

Alternately stated, given $\psi(\vec{r},t_0)$, then

$$c_n(t_0) = \int d^3r_0\,\psi_n^*(\vec{r}_0)\psi(\vec{r}_0,t_0)$$

and $\psi(\vec{r},t)$ at all other times is given by

$$\psi(\vec{r},t) = \sum_n{}' C_n(t_0)\, e^{-i\frac{E_n(t-t_0)}{\hbar}}\, \psi_n(\vec{r})$$

$$= \sum_n{}' \int d^3r_0\, \psi_n^*(\vec{r}_0)\psi(\vec{r}_0,t_0)\, e^{-i\frac{E_n(t-t_0)}{\hbar}}\, \psi_n(\vec{r})$$

$$= \int d^3r_0 \left[\sum_n{}' \psi_n^*(\vec{r}_0)\psi_n(\vec{r})\, e^{-i\frac{E_n(t-t_0)}{\hbar}}\right]\psi(\vec{r}_0,t_0)$$

For times $t > t_0$, that is t later than t_0, we <u>define</u> this as

$$\equiv \int d^3r_0\, K(\vec{r},t;\vec{r}_0,t_0)\, \psi(\vec{r}_0,t_0)$$

The quantity in brackets, K, is the kernel or the Green function for the Schrödinger equation. It is given by

$$K(\vec{r},t;\vec{r}_0,t_0) \equiv \Theta(t-t_0) \sum_n{}' \psi_n^*(\vec{r}_0)\psi_n(\vec{r})\, e^{-i\frac{E_n(t-t_0)}{\hbar}}$$

where the step function $\Theta(t-t_0)$ is defined as

$$\Theta(t-t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases}$$

it ensures that $t > t_0$ and that $K = 0$ for $t < t_0$. The interpretation of the above equation for $\psi(\vec{r},t)$ is that given the state everywhere in space $(\vec{r}_0)$ at an earlier time t_0, we can find the state at any point in space ($\vec{r}$) at a later time t by letting $\psi(\vec{r}_0, t_0)$ propagate via K through space and time via Huygens principle. K is also known as the propagator. To see that K is the Schrödinger Green function consider

$$\left(i\hbar\frac{\partial}{\partial t} - H(\vec{r})\right)K(\vec{r},t;\vec{r}_0,t_0)$$

$$= i\hbar\left(\frac{\partial}{\partial t}\Theta(t-t_0)\right)\sum_n \psi_n^*(\vec{r}_0)\psi_n(\vec{r})e^{-i\frac{E_n(t-t_0)}{\hbar}}$$

$$+\Theta(t-t_0)\sum_n \psi_n^*(\vec{r}_0)\psi_n(\vec{r})\, i\hbar\frac{\partial}{\partial t}e^{-i\frac{E_n(t-t_0)}{\hbar}}$$

$$-\Theta(t-t_0)\sum_n \psi_n^*(\vec{r}_0)\left(H(\vec{r})\psi_n(\vec{r})\right)e^{-i\frac{E_n(t-t_0)}{\hbar}}$$

$$= i\hbar\delta(t-t_0)\sum_n \psi_n^*(\vec{r}_0)\psi_n(\vec{r})e^{-i\frac{E_n(t-t_0)}{\hbar}}$$

$$+\Theta(t-t_0)\sum_n \psi_n^*(\vec{r}_0)\psi_n(\vec{r})e^{\frac{-iE_n(t-t_0)}{\hbar}}\underbrace{(E_n-E_n)}_{\to 0}$$

Since $t=t_0$ we get in the first term

$$= i\hbar\delta(t-t_0)\underbrace{\sum_n \psi_n^*(\vec{r}_0)\psi_n(\vec{r})}_{=\delta^3(\vec{r}-\vec{r}_0) \text{ by completeness}}$$

$$= i\hbar\delta(t-t_0)\delta^3(\vec{r}-\vec{r}_0)$$

Thus K is the Green function of $\left[i\hbar\frac{\partial}{\partial t} - H(\vec{r})\right]$. Further K obeys the boundary condition $K=0$ for $t<t_0$ by definition. See page -90'-

We can apply this time evolution over and over again to go from $\psi(\vec{r}_0,t_0) \rightarrow \psi(\vec{r}_1,t_1) \rightarrow \psi(\vec{r}_2,t_2) \rightarrow \ldots \rightarrow \psi(\vec{r}_n,t_n) \rightarrow \psi(\vec{r},t)$ with $t > t_n > t_{n-1} > \ldots > t_1 > t_0$,

thus

$$\psi(\vec{r},t) = \int d^3r_0\, d^3r_1\, d^3r_2 \cdots d^3r_n \times$$

$$\times K(\vec{r},t;\vec{r}_n,t_n)K(\vec{r}_n,t_n;\vec{r}_{n-1},t_{n-1})\cdots\times$$

$$\cdots\times K(\vec{r}_2,t_2;\vec{r}_1,t_1)K(\vec{r}_1,t_1;\vec{r}_0,t_0)\psi(\vec{r}_0,t_0).$$

For very short time intervals we can find an integral representation for the Green function. Let $t_k - t_{k-1} = \epsilon$ be small, then

The energy eigenfunction expansion of the wavefunction is valid for times earlier and later than t_0

$$\psi(\vec{r},t) = \sum_n{}' C_n(t_0)\, e^{-\frac{iE_n(t-t_0)}{\hbar}}\, \psi_n(\vec{r})\,.$$

When we use orthonormality of the $\psi_n(\vec{r})$ to find $C_n(t_0)$

$$C_n(t_0) = \int d^3r_0\, \psi_n^*(\vec{r}_0)\, \psi(\vec{r}_0,t_0)\,,$$

we have for all times t

$$\psi(\vec{r},t) = \int d^3r_0 \left[\sum_n{}' \psi_n^*(\vec{r}_0)\psi_n(\vec{r})\, e^{-\frac{iE_n(t-t_0)}{\hbar}} \right] \times$$

$$\times\, \psi(\vec{r}_0,t_0)\,.$$

The Kernel or Green function K is defined by

$$K(\vec{r},t;\vec{r}_0,t_0) \equiv \theta(t-t_0) \sum_n{}' \psi_n^*(\vec{r}_0)\psi_n(\vec{r})\, e^{-\frac{iE_n(t-t_0)}{\hbar}}\,,$$

That is for times $t > t_0$, it gives directly the time evolution of the wave function. By definition for $t_0 > t$, $K = 0$.

That is, for $t \to t_0^+$, $K(\vec{r}, t_0, \vec{r}_0, t_0) = \delta^3(\vec{r}-\vec{r}_0)$, K initially begins as a Dirac delta-function in position, it then propagates according to the Schrödinger equation for $t > t_0$ since

$$\left(i\hbar\frac{\partial}{\partial t} - H_{\vec{r}}\right)K = i\hbar\,\delta(t-t_0)\,\delta^3(\vec{r}-\vec{r}_0)$$

$$(= 0 \text{ for } t > t_0).$$

Even though K directly describes forward time evolution, $t > t_0$, we can use it to evolve backwards in time before t_0. In particular if $t < t_0$ $\psi(\vec{r}, t)$ is still given by

$$\psi(\vec{r}, t) = \int d^3r_0 \left[\sum_n \psi_n^*(\vec{r}_0)\psi_n(\vec{r})\, e^{-\frac{iE_n(t-t_0)}{\hbar}}\right]\psi(\vec{r}_0, t_0).$$

The term is brackets is nothing but K^*, that is for $t < t_0$ we have simply

$$K^*(\vec{r}_0, t_0; \vec{r}, t) = \Theta(t_0 - t)\sum_n \psi_n^*(\vec{r}_0)\psi_n(\vec{r}) \times$$
$$\times\, e^{-\frac{iE_n(t-t_0)}{\hbar}}.$$

Thus for $t < t_0$

$$\psi(\vec{r},t) = \int d^3r_0\, K^*(\vec{r}_0,t_0;\vec{r},t)\,\psi(\vec{r}_0,t_0),$$

that is K^* gives the time evolution backwards in time for $t < t_0$.

Since $K(\vec{r},t;\vec{r}_0,t_0)$ takes the system forward in time from t_0 to $t > t_0$ and $K^*(\vec{r}_0,t_0;\vec{r}',t')$ evolves the system back from t_0 to $t' < t_0$ we have that to go from $t_0 \to t \to t'$ with $t > t' > t_0$ is accomplished by

$$\int d^3r\, K^*(\vec{r},t;\vec{r}',t')\,K(\vec{r},t;\vec{r}_0,t_0) = K(\vec{r}',t';\vec{r}_0,t_0),$$

as compared to evolving forward in time from $t_0 \to t \to t'$ when $t' > t > t_0$

$$\int d^3r\, K(\vec{r}',t';\vec{r},t)\,K(\vec{r},t;\vec{r}_0,t_0) = K(\vec{r}',t';\vec{r}_0,t_0).$$

To reinforce this interpretation, we show that it follows directly from the conservation of probability

$$\int d^3r \, |\psi(\vec{r},t)|^2 = \int d^3r_0 \, |\psi(\vec{r}_0,t_0)|^2,$$

substituting on the LHS

$$\psi(\vec{r},t) = \int d^3r_0 \, K(\vec{r},t;\vec{r}_0,t_0)\, \psi(\vec{r}_0,t_0)$$

we get

$$\int d^3r \, K^*(\vec{r},t;\vec{r}_0{}',t_0)\, K(\vec{r},t;\vec{r}_0,t_0) = \delta^3(\vec{r}_0{}' - \vec{r}_0).$$

Again it is seen that $K^*(\vec{r}_0,t_0;\vec{r},t)$ just undoes the work of $K(\vec{r},t;\vec{r}_0,t_0)$.

Note multiplying the above by $K(\vec{r}_0,t_0;\vec{r}',t')$ and integrating over $\vec{r}_0$ yields
$(t > t_0 > t')$

$$K(\vec{r}_0{}',t_0;\vec{r}',t') = \int d^3r \int d^3r_0 \, K^*(\vec{r},t;\vec{r}_0{}',t_0) \times$$

$$\times K(\vec{r},t;\vec{r}_0,t_0)\, K(\vec{r}_0,t_0;\vec{r}',t')$$

$$= \int d^3r \, K^*(\vec{r},t;\vec{r}_0{}',t_0)\, K(\vec{r},t;\vec{r}',t')$$

which is just our previous formula on page -90'''-.

$$K(\vec{r}_k, t_k; \vec{r}_{k-1}, t_{k-1}) = \sum_n{}' \psi_n^*(\vec{r}_{k-1})\psi_n(\vec{r}_k) \times$$
$$\times \underbrace{e^{-iE_n(t_k - t_{k-1})/\hbar}}_{= e^{-i\epsilon E_n/\hbar} \approx 1 - i\frac{\epsilon}{\hbar}E_n}$$

$$= \sum_n{}' \psi_n^*(\vec{r}_{k-1})\left[1 - \frac{i\epsilon}{\hbar}\left(-\frac{\hbar^2}{2m}\nabla_k^2 + V(\vec{r}_k)\right)\right]\psi_n(\vec{r}_k)$$

Since $H(\vec{r}_k)\psi_n(\vec{r}_k) = E_n\psi_n(\vec{r}_k)$

$$= \left[1 - \frac{i\epsilon}{\hbar}\left(-\frac{\hbar^2}{2m}\nabla_k^2 + V(\vec{r}_k)\right)\right]\sum_n{}' \psi_n^*(\vec{r}_{k-1})\psi_n(\vec{r}_k)$$

$$= \left[1 - \frac{i\epsilon}{\hbar}\left(-\frac{\hbar^2}{2m}\nabla_k^2 + V(\vec{r}_k)\right)\right]\delta^3(\vec{r}_k - \vec{r}_{k-1})$$

$$= \left[1 - \frac{i\epsilon}{\hbar}\left(-\frac{\hbar^2}{2m}\nabla_k^2 + V(\vec{r}_k)\right)\right]\int \frac{d^3p_k}{(2\pi\hbar)^3} e^{+i\vec{p}_k\cdot(\vec{r}_k - \vec{r}_{k-1})/\hbar}$$

$$= \int \frac{d^3p_k}{(2\pi\hbar)^3} e^{+\frac{i}{\hbar}\vec{p}_k\cdot(\vec{r}_k - \vec{r}_{k-1})}\left[1 - \frac{i\epsilon}{\hbar}\left(\frac{\vec{p}_k^{\,2}}{2m} + V(\vec{r}_k)\right)\right]$$

$$= \int \frac{d^3p_k}{(2\pi\hbar)^3} e^{+\frac{i}{\hbar}\vec{p}_k\cdot(\vec{r}_k - \vec{r}_{k-1})} e^{-i\frac{\epsilon}{\hbar}\left(\frac{\vec{p}_k^{\,2}}{2m} + V(\vec{r}_k)\right)}$$

$$= K(\vec{r}_k, t_k; \vec{r}_{k-1}, t_{k-1}) \quad \text{with } t_k - t_{k-1} = \epsilon$$

Putting all these factors together, we find

$$K(\vec{r},t;\vec{r}_0,t_0) \approx \int d^3r_1 \cdots d^3r_n \int \frac{d^3p_1}{(2\pi\hbar)^3} \cdots \frac{d^3p_{n+1}}{(2\pi\hbar)^3} \times$$
$$\times\, e^{-i\frac{\epsilon}{\hbar}\sum_{k=1}^{n+1}\left[\frac{\vec{p}_k^{\,2}}{2m} + V(\vec{r}_k) - \vec{p}_k\cdot\frac{(\vec{r}_k-\vec{r}_{k-1})}{\epsilon}\right]}$$

where we let $\vec{r} = r_{n+1}$, $t = t_{n+1}$.

In the limit $n \to \infty$, $\epsilon \to 0$ this becomes exact

$$K(\vec{r},t;\vec{r}_0,t_0) = \lim_{\substack{n\to\infty \\ \epsilon\to 0}} \int d^3r_1 \cdots d^3r_n \int \frac{d^3p_1}{(2\pi\hbar)^3} \cdots \frac{d^3p_{n+1}}{(2\pi\hbar)^3} \times$$

with $(n+1)\epsilon = (t-t_0)$

$$\times\, e^{-i\frac{\epsilon}{\hbar}\sum_{k=1}^{n+1}\left[\frac{\vec{p}_k^{\,2}}{2m} + V(\vec{r}_k) - \vec{p}_k\cdot\frac{(\vec{r}_k-\vec{r}_{k-1})}{\epsilon}\right]}$$

This infinite product of integrals is called a Feynman path integral, or functional integral. It is a new type of integral. This infinite product of integrals is often denoted by the symbol

$$\int [dp][dx] \quad \text{or} \quad \int \mathcal{D}p\,\mathcal{D}x$$

As $\epsilon \to 0$ and $n \to \infty$ we see that $\frac{\vec{r}_k - \vec{r}_{k-1}}{\epsilon} = \dot{\vec{r}}(t_k)$ while $\vec{p}_k = \vec{p}(t_k)$ so that the exponent becomes

$$-i\frac{\epsilon}{\hbar}\sum_{k=1}^{n+1}\left[\frac{\vec{p}_k^{\,2}}{2m} + V(\vec{r}_k) - \vec{p}_k\cdot\frac{(\vec{r}_k - \vec{r}_{k-1})}{\epsilon}\right]$$

$$= +\frac{i}{\hbar}\int_{t_0}^{t} dt\left[\vec{p}\cdot\dot{\vec{r}} - H(\vec{r},\vec{p})\right]$$

with $H(\vec{r},\vec{p}) = \frac{\vec{p}^{\,2}}{2m} + V(\vec{r})$ the classical Hamiltonian.

Hence we write the functional integral representation for the Green function as

$$K(\vec{r},t;\vec{r}_0,t_0) = \int_{\vec{r}(t_0)=\vec{r}_0}^{\vec{r}(t)=\vec{r}} [dp][dx]\, e^{+\frac{i}{\hbar}\int_{t_0}^{t} dt\left[\vec{p}\cdot\dot{\vec{r}} - H(\vec{r},\vec{p})\right]}$$

where it is understood that this stands for the infinite product of integrals on page -92-.

Returning to this latticized integral, we perform the momentum integrals

Consider

$$\int \frac{d^3 p_k}{(2\pi\hbar)^3}\, e^{-i\frac{\epsilon}{\hbar}\left(\frac{\vec{p}_k^{\,2}}{2m} - \vec{p}_k\cdot\frac{(\vec{r}_k-\vec{r}_{k-1})}{\epsilon}\right)}$$

$\vec{p}_k = p_{kx}\hat{i} + p_{ky}\hat{j} + p_{kz}\hat{k}$ (so $p = p_{kx}$)

$$= \left[\int \frac{dp}{(2\pi\hbar)}\, e^{-i\frac{\epsilon}{\hbar}\left(\frac{p^2}{2m} - p\frac{(x_k-x_{k-1})}{\epsilon}\right)}\right]_x \left[\;\right]_y \left[\;\right]_z .$$

But recall

$$\int \frac{dp}{(2\pi\hbar)}\, e^{\frac{-i\epsilon}{\hbar 2m}(p-a)^2} = \frac{1}{(2\pi\hbar)}\sqrt{\frac{2\pi\hbar m}{i\epsilon}} = \sqrt{\frac{m}{2\pi\hbar i\epsilon}}$$

$$= \int \frac{dp}{(2\pi\hbar)}\, e^{-\frac{i\epsilon}{\hbar}\left(\frac{p^2}{2m} - \frac{ap}{m} + \frac{a^2}{2m}\right)}$$

$$= e^{-i\frac{\epsilon a^2}{2m\hbar}} \int \frac{dp}{(2\pi\hbar)}\, e^{-\frac{i\epsilon}{\hbar}\left(\frac{p^2}{2m} - \frac{a}{m}p\right)}$$

Hence

$$\int \frac{d^3 p_k}{(2\pi\hbar)^3}\, e^{-i\frac{\epsilon}{\hbar}\left(\frac{\vec{p}_k^{\,2}}{2m} - \vec{p}_k\cdot\frac{(\vec{r}_k-\vec{r}_{k-1})}{\epsilon}\right)}$$

$$= \left(\frac{m}{2\pi\hbar i\epsilon}\right)^{3/2} e^{+\frac{i\epsilon}{\hbar}\frac{m}{2}\frac{(\vec{r}_k-\vec{r}_{k-1})^2}{\epsilon^2}}$$

$$= \left(\frac{m}{2\pi\hbar i\epsilon}\right)^{3/2} e^{+\frac{i\epsilon}{\hbar}\frac{1}{2}m\dot{\vec{r}}_k^{\,2}}$$

Substituting this into the functional integral yields

$$K(\vec{r},t;\vec{r}_0,t_0) = \lim_{\substack{n\to\infty \\ \epsilon\to 0}} \int \frac{d^3r_1}{\left(\frac{2\pi\hbar i\epsilon}{m}\right)^{3/2}} \cdots \frac{d^3r_n}{\left(\frac{2\pi\hbar i\epsilon}{m}\right)^{3/2}} \times$$

with $(n+1)\epsilon = (t-t_0)$

$$\times \frac{1}{\left(\frac{2\pi\hbar i\epsilon}{m}\right)^{3/2}} \, e^{+\frac{i}{\hbar}\epsilon\sum_{k=1}^{n+1}\left[\frac{1}{2}m\frac{(\vec{r}_k-\vec{r}_{k-1})^2}{\epsilon^2} - V(\vec{r}_k)\right]}$$

The bracketed terms in the exponent is just the Lagrangian for the classical trajectory $\vec{r}(t)$, $\dot{\vec{r}}(t)$, beginning at t_0 at $\vec{r}_0$, $\dot{\vec{r}}_0$ and ending at t at $\vec{r}$, $\dot{\vec{r}}$

$$L(\vec{r}(t),\dot{\vec{r}}(t)) = \frac{1}{2}m\dot{\vec{r}}(t)^2 - V(\vec{r}(t)) .$$

Thus in the limit we write this integral as

$$\boxed{K(\vec{r},t;\vec{r}_0,t_0) = \int_{\vec{r}(t_0)=\vec{r}_0}^{\vec{r}(t)=\vec{r}} [dr]\, e^{+\frac{i}{\hbar}\int_{t_0}^{t} dt\, L(\vec{r}(t),\dot{\vec{r}}(t))}}$$

Again the Feynman path integral

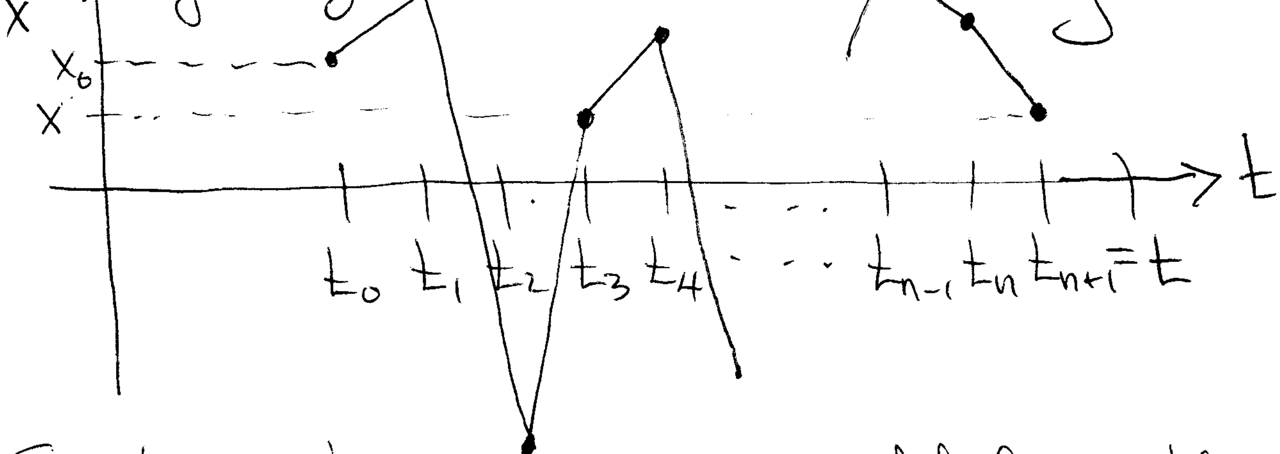
$\int\{dr\}$ stands for the infinite product of integrals

$$\int[dr] = \lim_{\substack{n\to\infty \\ \epsilon\to 0}} \int \frac{d^3r_1}{\left(\frac{2\pi\hbar i\epsilon}{m}\right)^{3/2}} \cdots \frac{d^3r_n}{\left(\frac{2\pi\hbar i\epsilon}{m}\right)^{3/2}} \left(\frac{1}{\frac{2\pi\hbar i\epsilon}{m}}\right)^{3/2}.$$

Hence the Green function is obtained by summing over all classical paths between $\vec{r}_0$ at t_0 and $\vec{r}$ at t weighted by the factor

$$e^{\frac{i}{\hbar}(\text{Action})}$$ for that path.

That is we can break up the classical trajectory into time slots between t_0 and t. Thus for $(n+1)$ divisions we have $t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = t$ and we sum over all possible values of the coordinate $\vec{r}(t)$ in each interval beginning with $\vec{r}_0$ at t_0 and ending with $\vec{r}$ at t



Such a trajectory is weighted in the

sum by $e^{+\frac{i}{\hbar}\int dt\, L}$. This gives a Lagrangian formulation of Q.M.

Thus Feynman replaces the Schrödinger equation with that Green function represented as a path integral as the dynamical starting point of quantum mechanics (Postulate 4)

Thus Feynman showed that

$$\psi(\vec{r},t) = \int d^3r_0\, K(\vec{r},t;\vec{r}_0,t_0)\, \psi(\vec{r}_0,t_0)$$

with

$$K(\vec{r},t;\vec{r}_0,t_0) = \int_{\vec{r}(t_0)=\vec{r}_0}^{\vec{r}(t)=\vec{r}} [dr]\, e^{+\frac{i}{\hbar}\int_{t_0}^{t} dt\, L(\vec{r}(t),\dot{\vec{r}}(t))}$$

with $L(\vec{r},\dot{\vec{r}}) = \vec{p}\cdot\dot{\vec{r}} - H(\vec{r},\vec{p})$, the classical Lagrangian.

Examples: Consider a free particle of mass m in one dimension. The classical Hamiltonian is $H = \frac{p^2}{2m}$ with $p = m\dot{x}$. So the classical Lagrangian is

$$L = p\dot{x} - H = \tfrac{1}{2} m \dot{x}^2.$$

The Green function is given by the path integral

$$K(x,t;x_0,t_0) = \int_{x(t_0)=x_0}^{x(t)=x} [dx]\, e^{\frac{i}{\hbar}\int_{t_0}^{t} dt\, \frac{1}{2} m \dot{x}(t)^2}$$

$$= \lim_{\substack{\epsilon\to 0\\ n\to\infty}} \int_{-\infty}^{+\infty} \frac{dx_1}{\sqrt{\frac{2\pi\hbar i\epsilon}{m}}} \cdots \int_{-\infty}^{+\infty} \frac{dx_n}{\sqrt{\frac{2\pi\hbar i\epsilon}{m}}} \frac{1}{\sqrt{\frac{2\pi\hbar i\epsilon}{m}}} \times$$

$$\times\, e^{+\frac{i\epsilon}{\hbar}\sum_{k=1}^{n+1} \frac{1}{2} m \frac{(x_k - x_{k-1})^2}{\epsilon^2}}$$

These are just Gaussian integrals, we can perform one integration after another just to keep repeating the Gaussian. Start with the x_1 integral

$$\frac{1}{\sqrt{\frac{2\pi\hbar i\epsilon}{m}}}\int_{-\infty}^{+\infty}\frac{dx_1}{\sqrt{\frac{2\pi\hbar i\epsilon}{m}}}\, e^{-\frac{m}{2i\hbar\epsilon}\left[(x_1-x_0)^2+(x_2-x_1)^2\right]}$$

$$=\frac{1}{\sqrt{\frac{2\pi\hbar i(2\epsilon)}{m}}}\, e^{-\frac{m}{2i\hbar(2\epsilon)}(x_2-x_0)^2}\,.$$

The x_2-integral becomes

$$\sqrt{\frac{m}{2\pi\hbar i(2\epsilon)}}\int_{-\infty}^{+\infty}dx_2\sqrt{\frac{m}{2\pi\hbar i(\epsilon)}}\, e^{-\frac{m}{2i\hbar\epsilon}\left[(x_3-x_2)^2+\frac{1}{2}(x_2-x_0)^2\right]}$$

$$=\sqrt{\frac{m}{2\pi\hbar i(3\epsilon)}}\, e^{-\frac{m}{2i\hbar(3\epsilon)}(x_3-x_0)^2}\,.$$

Thus we continue until the x_n is performed with $x_{n+1}=x$ we find

$$\sqrt{\frac{m}{2\pi\hbar i(n+1)\epsilon}}\, e^{-\frac{m}{2i\hbar(n+1)\epsilon}(x_{n+1}-x_0)^2}\,.$$

Recall that $(n+1)\epsilon = t-t_0$, so we find the <u>free particle Green</u> function

$$\boxed{K(x,t;x_0,t_0) = \theta(t-t_0)\sqrt{\frac{m}{2\pi i\hbar(t-t_0)}}\; e^{+\frac{i}{\hbar}\frac{m}{2}\frac{(x-x_0)^2}{(t-t_0)}}}$$

Now suppose we have a Gaussian wave packet at $t_0 = 0$

$$\psi(x_0, t_0=0) = \frac{A}{2\sqrt{\pi}\,a}\, e^{ik_0x_0}\, e^{-\frac{x_0^2}{4a^2}}$$

(recall page -34-), then at t we found that

$$\psi(x,t) = \frac{A}{2\sqrt{\pi}}\sqrt{\frac{1}{a^2+\frac{i\hbar t}{2m}}} \;\times\; e^{i\left[k_0x - \frac{\hbar k_0^2}{2m}t\right]}\; e^{-\frac{\left(x-\frac{\hbar k_0}{m}t\right)^2}{4\left(a^2+\frac{i\hbar t}{2m}\right)}}.$$

$\left(\text{Note if } \int dx\,|\psi(x,t)|^2 = 1 = \frac{|A|^2\sqrt{\pi}}{\sqrt{2}\,a} \Rightarrow A = \left(a\sqrt{8\pi}\right)^{1/2}\right)$

We should obtain this same result by using the free particle Green function K, to evolve $\psi(x_0,t_0)$ in time to $\psi(x,t)$

That is

$$\psi(x,t) = \int_{-\infty}^{+\infty} dx_0 \sqrt{\frac{m}{2\pi\hbar i t}}\, e^{\frac{i}{\hbar}\frac{m}{2}\frac{(x-x_0)^2}{t}}\, \psi(x_0,t_0)$$

$$= \int_{-\infty}^{+\infty} dx_0 \sqrt{\frac{m}{2\pi\hbar i t}}\, e^{\frac{i}{\hbar}\frac{m}{2}\frac{(x-x_0)^2}{t}}\, \frac{A}{2a\sqrt{\pi}}\, e^{ik_0x_0 - \frac{x_0^2}{4a^2}}$$

$$= \frac{A}{2a\sqrt{\pi}} \sqrt{\frac{m}{2\pi\hbar i t}}\, e^{\frac{i}{\hbar}\frac{m}{2}\frac{x^2}{t}} \times$$

$$\times \int_{-\infty}^{+\infty} dx_0\, e^{-\left(\frac{1}{4a^2} - \frac{im}{2\hbar t}\right)x_0^2}\, e^{-\left(\frac{im}{\hbar t}x - ik_0\right)x_0} .$$

Once again we have a Gaussian integral to perform. As usual we do it by completing the square

$$\int_{-\infty}^{+\infty} dx\, e^{-ax^2 - bx} = \int_{-\infty}^{+\infty} dx\, e^{-a\left(x+\frac{b}{2a}\right)^2 + \frac{b^2}{4a}}$$

$$= e^{+\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}} .$$

This gives

$$\psi(x,t) = \frac{A}{2a\sqrt{\pi}}\sqrt{\frac{m}{2\pi\hbar i t}}\; e^{\frac{i}{\hbar}\frac{m}{2}\frac{x^2}{t}} \times$$

$$\times \sqrt{\frac{\pi}{\frac{1}{4a^2}-\frac{im}{2\hbar t}}}\; e^{\frac{\left(\frac{imx}{\hbar t}-ik_0\right)^2}{4\left(\frac{1}{4a^2}-\frac{im}{2\hbar t}\right)}}$$

$$= \frac{A}{2a\sqrt{\pi}}\sqrt{\frac{m}{2\pi\hbar i t}}\sqrt{\frac{4\pi a^2\frac{i\hbar t}{2m}}{a^2+\frac{i\hbar t}{2m}}} \times$$

$$\times e^{\frac{i}{\hbar}\frac{m}{2}\frac{x^2}{t}}\; e^{\frac{-\left(\frac{m^2x^2}{\hbar^2t^2}+k_0^2-\frac{2mk_0x}{\hbar t}\right)}{-i4\left(\frac{i2\hbar t+m4a^2}{4a^2 2\hbar t}\right)}}$$

$$= \frac{A}{2\sqrt{\pi}}\sqrt{\frac{1}{a^2+\frac{i\hbar t}{2m}}}\; \exp\left[\frac{-1}{4\left(a^2+i\frac{\hbar t}{2m}\right)} \times\right.$$

$$\left.\times\left(\frac{2i\hbar k_0^2a^2t}{m}-4ia^2k_0x+\cancel{\frac{2imx^2a^2}{\hbar t}}-\cancel{i\frac{2mx^2a^2}{\hbar t}}+x^2\right)\right]$$

$$= \frac{A}{2\sqrt{\pi}}\sqrt{\frac{1}{a^2+\frac{i\hbar t}{2m}}}\; e^{\frac{4ik_0x}{4\left(a^2+\frac{i\hbar t}{2m}\right)}\left(\frac{i\hbar t}{2m}+a^2\right)} \times$$

$$\times e^{\frac{-1}{4\left(a^2+\frac{i\hbar t}{2m}\right)}\left(x^2+\frac{2i\hbar t a^2k_0^2}{m}-\frac{2xk_0\hbar t}{m}\right)}$$

Thus we finally obtain

$$\psi(x,t) = \frac{A}{2\sqrt{\pi}} \frac{1}{\sqrt{a^2 + \frac{i\hbar t}{2m}}} e^{i\left(k_0 x - \frac{\hbar k_0^2 t}{2m}\right)} \times$$

$$\times\, e^{-\frac{\left(x - \frac{\hbar k_0 t}{m}\right)^2}{4\left(a^2 + \frac{i\hbar t}{2m}\right)}}$$

in agreement with our previous result obtained from Schrödinger's equation directly (i.e. as an expansion in terms of plane wave solutions to Schrödinger's equation).

Similarly one could calculate the propagation kernel K for other Hamiltonia, for instance the one-dimensional harmonic oscillator Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 .$$

The Lagrangian is simply

$$L = p\dot{x} - H = \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2} x^2 .$$

The Green function is then

$$K(x,t;x_0 t_0) = \int_{x(t_0)=x_0}^{x(t)=x} [dx]\, e^{\frac{i}{\hbar}\int_{t_0}^{t} dt \left(\frac{1}{2} m \dot{x}^2(t) - \frac{m\omega^2}{2} x^2(t)\right)}$$

$$= \sqrt{\frac{m\omega}{2\pi\hbar i \sin\omega(t-t_0)}} \times \Theta(t-t_0) \times e^{\frac{i m\omega}{\hbar\, 2\sin\omega(t-t_0)}\left[(x^2+x_0^2)\cos\omega(t-t_0) - 2xx_0\right]}.$$

Note for $\omega \to 0$ we just recover the free particle kernel of page - 100 -.

4.10. Feynman's Path Integral Formulation of Quantum Mechanics Revisited

As we have found earlier, Feynman was able to represent the time evolution of a quantum mechanical system by means of a kernel or Green function which was written as the sum over all possible paths in configuration space each weighted by $\exp(\frac{i}{\hbar}\text{Action})$ for the path. That is the wavefunction at time t and position $\vec{r}$, $\psi(\vec{r},t)$, for the system was related to $\psi(\vec{r}_0,t_0)$ the wavefunction at earlier time t_0 and all other positions $\vec{r}_0$ by the Green function equation

$$\psi(\vec{r},t) = \int d^3r_0\, K(\vec{r},t;\vec{r}_0,t_0)\,\psi(\vec{r}_0,t_0)$$

with the kernel given as the sum over energy eigenstates as

$$K(\vec{r},t;\vec{r}_0,t_0) = \Theta(t-t_0)\sum_n \psi_n^*(\vec{r}_0)\psi_n(\vec{r})\,e^{-i\frac{E_n(t-t_0)}{\hbar}}.$$

With repeated application of this formula for smaller and smaller time slices, Feynman was able to represent K as an integral of a new type

$$K(\vec{r},t;\vec{r}_0,t_0)=\int_{\vec{r}(t_0)=\vec{r}_0}^{\vec{r}(t)=\vec{r}}[dp][dx]\,e^{+\frac{i}{\hbar}\int_{t_0}^{t}dt\left[\vec{p}\cdot\dot{\vec{r}}-H(\vec{r},\vec{p})\right]}$$

with $H(\vec{r},\vec{p})$ the classical Hamiltonian, for instance

$$H(\vec{r},\vec{p})=\frac{\vec{p}^{\,2}}{2m}+V(\vec{r})\ .$$

The Gaussian momentum integrals can be performed in such a case to obtain an integral over all paths

$$K(\vec{r},t;\vec{r}_0,t_0)=\int_{\vec{r}(t_0)=\vec{r}_0}^{\vec{r}(t)=\vec{r}}[dr]\,e^{+\frac{i}{\hbar}\int_{t_0}^{t}dt\,L(\vec{r}(t),\dot{\vec{r}}(t))}$$

where L is the classical Lagrangian

$$L(\vec{r}(t),\dot{\vec{r}}(t))=\tfrac{1}{2}m\dot{\vec{r}}(t)^2-V(\vec{r}(t))\ .$$

$$=\vec{p}\cdot\dot{\vec{r}}-H(\vec{r},\vec{p})\ .$$

Since the wavefunction $\psi(\vec{r},t)$ is simply the inner product

$$\psi(\vec{r},t)=\langle\vec{r}\,|\,\psi(t)\rangle$$ we

as well can re-formulate the dynamical laws of quantum mechanics (Postulate 4) to be that of Feynman in our abstract approach.

For simplicity we work in one-dimension with the position operator denoted by $Q\ (=X)$ and its eigenstates $|q\rangle$ with eigenvalues q; $Q|q\rangle = q|q\rangle$.

In the Schrödinger picture formulation of quantum mechanics the states depend on time while the operators are time independent; thus $|\psi(t)\rangle_s$ obey the Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle_s = H(Q,P)|\psi(t)\rangle_s$$

and, for instance, $\frac{d}{dt}Q_s = 0$. In the coordinate basis, $Q_s|q\rangle = q|q\rangle$, the vectors $|q\rangle$ are time independent eigenvectors of Q_s, normalized so that

$$\langle q'|q\rangle = \delta(q'-q) \quad \text{and}$$

complete

$$\int_{-\infty}^{+\infty} dq\, |q\rangle\langle q| = 1 .$$

The solution to Schrödinger's equation is simply

$$|\psi(t)\rangle_S = e^{-\frac{iHt}{\hbar}} |\psi(0)\rangle_S .$$

The initial Schrödinger state $|\psi(0)\rangle_S$ is time independent and provides the bridge to the Heisenberg picture formulation of quantum mechanics. The Heisenberg state $|\psi\rangle_H$ is simply

$$|\psi\rangle_H \equiv |\psi(0)\rangle_S = e^{+\frac{i}{\hbar}Ht} |\psi(t)\rangle_S .$$

Since $e^{+\frac{i}{\hbar}Ht}$ is unitary we have

$${}_S\langle\psi(t)|Q_S|\phi(t)\rangle_S = {}_H\langle\psi| e^{+\frac{iHt}{\hbar}} Q_S e^{-\frac{i}{\hbar}Ht} |\phi\rangle_H$$

$$\equiv {}_H\langle\psi| Q_H(t) |\phi\rangle_H ,$$

The Heisenberg picture operators depend on time

$$Q_H(t) = e^{+\frac{i}{\hbar}Ht} \underbrace{Q_S}_{=Q_H(0)} e^{-\frac{i}{\hbar}Ht} .$$

The eigenstates of the position operator $Q_H(t)$ at time t are given in terms of the moving coordinate vectors

$$|q,t\rangle \equiv e^{+\frac{i}{\hbar}Ht} |q\rangle .$$

Hence

$$Q_H(t)|q,t\rangle = e^{+\frac{i}{\hbar}Ht} Q_S e^{-\frac{i}{\hbar}Ht} e^{+\frac{i}{\hbar}Ht} |q\rangle$$

$$= e^{+\frac{i}{\hbar}Ht} \underbrace{Q_S|q\rangle}_{=q|q\rangle}$$

$$= q\, e^{+\frac{i}{\hbar}Ht} |q\rangle$$

$$= q|q,t\rangle .$$

$|q,t\rangle$ is a Heisenberg picture state; t is just a label for the moving axes.

Similarly the normalization of the states $|q,t\rangle$ is

$$\langle q',t|q,t\rangle = \langle q'|q\rangle = \delta(q'-q)$$

and Completeness becomes

$$e^{\frac{i}{\hbar}Ht}\left[1 = \int_{-\infty}^{+\infty} dq\, |q\rangle\langle q|\right] e^{-\frac{i}{\hbar}Ht}$$

$$\Rightarrow \quad 1 = \int_{-\infty}^{+\infty} dq\, |q,t\rangle\langle q,t|$$

(i.e. $\;{}_S\langle \psi(t)|\phi(t)\rangle_S = {}_H\langle \psi|\phi\rangle_H$

$$= \int dq\, {}_S\langle \psi(t)|q\rangle\langle q|\phi(t)\rangle_S$$

$$= \int dq\, {}_H\langle \psi|e^{\frac{iHt}{\hbar}}|q\rangle\langle q|e^{-\frac{iHt}{\hbar}}|\phi\rangle_H$$

$$= \int dq\, {}_H\langle \psi|q,t\rangle\langle q,t|\phi\rangle_H$$

$\Rightarrow \int dq\, |q,t\rangle\langle q,t| = 1.$)

The kernel or Green function defined by

$$K(q,t;q_0,t_0) \equiv \langle q,t | q_0,t_0 \rangle$$

controls the time evolution of the states, all dynamics is contained in K.

$$\begin{aligned}\psi(q,t) &= \langle q | \psi(t) \rangle_S \\ &= \langle q,t | \psi \rangle_H \\ &= \int dq_0 \langle q,t | q_0,t_0 \rangle \underbrace{\langle q_0,t_0 | \psi \rangle_H}_{= \langle q_0 | \psi(t_0) \rangle_S} \\ &= \int dq_0 \langle q,t | q_0,t_0 \rangle \psi(q_0,t_0)\end{aligned}$$

$$\boxed{\psi(q,t) = \int dq_0 \, K(q,t;q_0,t_0) \, \psi(q_0,t_0)}$$

Note for $t \to t_0$ $K(q,t;q_0,t) = \langle q t | q_0, t \rangle = \delta(q_0 - q)$ by the normalization of the $|q,t\rangle$ states.

Also

$$K(q,t;q_0 t_0) = \langle q,t | q_0, t_0 \rangle$$

$$= \langle q | e^{-\frac{i}{\hbar}H(t-t_0)} | q_0 \rangle$$

$$= \langle q | U(t,t_0) | q_0 \rangle$$

The Kernel is the $\{|q\rangle\}$ basis matrix elements of the time evolution operator U. For early times $t_0 \to -\infty$ U relates the in-coming states to the late time $t \to +\infty$ out-going states, it is the scattering operator $S = U(t_0, -\infty)$, and K is just its position matrix elements.

As before if we know the energy eigenvectors

$$H|n\rangle = E_n |n\rangle$$

with normalization $\langle m|n\rangle = \delta_{mn}$ and

completeness $\sum_n |n\rangle\langle n| = 1$

we can represent the time evolution as an energy sum

$$\psi(q,t) = \langle q | \psi(t) \rangle_s$$

$$= \sum_n \langle q | n \rangle \langle n | \psi(t) \rangle_s$$

$$= \sum_n \langle q | n \rangle \langle n | e^{-\frac{i}{\hbar}H(t-t_0)} | \psi(t_0) \rangle_s$$

$$= \int dq_0 \sum_n \langle q | n \rangle \underbrace{\langle n | e^{-\frac{i}{\hbar}H(t-t_0)} | q_0 \rangle}_{= e^{-\frac{i}{\hbar}E_n(t-t_0)} \langle n | q_0 \rangle} \langle q_0 | \psi(t_0) \rangle_s$$

$$= \int dq_0 \left[\sum_n \langle q | n \rangle e^{-\frac{i}{\hbar}E_n(t-t_0)} \langle n | q_0 \rangle \right] \psi(q_0,t_0)$$

denoting the energy eigenwavefunctions as

$\psi_n(q) \equiv \langle q | n \rangle$ we have

$$\psi(q,t) = \int dq_0 \left(\sum_n \psi_n(q) e^{-\frac{i}{\hbar}E_n(t-t_0)} \psi_n^*(q_0) \right) \psi(q_0,t_0)$$

Hence the energy representation of the Kernel is

$$K(q,t;q_0,t_0) = \sum_n \psi_n(q)\psi_n^*(q_0)e^{-\frac{i}{\hbar}E_n(t-t_0)}$$

as we found earlier. (Note $t \to t_0$

$$K = \sum_n \psi_n(q)\psi_n^*(q_0) = \delta(q_0-q) \text{ by}$$

completeness of the energy eigenstates

$$\langle q|\left(\sum_n |n\rangle\langle n| = 1\right)|q_0\rangle$$

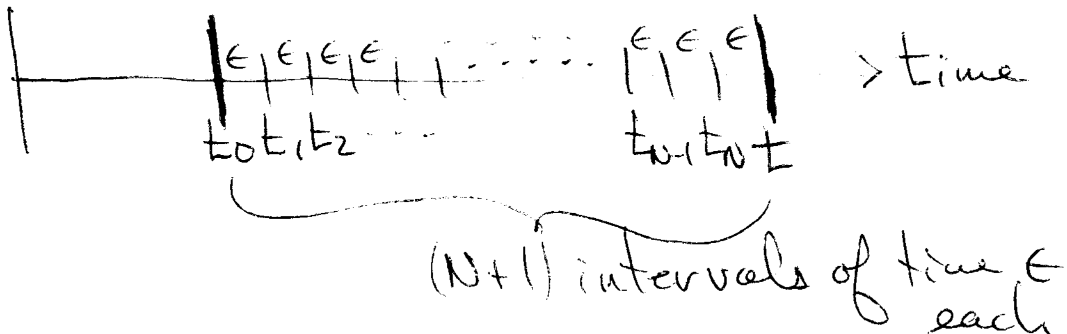
$$\Rightarrow \quad \sum_n \langle q|n\rangle\langle n|q_0\rangle = \langle q|q_0\rangle = \delta(q-q_0)$$

$$= \sum_n \psi_n(q)\psi_n^*(q_0) = \delta(q-q_0).)$$

Of course we could proceed as before to derive the Feynman path integral representation for K from the above energy representation, but let's return to the definition of K as the inner product of position eigenstates at different times and exploit our completeness relation for these states directly,

$K(q,t;q_0,t_0) \equiv \langle q,t|q_0,t_0\rangle$. Let's divide the time interval (t_0,t) into $(N+1)$ short intervals ϵ

$$t > t_N > t_{N-1} > \cdots > t_1 > t_0$$



At each time t_i, $i=1\ldots N$, we insert 1 as a sum over the complete set of states $\{|q,t_i\rangle\}$

$$\langle q,t|q_0,t_0\rangle = \int\left(\prod_{i=1}^{N} dq_i\right)\langle q,t|q_N,t_N\rangle\langle q_N t_N|q_{N-1},t_{N-1}\rangle$$

$$\cdots\langle q_2 t_2|q_1,t_1\rangle\langle q_1,t_1|q_0,t_0\rangle$$

As previously for small time intervals ϵ we can evaluate the Kernel

$$\langle q_{i+1}, t_{i+1} | q_i, t_i \rangle = \langle q_{i+1} | e^{-\frac{i}{\hbar} H (t_{i+1} - t_i)} | q_i \rangle$$

$$= \langle q_{i+1} | e^{-\frac{i}{\hbar} H \epsilon} | q_i \rangle$$

$$= \langle q_{i+1} | (1 - \tfrac{i}{\hbar} H \epsilon) | q_i \rangle + O(\epsilon^2) .$$

The Hamiltonian has the form

$$H = H(P, Q) = \frac{1}{2m} P^2 + V(Q)$$

so that

$$\langle q_{i+1} | H | q_i \rangle = \frac{1}{2m} \langle q_{i+1} | P^2 | q_i \rangle + \langle q_{i+1} | V(Q) | q_i \rangle$$

$$= \frac{1}{2m} \langle q_{i+1} | P^2 | q_i \rangle + V(q_i) \delta(q_{i+1} - q_i) .$$

In order to evaluate the P^2 term, consider a complete set of momentum eigenstates $|p\rangle$

$$P|p\rangle = p|p\rangle \quad \text{with}$$

continuum normalization $\langle p' | p \rangle = \delta(p' - p) / (2\pi\hbar)$

and completeness $\displaystyle\int_{-\infty}^{+\infty} \frac{dp}{(2\pi\hbar)} |p\rangle\langle p| = 1.$

The momentum operator has the matrix elements

$$\langle q|P = \frac{\hbar}{i}\frac{\partial}{\partial q}\langle q| \quad ; \quad \text{that is}$$

$$\langle q|\underbrace{P|p\rangle}_{=p|p\rangle} = \frac{\hbar}{i}\frac{\partial}{\partial q}\langle q|p\rangle$$

$\Rightarrow$ The differential equation

$$p\langle q|p\rangle = \frac{\hbar}{i}\frac{\partial}{\partial q}\langle q|p\rangle$$

$$\Rightarrow \quad \langle q|p\rangle = N e^{\frac{i}{\hbar}pq} \qquad \text{as we know.}$$

Since we have normalization

$$\langle p'|p\rangle = (2\pi\hbar)\delta(p'-p) = \int dq \langle p'|q\rangle\langle q|p\rangle$$

$$= \int dq\, N^2 e^{\frac{i}{\hbar}(p-p')q}$$

$$= \hbar N^2 \int \frac{dq}{\hbar} e^{\frac{i}{\hbar}(p-p')q} = \hbar N^2 (2\pi)\delta(p-p')$$

$$\Rightarrow N = 1.$$

Hence

$$\langle q_{i+1}|P^2|q_i\rangle = \int_{-\infty}^{+\infty} \frac{dp}{(2\pi\hbar)} \langle q_{i+1}|p\rangle\langle p|P^2|q_i\rangle$$

$$= \int \frac{dp}{(2\pi\hbar)} p^2 e^{\frac{i}{\hbar}p(q_{i+1}-q_i)}$$

and

$$\langle q_{i+1}|H|q_i\rangle = \int_{-\infty}^{+\infty} \frac{dp_i}{(2\pi\hbar)}\left[\frac{p_i^2}{2m} + V\left(\frac{q_{i+1}+q_i}{2}\right)\right] e^{\frac{i}{\hbar}p_i(q_{i+1}-q_i)}$$

where $\delta(q_{i+1}-q_i) = \int_{-\infty}^{+\infty} \frac{dp_i}{(2\pi\hbar)} e^{\frac{i}{\hbar}p_i(q_{i+1}-q_i)}$

So

$$\langle q_{i+1}|\underbrace{H(P,Q)}_{\substack{\text{quantum}\\ \text{operator}\\ \text{Hamiltonian}}}|q_i\rangle = \int_{-\infty}^{+\infty} \frac{dp_i}{(2\pi\hbar)} \underbrace{H\left(p_i, \frac{q_{i+1}+q_i}{2}\right)}_{\substack{\text{classical}\\ \text{function}\\ \text{Hamiltonian}}} e^{\frac{i}{\hbar}p_i(q_{i+1}-q_i)}$$

and

$\langle q_{i+1}t_{i+1}|q_i,t_i\rangle$

$$= \int_{-\infty}^{+\infty} \frac{dp_i}{(2\pi\hbar)} e^{\frac{i}{\hbar}p_i(q_{i+1}-q_i)} e^{-\frac{i}{\hbar}H\left(p_i, \frac{q_{i+1}+q_i}{2}\right)\epsilon} + O(\epsilon^2)$$

So for each time slice we plug in a similar expression to obtain

$$\langle q,t|q_0,t_0\rangle = \int_{-\infty}^{+\infty}\left(\prod_{i=1}^{N} dq_i\right)\left(\prod_{j=0}^{N}\frac{dp_j}{(2\pi\hbar)}\right) \times e^{\frac{i}{\hbar}\left[\sum_{i=0}^{N} p_i(q_{i+1}-q_i) - \epsilon H(p_i, \frac{1}{2}(q_{i+1}+q_i))\right]} + O(\epsilon^2)$$

where we have defined $q = q_{N+1}$ and recall $t-t_0 = (N+1)\epsilon$.

As usual we consider more and more $N\to\infty$ shorter time intervals $\epsilon\to 0$ so that $t-t_0 = (N+1)\epsilon$ is fixed and write this ∞ product of integrals as a Feynman functional integral

$$\langle q,t|q_0,t_0\rangle = \int_{q_0}^{q}[dq][dp]\,e^{\frac{i}{\hbar}\int_{t_0}^{t} dt\,[p\dot{q} - H(p,q)]}$$

$$\equiv \lim_{\substack{N\to\infty \\ \epsilon\to 0 \\ t-t_0=(N+1)\epsilon}} \int_{-\infty}^{+\infty} dq_1\cdots dq_N \int_{-\infty}^{+\infty}\frac{dp_0}{2\pi\hbar}\cdots\frac{dp_N}{2\pi\hbar}\, e^{\frac{i}{\hbar}\epsilon\sum_{i=0}^{N}\left[p_i\frac{(q_{i+1}-q_i)}{\epsilon} - H\left(p_i, \frac{q_{i+1}+q_i}{2}\right)\right]}$$

Since $p\dot{q}-H$ looks like the Lagrangian L, we can re-express the functional integral in terms of summing over all classical paths by performing the momentum functional integrals. Since they are Gaussian integrals, we find

$$\cdots \int_{-\infty}^{+\infty} \frac{dp_i}{2\pi\hbar}\, e^{\frac{i}{\hbar} p_i (q_{i+1}-q_i) - \frac{i}{\hbar}\epsilon \frac{p_i^2}{2m}} \cdots$$

$$= \cdots \frac{1}{\sqrt{\frac{2\pi i \epsilon}{m\hbar}}}\, e^{\frac{i}{\hbar}\frac{m}{2}\frac{(q_{i+1}-q_i)^2}{\epsilon}} \cdots .$$

This yields

$$\boxed{\begin{aligned} &\langle q, t | q_0, t_0 \rangle \\ &= \lim_{\substack{N\to\infty \\ \epsilon\to 0 \\ t-t_0=(N+1)\epsilon}} \int_{-\infty}^{+\infty} \frac{dq_1}{\sqrt{\frac{2\pi i\epsilon}{m\hbar}}} \cdots \frac{dq_N}{\sqrt{\frac{2\pi i\epsilon}{m\hbar}}}\, \frac{1}{\sqrt{\frac{2\pi i\epsilon}{m\hbar}}} \times \\ &\times e^{\frac{i}{\hbar}\sum_{i=0}^{N} \epsilon\left[\frac{m}{2}\left(\frac{q_{i+1}-q_i}{\epsilon}\right)^2 - V\left(\frac{q_{i+1}+q_i}{2}\right)\right]} \end{aligned}}$$

As before for $\epsilon = dt$; $\frac{q_{i+1}-q_i}{\epsilon} = \frac{dq}{dt}$

The timeslice sum in the exponent becomes the time integral

$$\sum_{i=0}^{N-1} dt \to \int_{t_0}^{t} dt$$

and

$$\frac{i}{\hbar}\sum_{i=0}^{N-1}\epsilon\left[\frac{m}{2}\left(\frac{q_{i+1}-q_i}{\epsilon}\right)^2 - V\left(\frac{q_i+q_{i+1}}{2}\right)\right]$$

$$\longrightarrow \frac{i}{\hbar}\int_{t_0}^{t} dt'\left[\frac{m}{2}\dot{q}(t')^2 - V(q(t'))\right].$$

This is just the classical action

$$S(q;t,t_0) \equiv \int_{t_0}^{t} dt'\, L(q(t'),\dot{q}(t'))$$

times $\frac{i}{\hbar}$ with classical Lagrangian

$$L(q(t),\dot{q}(t)) = \frac{1}{2}m\dot{q}^2(t) - V(q(t)).$$

Thus we finally obtain the form of the Green function or Kernel in quantum mechanics as a Feynman path integral

$$\langle q,t|q_0,t_0\rangle = \int_{q(t_0)=q_0}^{q(t)=q} [dq]\, e^{\frac{i}{\hbar}S(q;t,t_0)}$$

where $S(q;t,t_0) = \int_{t_0}^{t} dt'\, L(q(t'),\dot{q}(t'))$.

This Feynman path integral is defined as the infinite product of integrals

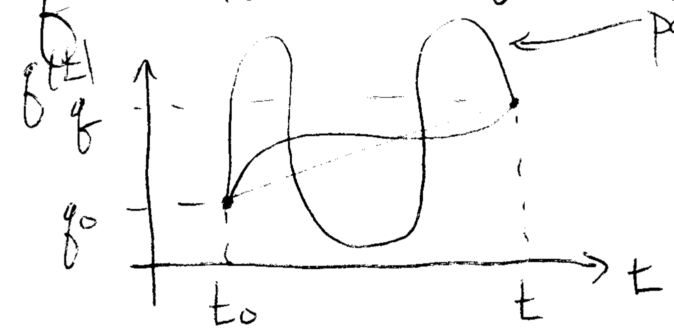
$$\langle q,t|q_0,t_0\rangle \equiv \lim_{\substack{N\to\infty \\ \epsilon\to 0 \\ t-t_0=(N+1)\epsilon}} \sqrt{\frac{m\hbar}{2\pi i\epsilon}} \int_{-\infty}^{+\infty} \frac{dq_1}{\sqrt{\frac{2\pi i\epsilon}{m\hbar}}} \cdots \frac{dq_N}{\sqrt{\frac{2\pi i\epsilon}{m\hbar}}} \times$$

$$\times\, e^{\frac{i}{\hbar}\sum_{i=0}^{N}\epsilon\left[\frac{m}{2}\left(\frac{q_{i+1}-q_i}{\epsilon}\right)^2 - V\left(\frac{q_i+q_{i+1}}{2}\right)\right]}$$

with the boundary conditions $q_{N+1}=q$ at time t and $q_0 = q_0$ at time t_0.

Cryptically then, we have that the Green function (Kernel or transformation function as it is sometimes called)

$\langle q,t|q_0,t_0\rangle = K(q,t;q_0,t_0)$ is just a sum over all paths $q(t)$ from q_0 at time t_0 to q at time t



with each path's contribution to the sum weighted by $e^{\frac{i}{\hbar}S}$ where

S is the classical action for such a path. So
$$K(q,t;q_0,t_0) = \sum_{\text{paths}} e^{\frac{i}{\hbar}S}.$$

If $\hbar \to 0$, only one path contributes, the path of stationary phase
$$\frac{\delta S}{\delta q(t)} = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0.$$
This is just the classical Newtonian trajectory which is a solution to the above Euler-Lagrange equation.