The Postulates Of Quantum Mechanics

Postulate 1: The State of the System

The set of all possible states of a physical system stand in a one-to-one correspondence with the vector directions (rays) in a Hilbert space $\mathcal{H}$. ($\mathcal{H}$ can be finite or infinite depending upon the system. To allow for continuous basis vectors we will extend the system and space to $\hat{\mathcal{H}}$.)

Since the state is described by the entire ray, we have that $|\psi>$ and $\lambda|\psi>$ with $\lambda \in \mathcal{C}$ describe the same state.

Postulate 2: Physical Observables and Hermitian Operators

The physical observables of a system stand in a one-to-one correspondence with the set of Hermitian operators on the state space $\mathcal{H}$. That is, to each measurable quantity $\mathcal{A}$, there corresponds a Hermitian operator A acting on $\mathcal{H}$. Out of the set of all Hermitian operators, there is a subset which consists of mutually commuting operators and are assumed to be complete (they form a CSCO; each A is assumed an observable).

Postulate 3: Measurement, Spectral Decomposition and Statistical Interpretation

a. The only possible result of the measurement of a physical observable $\mathcal{A}$ is one of the (real) eigenvalues of the corresponding Hermitian operator A.

b. Let $\{|\phi_k>\}$ be the simultaneous eigenstates of a CSCO so that

$$A|\phi_k>= a_k|\phi_k> \quad , \quad \text{etc.} \tag{1}$$

These states form an orthonormal basis for the state space $\mathcal{H}$.

c. For a system in state $|\psi>$ (with $<\psi|\psi>= 1$) the probability of measuring the value a_k for the physical observable $\mathcal{A}$ is

$$P_k = |<\phi_k|\psi>|^2 \quad . \tag{2}$$

d. Immediately following the measurement, the system is in the state $|\phi_k>$.

To be more explicit:

Discrete Orthonormal Basis

If the orthonormal basis is a discrete basis, then the simultaneous eigenvectors, $|\phi_{ab\ldots}>$, of the CSCO $\{A, B, \ldots\}$ obey

$$\begin{aligned} <\phi_{a'b'\ldots}|\phi_{ab\ldots}> &= \delta_{a'a}\delta_{b'b}\cdots \\ \sum_{a,b,\ldots} |\phi_{ab\ldots}><\phi_{ab\ldots}| &= 1, \end{aligned} \tag{3}$$

where

$$\begin{aligned} A|\phi_{ab\ldots}> &= a|\phi_{ab\ldots}> \\ B|\phi_{ab\ldots}> &= b|\phi_{ab\ldots}> \quad , \quad \text{etc.}, \end{aligned} \tag{4}$$

with the eigenvalues $\{a, b, \ldots\}$ taking discrete values (in 1-1 correspondence with the integers).

For an arbitrary state vector $|\psi>$, we have the expansion in terms of the $\{|\phi_{ab\ldots}>\}$ basis

$$|\psi>=\sum_{a,b,\ldots} \psi_{ab\ldots}|\phi_{ab\ldots}>, \tag{5}$$

with

$$\psi_{ab\ldots}=<\phi_{ab\ldots}|\psi> . \tag{6}$$

This implies

$$\begin{aligned} A|\psi> &= \sum_{a,b,\ldots} a \quad \psi_{ab\ldots}|\phi_{ab\ldots}> \\ B|\psi> &= \sum_{a,b,\ldots} b \quad \psi_{ab\ldots}|\phi_{ab\ldots}>, \\ \text{etc.} \end{aligned} \tag{7}$$

The probability of finding the values $a, b, \ldots$ for the system in state $|\psi>$ when $\mathcal{A}, \mathcal{B}, \ldots$ are measured is

$$P_{ab\ldots} = |<\phi_{ab\ldots}|\psi>|^2 . \tag{8}$$

Note that

$$\begin{aligned}
\sum_{a,b,\ldots} P_{ab\ldots} &= \sum_{a,b,\ldots} | < \phi_{ab\ldots}|\psi > |^2 \\
&= \sum_{a,b,\ldots} < \psi|\phi_{ab\ldots} >< \phi_{ab\ldots}|\psi > \\
&= < \psi| \underbrace{\sum_{a,b,\ldots} |\phi_{ab\ldots} >< \phi_{ab\ldots}|}_{=1} |\psi > \\
&= < \psi|\psi > \\
&= 1 \ ,
\end{aligned} \tag{9}$$

as required of a probability. Also the expectation value of $A, B, \ldots$ in state $|\psi >$ is

$$\begin{aligned}
< \psi|A|\psi > &= \sum_{a,b,\ldots} a \underbrace{\psi_{ab\ldots}}_{=<\phi_{ab\ldots}|\psi>} < \psi|\phi_{ab\ldots} > \\
&= \sum_{a,b,\ldots} a \ | < \phi_{ab\ldots}|\psi > |^2 \\
&= \sum_{a,b,\ldots} a \ P_{ab\ldots} \ , \quad \text{etc.}
\end{aligned} \tag{10}$$

Continuous Orthonormal Basis

On the otherhand if the orthonormal basis is continuous, then

$$\begin{aligned}
< \phi_{\alpha'\beta'\ldots}|\phi_{\alpha\beta\ldots} > &= \delta(\alpha' - \alpha)\delta(\beta' - \beta)\cdots \\
\int d\alpha d\beta \cdots |\phi_{\alpha\beta\ldots} >< \phi_{\alpha\beta\ldots}| &= 1,
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
A|\phi_{\alpha\beta\ldots} > &= \alpha|\phi_{\alpha\beta\ldots} > \\
B|\phi_{\alpha\beta\ldots} > &= \beta|\phi_{\alpha\beta\ldots} >, \\
&\text{etc.}
\end{aligned} \tag{12}$$

with the eigenvalues $\{\alpha, \beta, \ldots\}$ taking on a continuum of values.

For an arbitrary ket vector $|\psi >$, the expansion in terms of the continuous basis $\{|\phi_{\alpha\beta\ldots} >\}$ is

$$|\psi >= \int d\alpha d\beta \cdots \psi(\alpha, \beta, \ldots)|\phi_{\alpha\beta\ldots} >, \tag{13}$$

with

$$\psi(\alpha, \beta, \ldots) = < \phi_{\alpha\beta\ldots} | \psi > . \tag{14}$$

This implies

$$\begin{aligned} A|\psi > &= \int d\alpha d\beta \cdots \alpha \quad \psi(\alpha, \beta, \ldots) | \phi_{\alpha\beta\ldots} > \\ B|\psi > &= \int d\alpha d\beta \cdots \beta \quad \psi(\alpha, \beta, \ldots) | \phi_{\alpha\beta\ldots} > \\ \text{etc.} \end{aligned} \tag{15}$$

The probability of measuring $\mathcal{A}$ in the range α to $\alpha + d\alpha$, $\mathcal{B}$ in the range β to $\beta + d\beta$, etc. is

$$d\mathcal{P}(\alpha, \beta, \ldots) = | < \phi_{\alpha\beta\ldots} | \psi > |^2 d\alpha d\beta \cdots \quad . \tag{16}$$

Note that

$$\begin{aligned} \int d\mathcal{P}(\alpha, \beta, \ldots) &= \int d\alpha d\beta \cdots | < \phi_{\alpha\beta\ldots} | \psi > |^2 \\ &= < \psi | \underbrace{\int d\alpha d\beta \cdots | \phi_{\alpha\beta\ldots} >< \phi_{\alpha\beta\ldots} |}_{=1} | \psi > \\ &= < \psi | \psi > \\ &= 1 , \end{aligned} \tag{17}$$

as required of a probability. Also the expectation value of $A, B, \ldots$ in state $|\psi >$ is

$$\begin{aligned} < \psi | A | \psi > &= \int d\alpha d\beta \cdots \alpha \quad \underbrace{\psi(\alpha, \beta, \ldots)}_{=< \phi_{\alpha\beta\ldots} | \psi >} < \psi | \phi_{\alpha\beta\ldots} > \\ &= \int d\alpha d\beta \cdots \alpha \quad | < \phi_{\alpha\beta\ldots} | \psi > |^2 \\ &= \int d\mathcal{P}(\alpha, \beta, \ldots) \alpha \\ \text{etc.} \end{aligned} \tag{18}$$

Postulate 4: Time Evolution and The Schrödinger Equation

The time evolution of the physical states is given by the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t) >= H(t) | \psi(t) > \quad , \tag{19}$$

where $H = H(t)$ is the Hermitian Hamiltonian operator. $H(t)$ is the total energy of the system.

For an isolated system, the Hamiltonian is time independent. Schrödinger's equation is simply

$$i\hbar\frac{d}{dt}|\psi(t)>= H|\psi(t)> \quad , \tag{20}$$

with the eigenvalues of H the possible energies of the system.

In general the observables may also have explicit time dependence $A = A(t)$. (For example, a charged particle can interact with an external time varying electromagnetic field.) For an isolated system, as is the Hamiltonian, the observables are time independent, $\frac{dA}{dt} = 0$.