

1.3.9. Feynman's Path Integral Formulation of Quantum Mechanics

As we have seen for time independent potentials, the stationary states of the Schrödinger equation form a complete set of functions. The time independent Schrödinger equation is the energy eigenvalue equation

$$H \psi_n(\vec{r}) = E_n \psi_n(\vec{r})$$

with $\{\psi_n\}$ orthonormal: $\int d^3r \psi_m^*(\vec{r}) \psi_n(\vec{r}) = \delta_{mn}$
and complete $\sum_n \psi_n^*(\vec{r}') \psi_n(\vec{r}) = \delta^3(\vec{r} - \vec{r}')$.

Hence any solution to the time dependent Schrödinger equation at time t $\psi(\vec{r}, t)$, may be expanded in terms of $\psi_n(\vec{r})$

$$\psi(\vec{r}, t) = \sum_n C_n(t) \psi_n(\vec{r})$$

where

$$C_n(t) = \int d^3r \psi_n^*(\vec{r}) \psi(\vec{r}, t)$$

from orthonormality of $\{\psi_n\}$. Since $\psi(\vec{r}, t)$ obeys the time dependent Schrödinger equation, we have

-87-

$$\Rightarrow i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = H \psi(\vec{r}, t)$$

$$\sum_n i\hbar \frac{d}{dt} C_n(t) \psi_n(\vec{r}) = \sum_n C_n(t) H \psi_n(\vec{r})$$

$$= \sum_n E_n C_n(t) \psi_n(\vec{r}).$$

Applying orthonormality yields $i\hbar \frac{d}{dt} C_n(t) = E_n C_n(t)$.
This has the solution

$$C_n(t) = e^{\frac{-i E_n (t - t_0)}{\hbar}} C_n(t_0),$$

where t_0 is some initial condition time, previously we chose $t_0 = 0$, but in general it is arbitrary. Thus

$$\psi(\vec{r}, t) = \sum_n C_n(t_0) e^{\frac{-i E_n (t - t_0)}{\hbar}} \psi_n(\vec{r}).$$

Alternately stated, given $\psi(\vec{r}, t_0)$, then

$$C_n(t_0) = \int d^3 r_0 \psi_n^*(\vec{r}_0) \psi(\vec{r}_0, t_0)$$

and $\psi(\vec{r}, t)$ at all other times is given by

-88-

$$\psi(\vec{r}, t) = \sum_n C_n(t_0) e^{\frac{-i E_n(t-t_0)}{\hbar}} \psi_n(\vec{r})$$

$$= \sum_n \int d^3 r_0 \psi_n^*(\vec{r}_0) \psi(\vec{r}_0, t_0) e^{\frac{-i E_n(t-t_0)}{\hbar}} \psi_n(\vec{r})$$

$$= \int d^3 r_0 \left[\sum_n \psi_n^*(\vec{r}_0) \psi_n(\vec{r}) e^{\frac{-i E_n(t-t_0)}{\hbar}} \right] \psi(\vec{r}_0, t_0)$$

For times $t > t_0$, that is t later than t_0 , we define this as

$$\psi \equiv \int d^3 r_0 K(\vec{r}, t; \vec{r}_0, t_0) \psi(\vec{r}_0, t_0)$$

The quantity in brackets, K , is the kernel the Green function for the Schrödinger equation. It is given by

$$K(\vec{r}, t; \vec{r}_0, t_0) = \Theta(t-t_0) \sum_n \psi_n^*(\vec{r}_0) \psi_n(\vec{r}) e^{\frac{-i E_n(t-t_0)}{\hbar}}$$

where the step function $\Theta(t-t_0)$ is defined as

$$\Theta(t-t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases}, \text{ it}$$

ensures that $t > t_0$ and that $K=0$ for $t < t_0$. The interpretation of the above equation for $\psi(\vec{r}, t)$ is that given the state everywhere in space ($\vec{r}_0$) at an

earlier time t_0 , we can find the state at any point in space ($\vec{r}$) at a later time t by letting $\psi(\vec{r}_0, t_0)$ propagate via K through space and time via Huygens principle. K is also known as the propagator. To see that K is the Schrödinger Green function consider

$$\begin{aligned} & (i\hbar \frac{\partial}{\partial t} - H(\vec{r})) K(\vec{r}, t; \vec{r}_0, t_0) \\ &= i\hbar \left(\frac{\partial}{\partial t} \Theta(t-t_0) \right) \sum_n \psi_n^*(\vec{r}_0) \psi_n(\vec{r}) e^{-i \frac{E_n(t-t_0)}{\hbar}} \\ & \quad + \Theta(t-t_0) \sum_n \psi_n^*(\vec{r}_0) \psi_n(\vec{r}) i\hbar \frac{\partial}{\partial t} e^{-i \frac{E_n(t-t_0)}{\hbar}} \\ & \quad - \Theta(t-t_0) \sum_n \psi_n^*(\vec{r}_0) (H(\vec{r}) \psi_n(\vec{r})) e^{-i \frac{E_n(t-t_0)}{\hbar}} \\ &= i\hbar \delta(t-t_0) \sum_n \psi_n^*(\vec{r}_0) \psi_n(\vec{r}) e^{-i \frac{E_n(t-t_0)}{\hbar}} \\ & \quad + \Theta(t-t_0) \sum_n \psi_n^*(\vec{r}_0) \psi_n(\vec{r}) e^{-i \frac{E_n(t-t_0)}{\hbar}} (E_n - E_n) \end{aligned}$$

Since $t = t_0$ we get in the first term

$$\begin{aligned} &= i\hbar \delta(t-t_0) \sum_n \psi_n^*(\vec{r}_0) \psi_n(\vec{r}) \\ &= i\hbar \delta(t-t_0) \delta^3(\vec{r} - \vec{r}_0) \quad \text{by completeness} \end{aligned}$$

Thus K is the Green function of $[i\hbar \frac{\partial}{\partial t} - H(\vec{r})]$. Further K obeys the boundary condition $K=0$ for $t < t_0$ by definition. See page - 90 -

We can apply this time evolution over and over again to go from $\psi(\vec{r}_0, t_0) \rightarrow \psi(\vec{r}_1, t_1) \rightarrow \psi(\vec{r}_2, t_2) \rightarrow \dots \rightarrow \psi(\vec{r}_n, t_n) \rightarrow \psi(\vec{r}, t)$ with $t > t_n > t_{n-1} > \dots > t_1 > t_0$,

thus

$$\begin{aligned} \psi(\vec{r}, t) = & \int d^3 r_0 d^3 r_1 d^3 r_2 \dots d^3 r_n \times \\ & \times K(\vec{r}, t; \vec{r}_n, t_n) K(\vec{r}_n, t_n; \vec{r}_{n-1}, t_{n-1}) \dots \times \\ & \dots \times K(\vec{r}_2, t_2; \vec{r}_1, t_1) K(\vec{r}_1, t_1; \vec{r}_0, t_0) \psi(\vec{r}_0, t_0). \end{aligned}$$

For very short time intervals we can find an integral representation for the Green function. Let $t_k - t_{k-1} = \epsilon$ be small, then

-90-

The energy eigenfunction expansion of the wavefunction is valid for times earlier and later than t_0

$$\psi(\vec{r}, t) = \sum_n C_n(t_0) e^{\frac{-i E_n(t-t_0)}{\hbar}} \psi_n(\vec{r})$$

When we use orthonormality of the $\psi_n(\vec{r})$ to find $C_n(t_0)$

$$C_n(t_0) = \int d^3 r_0 \psi_n^*(\vec{r}_0) \psi(\vec{r}_0, t_0)$$

we have for all times t

$$\psi(\vec{r}, t) = \int d^3 r_0 \left[\sum_n \psi_n^*(\vec{r}_0) \psi_n(\vec{r}) e^{\frac{-i E_n(t-t_0)}{\hbar}} \right] \times \psi(\vec{r}_0, t_0)$$

The kernel or Green function K is defined by

$$K(\vec{r}, t; \vec{r}_0, t_0) \equiv \theta(t-t_0) \sum_n \psi_n^*(\vec{r}_0) \psi_n(\vec{r}) e^{\frac{-i E_n(t-t_0)}{\hbar}}$$

That is for times $t > t_0$, it gives directly the time evolution of the wavefunction. By definition for $t_0 > t$, $K = 0$.

- That is, for $t \rightarrow t_0^+$, $K(\vec{r}, t_0, \vec{r}_0, t_0) = \delta^3(\vec{r} - \vec{r}_0)$,
 K initially begins as a Dirac delta-function in position, it then propagates according to the Schrödinger equation for $t > t_0$
 Since

$$(i\hbar \frac{\partial}{\partial t} - H_{\vec{r}}) K = i\hbar \delta(t - t_0) \delta^3(\vec{r} - \vec{r}_0)$$

$$(\text{= } 0 \text{ for } t > t_0).$$

- Even though K directly describes forward time evolution, $t > t_0$, we can use it to evolve backwards in time before t_0 . In particular if $t < t_0$ $\psi(\vec{r}, t)$ is still given by

$$\psi(\vec{r}, t) = \int d^3 r_0 \left[\sum_n \psi_n^*(\vec{r}_0) \psi_n(\vec{r}) e^{\frac{-iE_n(t-t_0)}{\hbar}} \right] \psi(\vec{r}_0, t_0)$$

The term in brackets is nothing but K^* , that is, for $t < t_0$ we have simply

$$K^*(\vec{r}_0, t_0; \vec{r}, t) = \Theta(t_0 - t) \sum_n \psi_n^*(\vec{r}_0) \psi_n(\vec{r}) \times e^{\frac{-iE_n(t-t_0)}{\hbar}}$$

-90'''-

Thus for $t < t_0$

$$\psi(\vec{r}, t) = \int d^3r_0 K^*(\vec{r}_0, t_0; \vec{r}, t) \psi(\vec{r}_0, t_0),$$

that is K^* gives the time evolution backwards in time for $t < t_0$.

Since $K(\vec{r}, t; \vec{r}_0, t_0)$ takes the system forward in time from t_0 to $t > t_0$ and $K^*(\vec{r}_0, t_0; \vec{r}', t')$ evolves the system back from t_0 to $t' < t_0$, we have that to go from $t_0 \rightarrow t \rightarrow t'$ with $t > t' > t_0$ is accomplished by

$$\int d^3r K^*(\vec{r}, t; \vec{r}', t') K(\vec{r}, t; \vec{r}_0, t_0) = K(\vec{r}', t'; \vec{r}_0, t_0),$$

as compared to evolving forward in time from $t_0 \rightarrow t \rightarrow t'$ when $t' > t > t_0$

$$\int d^3r K(\vec{r}', t'; \vec{r}, t) K(\vec{r}, t; \vec{r}_0, t_0) = K(\vec{r}', t'; \vec{r}_0, t_0).$$

To reinforce this interpretation, we show that it follows directly from the conservation of probability

-90'''-

$$\int d^3r |\psi(\vec{r}, t)|^2 = \int d^3r_0 |\psi(\vec{r}_0, t_0)|^2,$$

Substituting on the LHS

$$\psi(\vec{r}, t) = \int d^3r_0 K(\vec{r}, t; \vec{r}_0, t_0) \psi(\vec{r}_0, t_0)$$

we get

$$\int d^3r K^*(\vec{r}, t; \vec{r}_0', t_0) K(\vec{r}, t; \vec{r}_0, t_0) = \delta^3(\vec{r}_0' - \vec{r}_0).$$

Again it is seen that $K^*(\vec{r}_0, t_0; \vec{r}, t)$ just undoes the work of $K(\vec{r}, t; \vec{r}_0, t_0)$.

Note multiplying the above by $K(\vec{r}_0, t_0; \vec{r}', t')$ and integrating over $\vec{r}_0$ yields
($t > t_0 > t'$)

$$K(\vec{r}_0', t_0; \vec{r}', t') = \int d^3r \left[\int d^3r_0 K^*(\vec{r}, t; \vec{r}_0', t_0) \times \right. \\ \left. \times K(\vec{r}, t; \vec{r}_0, t_0) K(\vec{r}_0, t_0; \vec{r}', t') \right]$$

$$= \int d^3r K^*(\vec{r}, t; \vec{r}_0', t_0) K(\vec{r}, t; \vec{r}', t')$$

which is just our previous formula on page
-90'''-

-91-

$$\begin{aligned}
 K(\vec{r}_k, t_k; \vec{r}_{k-1}, t_{k-1}) &= \sum_n \psi_n^*(\vec{r}_{k-1}) \psi_n(\vec{r}_k) \times \\
 &\quad \times e^{-i E_n (t_k - t_{k-1}) / \hbar} \\
 &= e^{-i \epsilon E_n / \hbar} \approx \left[1 - i \frac{\epsilon}{\hbar} E_n \right] \\
 &= \sum_n \psi_n^*(\vec{r}_{k-1}) \left[1 - i \frac{\epsilon}{\hbar} \left(-\frac{\hbar^2}{2m} \nabla_k^2 + V(\vec{r}_k) \right) \right] \psi_n(\vec{r}_k)
 \end{aligned}$$

Since $H(\vec{r}_k) \psi_n(\vec{r}_k) = E_n \psi_n(\vec{r}_k)$

$$\begin{aligned}
 &= \left[1 - i \frac{\epsilon}{\hbar} \left(-\frac{\hbar^2}{2m} \nabla_k^2 + V(\vec{r}_k) \right) \right] \sum_n \psi_n^*(\vec{r}_{k-1}) \psi_n(\vec{r}_k) \\
 &= \left[1 - i \frac{\epsilon}{\hbar} \left(-\frac{\hbar^2}{2m} \nabla_k^2 + V(\vec{r}_k) \right) \right] \delta^3(\vec{r}_k - \vec{r}_{k-1}) \\
 &= \left[1 - i \frac{\epsilon}{\hbar} \left(-\frac{\hbar^2}{2m} \nabla_k^2 + V(\vec{r}_k) \right) \right] \int \frac{d^3 p_k}{(2\pi\hbar)^3} e^{\frac{i \vec{p}_k \cdot (\vec{r}_k - \vec{r}_{k-1})}{\hbar}} \\
 &= \int \frac{d^3 p_k}{(2\pi\hbar)^3} e^{\frac{i}{\hbar} \vec{p}_k \cdot (\vec{r}_k - \vec{r}_{k-1})} \left[1 - i \frac{\epsilon}{\hbar} \left(\frac{\vec{p}_k^2}{2m} + V(\vec{r}_k) \right) \right] \\
 &= \int \frac{d^3 p_k}{(2\pi\hbar)^3} e^{\frac{i}{\hbar} \vec{p}_k \cdot (\vec{r}_k - \vec{r}_{k-1})} e^{-i \frac{\epsilon}{\hbar} \left(\frac{\vec{p}_k^2}{2m} + V(\vec{r}_k) \right)} \\
 &= K(\vec{r}_k, t_k; \vec{r}_{k-1}, t_{k-1}) \quad \text{with } t_k - t_{k-1} = \epsilon
 \end{aligned}$$

Putting all these factors together, we find

$$K(\vec{r}, t; \vec{r}_0, t_0) \approx \int d^3 r_1 \dots d^3 r_n \left[\frac{d^3 p_1}{(2\pi\hbar)^3} \dots \frac{d^3 p_{n+1}}{(2\pi\hbar)^3} \times \right. \\ \left. \times e^{-i \frac{\epsilon}{\hbar} \sum_{k=1}^{n+1} \left[\frac{\vec{p}_k^2}{2m} + V(\vec{r}_k) - \vec{p}_k \cdot \frac{(\vec{r}_k - \vec{r}_{k-1})}{\epsilon} \right]} \right]$$

where we let $\vec{r} = \vec{r}_{n+1}$, $t = t_{n+1}$.

In the limit $n \rightarrow \infty$, $\epsilon \rightarrow 0$ this becomes exact

$$K(\vec{r}, t; \vec{r}_0, t_0) = \lim_{\substack{n \rightarrow \infty \\ \epsilon \rightarrow 0}} \left(\int d^3 r_1 \dots d^3 r_n \left[\frac{d^3 p_1}{(2\pi\hbar)^3} \dots \frac{d^3 p_{n+1}}{(2\pi\hbar)^3} \times \right. \right. \\ \left. \left. \times e^{-i \frac{\epsilon}{\hbar} \sum_{k=1}^{n+1} \left[\frac{\vec{p}_k^2}{2m} + V(\vec{r}_k) - \vec{p}_k \cdot \frac{(\vec{r}_k - \vec{r}_{k-1})}{\epsilon} \right]} \right] \right) \\ \text{with } (n+1)\epsilon = (t - t_0)$$

This infinite product of integrals is called a Feynman path integral, or functional integral. It is a new type of integral. This infinite product of integrals is often denoted by the symbol $\int [dp][dx]$ or $\int dp dx$

As $\epsilon \rightarrow 0$ and $n \rightarrow \infty$ we see that $\frac{\vec{r}_k - \vec{r}_{k-1}}{\epsilon} = \dot{\vec{r}}(t_k)$ while $\vec{p}_k = \vec{p}(t_k)$

So that the exponent becomes

$$-i \frac{\epsilon}{\hbar} \sum_{k=1}^{n+1} \left[\frac{\vec{p}_k^2}{2m} + V(\vec{r}_k) - \vec{p}_k \cdot \frac{(\vec{r}_k - \vec{r}_{k-1})}{\epsilon} \right]$$

$$= +i \frac{\epsilon}{\hbar} \int_{t_0}^t dt [\vec{p} \cdot \dot{\vec{r}} - H(\vec{r}, \vec{p})]$$

with $H(\vec{r}, \vec{p}) = \frac{\vec{p}^2}{2m} + V(\vec{r})$ the classical Hamiltonian.

Hence we write the functional integral representation for the Green function as

$$K(\vec{r}, t; \vec{r}_0, t_0) = \int_{\substack{\vec{r}(t) = \vec{r} \\ \vec{r}(t_0) = \vec{r}_0}} [dp][dx] e^{+i \frac{\epsilon}{\hbar} \int_{t_0}^t dt [\vec{p} \cdot \dot{\vec{r}} - H(\vec{r}, \vec{p})]}$$

where it is understood that this stands for the infinite product of integrals on page-92-.

Returning to this latticized integral, we perform the momentum integrals

Consider

$$\int \frac{d^3 p_k}{(2\pi\hbar)^3} e^{-i\frac{\epsilon}{\hbar} \left(\frac{\vec{p}_k^2}{2m} - \vec{p}_k \cdot \frac{(\vec{r}_k - \vec{r}_{k-1})}{\epsilon} \right)}$$

$\vec{p}_k = \vec{p}_{kx} \hat{i} + \vec{p}_{ky} \hat{j} + \vec{p}_{kz} \hat{k}$
(So $p = p_k$)

$$= \left[\int \frac{dp}{(2\pi\hbar)} e^{-i\frac{\epsilon}{\hbar} \left(\frac{p^2}{2m} - p \frac{(x_k - x_{k-1})}{\epsilon} \right)} \right]_x \left[\right]_y \left[\right]_z.$$

But recall

$$\int \frac{dp}{(2\pi\hbar)} e^{-i\frac{\epsilon}{\hbar 2m} (p-a)^2} = \frac{1}{(2\pi\hbar)} \sqrt{\frac{2\pi\hbar m}{i\epsilon}} = \sqrt{\frac{m}{2\pi\hbar i\epsilon}}$$

$$= \int \frac{dp}{(2\pi\hbar)} e^{-i\frac{\epsilon}{\hbar} \left(\frac{p^2}{2m} - \frac{ap}{m} + \frac{a^2}{2m} \right)}$$

$$= e^{-i\frac{\epsilon a^2}{2m\hbar}} \int \frac{dp}{(2\pi\hbar)} e^{-i\frac{\epsilon}{\hbar} \left(\frac{p^2}{2m} - \frac{a}{m} p \right)}$$

Hence

$$\int \frac{d^3 p_k}{(2\pi\hbar)^3} e^{-i\frac{\epsilon}{\hbar} \left(\frac{\vec{p}_k^2}{2m} - \vec{p}_k \cdot \frac{(\vec{r}_k - \vec{r}_{k-1})}{\epsilon} \right)}$$

$$= \left(\frac{m}{2\pi\hbar i\epsilon} \right)^{3/2} e^{+i\frac{\epsilon}{\hbar} \frac{m}{2} \frac{(\vec{r}_k - \vec{r}_{k-1})^2}{\epsilon^2}}$$

$$= \left(\frac{m}{2\pi\hbar i\epsilon} \right)^{3/2} e^{+i\frac{\epsilon}{\hbar} \frac{1}{2} m \dot{\vec{r}}_k^2}$$

-95-

Substituting this into the functional integral yields

$$K(\vec{r}, t; \vec{r}_0, t_0) = \lim_{\substack{n \rightarrow \infty \\ \epsilon \rightarrow 0}} \int \frac{d^3 r_1}{\left(\frac{2\pi\hbar i \epsilon}{m}\right)^{3/2}} \cdots \frac{d^3 r_n}{\left(\frac{2\pi\hbar i \epsilon}{m}\right)^{3/2}} \times$$

with $(n+1)\epsilon = (t - t_0)$

$$\times \frac{1}{\left(\frac{2\pi\hbar i \epsilon}{m}\right)^{3/2}} e^{+ \frac{i}{\hbar} \epsilon \sum_{k=1}^{n+1} \left[\frac{1}{2} m \frac{(\vec{r}_k - \vec{r}_{k-1})^2}{\epsilon^2} - V(\vec{r}_k) \right]}$$

The bracketed terms in the exponent is just the Lagrangian for the classical trajectory $\vec{r}(t), \dot{\vec{r}}(t)$, beginning at t_0 at $\vec{r}_0, \dot{\vec{r}}_0$ and ending at t at $\vec{r}, \dot{\vec{r}}$

$$L(\vec{r}(t), \dot{\vec{r}}(t)) = \frac{1}{2} m \dot{\vec{r}}(t)^2 - V(\vec{r}(t))$$

Thus in the limit we write this integral as

$$K(\vec{r}, t; \vec{r}_0, t_0) = \int_{\substack{\vec{r}(t) = \vec{r} \\ \vec{r}(t_0) = \vec{r}_0}} [d\vec{r}] e^{+ \frac{i}{\hbar} \int_{t_0}^t dt L(\vec{r}(t), \dot{\vec{r}}(t))}$$

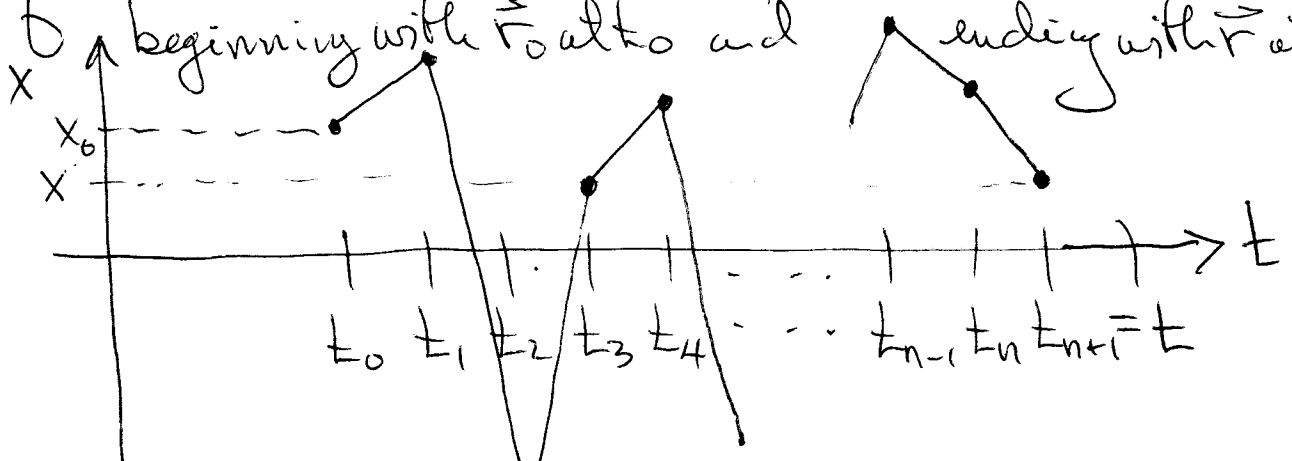
Again the Feynman path integral

$\int [dr]$ stands for the infinite product of integrals

$$\int [dr] = \lim_{\substack{n \rightarrow \infty \\ \epsilon \rightarrow 0}} \int \frac{d^3 r_1}{\left(\frac{2\pi\hbar i \epsilon}{m}\right)^{3/2}} \cdots \frac{d^3 r_n}{\left(\frac{2\pi\hbar i \epsilon}{m}\right)^{3/2}} \left(\frac{1}{\frac{2\pi\hbar i \epsilon}{m}}\right)^{3/2}.$$

Hence the Green function is obtained by summing over all classical paths between $\vec{r}_0$ at t_0 and $\vec{r}$ at t weighted by the factor $e^{\frac{i}{\hbar}(\text{Action})}$ for that path.

That is we can break up the classical trajectory into time slots between t_0 and t . Thus for $(n+1)$ divisions we have $t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = t$ and we sum over all possible values of the coordinate $\vec{r}(t)$ in each interval beginning with $\vec{r}_0$ at t_0 and ending with $\vec{r}$ at t .



Such a trajectory is weighted in the

-97-

sum by $e^{+\frac{i}{\hbar} \int dt L}$. This gives a Lagrangian formulation of Q.M.

Thus Feynman replaces the Schrödinger equation with the Green function represented as a path integral as the dynamical start in part of quantum mechanics (Postulate 4). Thus Feynman showed that

$$\psi(\vec{r}, t) = \int d^3 r_0 K(\vec{r}, t; \vec{r}_0, t_0) \psi(\vec{r}_0, t_0)$$

with

$$\vec{r}(t) = \vec{r} + \frac{i}{\hbar} \int_{t_0}^t dt L(\vec{r}(t), \dot{\vec{r}}(t))$$

$$K(\vec{r}, t; \vec{r}_0, t_0) = \int [d\vec{r}] e^{\frac{i}{\hbar} \int_{t_0}^t dt L(\vec{r}(t), \dot{\vec{r}}(t))}$$

$$\vec{r}(t_0) = \vec{r}_0$$

with $L(\vec{r}, \dot{\vec{r}}) = \vec{p} \cdot \dot{\vec{r}} - H(\vec{r}, \vec{p})$, the classical Lagrangian.

Examples: Consider a free particle of mass m in one dimension. The classical Hamiltonian is $H = \frac{p^2}{2m}$ with $p = m\dot{x}$. So the classical Lagrangian is

$$L = p\dot{x} - H = \frac{1}{2}m\dot{x}^2. \text{ The}$$

Green function is given by the path integral

$$K(x, t; x_0, t_0) = \int_{x(t_0)=x_0}^{x(t)=x} [dx] e^{\frac{i}{\hbar} \int_{t_0}^t dt \frac{1}{2} m \dot{x}(t)^2}$$

$$= \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{-\infty}^{+\infty} \frac{dx_1}{\sqrt{\frac{2\pi\hbar i \epsilon}{m}}} \cdots \int_{-\infty}^{+\infty} \frac{dx_n}{\sqrt{\frac{2\pi\hbar i \epsilon}{m}}} \frac{1}{\sqrt{\frac{2\pi\hbar i \epsilon}{m}}} \times$$

$$\times e^{\frac{i}{\hbar} \sum_{k=1}^{n+1} \frac{1}{2} m \frac{(x_k - x_{k-1})^2}{\epsilon^2}}$$

These are just Gaussian integrals, we can perform one integration after another just to keep repeating the Gaussian. Start with the x_1 integral

$$\begin{aligned}
 & \frac{1}{\sqrt{\frac{2\pi\hbar i\epsilon}{m}}} \int_{-\infty}^{+\infty} \frac{dx_1}{\sqrt{\frac{2\pi\hbar i\epsilon}{m}}} e^{-\frac{m}{2i\hbar\epsilon} [(x_1 - x_0)^2 + (x_2 - x_1)^2]} \\
 &= \frac{1}{\sqrt{\frac{2\pi\hbar i(2\epsilon)}{m}}} e^{-\frac{m}{2i\hbar(2\epsilon)} (x_2 - x_0)^2}
 \end{aligned}$$

The x_2 -integral becomes

$$\begin{aligned}
 & \frac{1}{\sqrt{\frac{2\pi\hbar i(2\epsilon)}{m}}} \int_{-\infty}^{+\infty} \frac{dx_2}{\sqrt{\frac{2\pi\hbar i\epsilon}{m}}} e^{-\frac{m}{2i\hbar\epsilon} [(x_3 - x_2)^2 + \frac{1}{2}(x_2 - x_0)^2]} \\
 &= \frac{1}{\sqrt{\frac{2\pi\hbar i(3\epsilon)}{m}}} e^{-\frac{m}{2i\hbar(3\epsilon)} (x_3 - x_0)^2}
 \end{aligned}$$

Thus we continue until the x_n is performed with $x_{n+1} = x$ we find

$$\frac{1}{\sqrt{\frac{2\pi\hbar i(n+1)\epsilon}{m}}} e^{-\frac{m}{2i\hbar(n+1)\epsilon} (x_{n+1} - x_0)^2}$$

Recall that $(n+1)\epsilon = t - t_0$, so

we find the free particle Green function

$$K(x,t; x_0, t_0) = \sqrt{\frac{m}{2\pi i \hbar (t-t_0)}} e^{+i \frac{m}{\hbar} \frac{(x-x_0)^2}{2(t-t_0)}} \Theta(t-t_0)$$

Now suppose we have a Gaussian wave packet at $t_0=0$

$$\psi(x_0, t_0=0) = \frac{A}{\sqrt{2\pi}a} e^{ik_0 x_0} e^{-\frac{x_0^2}{4a^2}}$$

(recall page -34-), then at t we found that

$$\psi(x,t) = \frac{A}{\sqrt{2\pi}} \sqrt{\frac{1}{a^2 + \frac{i\hbar t}{2m}}} e^{i\left[k_0 x - \frac{\hbar k_0^2}{2m} t\right]} e^{-\frac{(x - \frac{\hbar k_0}{m} t)^2}{4(a^2 + \frac{i\hbar t}{2m})}}$$

(Note if $\int dx |\psi(x,t)|^2 = 1 = \frac{|A|^2 \sqrt{\pi}}{\sqrt{2} a} \Rightarrow A = (a\sqrt{8\pi})^{1/2}$)

We should obtain this same result by using the free particle Green function K , to evolve $\psi(x_0, t_0)$ in time to $\psi(x, t)$

-101-

That is

$$\psi(x,t) = \int_{-\infty}^{\infty} dx_0 \sqrt{\frac{m}{2\pi\hbar i t}} e^{\frac{i}{\hbar} \frac{m}{2} \frac{(x-x_0)^2}{t}} \psi(x_0, t_0)$$

$$= \int_{-\infty}^{\infty} dx_0 \sqrt{\frac{m}{2\pi\hbar i t}} e^{\frac{i}{\hbar} \frac{m}{2} \frac{(x-x_0)^2}{t}} \frac{A}{2a\sqrt{\pi}} e^{ik_0 x_0 - \frac{x_0^2}{4a^2}}$$

$$= \frac{A}{2a\sqrt{\pi}} \sqrt{\frac{m}{2\pi\hbar i t}} e^{\frac{i}{\hbar} \frac{m}{2} \frac{x^2}{t}} \times \int_{-\infty}^{\infty} dx_0 e^{-\left(\frac{1}{4a^2} - \frac{im}{2\hbar t}\right)x_0^2 - \left(\frac{im}{\hbar t}x - ik_0\right)x_0}$$

Once again we have a Gaussian integral to perform. As usual we do it by completing the square

$$\int_{-\infty}^{\infty} dx e^{-ax^2 + bx} = \int_{-\infty}^{\infty} dx e^{-a\left(x + \frac{b}{2a}\right)^2 + \frac{b^2}{4a}} = e^{+\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}}$$

This gives

-102-

$$\psi(x,t) = \frac{A}{2a\sqrt{\pi}} \sqrt{\frac{m}{2\pi\hbar i t}} e^{\frac{i}{\hbar} \frac{m}{2} \frac{x^2}{t}} \times \sqrt{\frac{\pi}{\frac{1}{4a^2} - \frac{im}{2\hbar t}}} e^{\frac{(\frac{imx}{\hbar t} - ik_0)^2}{4(\frac{1}{4a^2} - \frac{im}{2\hbar t})}}$$

$$= \frac{A}{2a\sqrt{\pi}} \sqrt{\frac{m}{2\pi\hbar i t}} \sqrt{\frac{4\pi a^2 \frac{\hbar t}{2m}}{a^2 + \frac{\hbar t}{2m}}} \times e^{\frac{i}{\hbar} \frac{m}{2} \frac{x^2}{t}} e^{-\left(\frac{m^2 x^2}{\hbar^2 t^2} + k_0^2 - \frac{2mk_0 x}{\hbar t}\right)} \times e^{-i4\left(\frac{i2\hbar t + m4a^2}{4a^2 2\hbar t}\right)}$$

$$= \frac{A}{2\sqrt{\pi}} \sqrt{\frac{1}{a^2 + \frac{\hbar t}{2m}}} \exp\left[\frac{-1}{4(a^2 + i\frac{\hbar t}{2m})} \times \left(\frac{2i\hbar k_0 a^2 t}{m} - 4ia^2 k_0 x + \frac{2imx^2 a^2}{\hbar t} - i\frac{2mx^2 a^2}{\hbar t} + x^2\right)\right]$$

$$= \frac{A}{2\sqrt{\pi}} \sqrt{\frac{1}{a^2 + \frac{\hbar t}{2m}}} e^{\frac{4ik_0 x}{4(a^2 + \frac{i\hbar t}{2m})} \left(\frac{i\hbar t}{2m} + a^2\right)} \times e^{\frac{-1}{4(a^2 + \frac{i\hbar t}{2m})} \left(x^2 + \frac{2i\hbar t a^2 k_0^2}{m} - \frac{2xk_0 \hbar t}{m}\right)}$$

Thus we finally obtain

$$\psi(x,t) = \frac{A}{2\sqrt{\pi}} \frac{1}{\sqrt{a^2 + \frac{i\hbar t}{2m}}} e^{i(k_0 x - \frac{\hbar k_0^2 t}{2m})} \times e^{-\frac{(x - \frac{\hbar k_0 t}{m})^2}{4(a^2 + \frac{i\hbar t}{2m})}}$$

in agreement with our previous result obtained from Schrödinger's equation directly (i.e. as an expansion in terms of plane wave solutions to Schrödinger's equation).

Similarly one could calculate the propagation kernel K for other Hamiltonians, for instance the one-dimensional harmonic oscillator Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2.$$

The Lagrangian is simply

$$L = p\dot{x} - H = \frac{m}{2} \dot{x}^2 - \frac{m\omega^2}{2} x^2.$$

The Green function is then

$$K(x, t; x_0, t_0) = \int_{x(t_0)=x_0}^{x(t)=x} [dx] e^{\frac{i}{\hbar} \int_{t_0}^t dt \left(\frac{1}{2} m \dot{x}^2 - \frac{m\omega^2}{2} x^2 \right)}$$

$$= \sqrt{\frac{m\omega}{2\pi\hbar i \sin\omega(t-t_0)}} \times \Theta(t-t_0) \times$$

$$\times e^{\frac{im\omega}{\hbar 2 \sin\omega(t-t_0)} [(x^2 + x_0^2) \cos\omega(t-t_0) - 2xx_0]}$$

Note for $\omega \rightarrow 0$ we just recover the free particle kernel of page -100-.