4.1) 2.) of the complex (real) numbers.

So the space of finite norm n-tuples of complex numbers is a Hilbert space for n-finite or infinite.

---

Thus the underlying mathematical structure of quantum mechanics begins to emerge. The wavefunctions $\psi(\vec{r})$ are in correspondence with the vectors of a Hilbert space.

As we know, instead of using the position probability amplitude $\psi(\vec{r})$ to describe the system, we could determine all physical quantities from the momentum probability amplitude

$$\varphi(\vec{k}) = \int d^3r \, e^{-i\vec{k}\cdot\vec{r}} \psi(\vec{r}) .$$

These momentum space wavefunctions are square-integrable since they correspond to a probability density in momentum space and they have an inner product

- 4.1)

$$(\varphi, \eta) \equiv \int \frac{d^3k}{(2\pi)^3}\, \varphi^*(\vec{k})\, \eta(\vec{k}),$$

they are just $L^2(\mathbb{R}^3)$ functions again with finite norm. The momentum space wavefunctions then also lie in correspondence with vectors in the same Hilbert space and are equally good objects to describe the states of the system as are the position space wavefunctions $\psi(\vec{r})$.

This is reminiscent of the case in Euclidean 3-space ($E^3 = \mathbb{R}^3$) where a vector $\vec{v}$ can be equally well described by its coordinates with respect to different bases. Each component representation of the vector being a completely equivalent description of the abstract geometric vector. Similarly, we see that the state of the system is described by an abstract vector in Hilbert space $\mathcal{H}$. Each description of the vector in terms of different bases, like the position space components

4.1) given by the wavefunctions $\psi(\vec{r})$ or the momentum space components given by the momentum wavefunctions $\varphi(\hbar)$, is an equivalent representation of the underlying vectors in the Hilbert space.

Just as the 3-vector $\vec{v}$ can be expanded in terms of different basis vectors

$$\vec{v} = \sum_{i=1}^{3} v_i \hat{e}^{(i)} = \sum_{i=1}^{3} v_i' \hat{e}'^{(i)}$$

where $v_i$ are the component of $\vec{v}$ in the $\hat{e}^{(i)}$-system and $v_i'$ are the components of $\vec{v}$ in the $\hat{e}'^{(i)}$-system, so too we can view the $\psi(\vec{r})$ and $\varphi(\hbar)$ as the components of the coordinate free, abstract vector $\psi$ in 2 different bases, the coordinate and momentum bases. And just as the 3-vector components $v_i$, $v_i'$ can be given in terms of the inner product of the basis vectors and $\vec{v}$

$$v_i = \hat{e}^{(i)} \cdot \vec{v} \quad ; \quad v_i' = \hat{e}'^{(i)} \cdot \vec{v}$$

4.1) So too we can represent the spatial and momentum wavefunctions, as we shall see.

In order to make clear these observations it is useful to introduce a notation for vectors and inner products that was developed by Dirac called Dirac (bra-ket) notation.

Instead of just calling each vector in the Hilbert space by $\psi$, $\phi$, etc. Dirac enclosed them in a 'ket notation and the vectors are called ket-vectors, so we have that

$|\psi\rangle$, $|\phi\rangle$ etc., denote the

vectors in our Hilbert space. The label inside $|\;\rangle$ denotes the element of the space under consideration. So in Dirac notation we write the sum of 2 vectors as $|\psi\rangle + |\phi\rangle$, and the multiplication of a vector by

4.1) a complex number $\lambda$ as $\lambda|\psi\rangle$. The ket-vector for $(\lambda\psi)$ is just this product, so in Dirac notation $|\lambda\psi\rangle \equiv \lambda|\psi\rangle$.

Given any vector space we can always define another vector space dual to it by considering the set of all linear functionals (maps) on the space.

A linear functional $\phi$ is a map of $\mathcal{H}$ into the complex numbers that is linear, for $|\chi\rangle, |\psi\rangle \in \mathcal{H}$

$$\phi(|\psi\rangle) \in \mathbb{C}$$

and $\phi(|\psi\rangle + \lambda|\chi\rangle) = \phi(|\psi\rangle) + \lambda\phi(|\chi\rangle)$.

Further, a bounded linear functional is a linear functional for which

$$|\phi(|\psi\rangle)| \leq M \, \||\psi\rangle\| \quad , \quad M < \infty,$$

4.1) where recall $\| |\psi\rangle \| = (|\psi\rangle, |\psi\rangle)^{1/2}$.

The <u>norm</u> of $\phi$ is defined as the minimum value of $M$ for all $|\psi\rangle \in \mathcal{H}$,

$$\|\phi\| \equiv M_\phi = \inf\{M \geq 0 : |\phi(|\psi\rangle)| \leq M \| |\psi\rangle \| \text{ for } \forall |\psi\rangle \in \mathcal{H}\}.$$

---

The set of all linear functionals on $\mathcal{H}$ defines a vector space called the <u>dual (space) to $\mathcal{H}$</u> and is denoted $\mathcal{H}^*$. That is, if $\phi$ and $\Omega$ are linear functionals, then

$$(\phi + \Omega)(|\psi\rangle) \equiv \phi(|\psi\rangle) + \Omega(|\psi\rangle)$$

addition of complex numbers.

and

$$\lambda\phi(|\psi\rangle) \quad \text{are again}$$

linear functionals. And the addition of functionals and their multiplication by complex numbers obey all the required

Properties of a vector space.



4.1.) (The set of all bounded linear functionals, in fact, form a normed vector space with norm defined by $\|\phi\| = M_\phi$.)

---

According to the notation of Dirac, the vectors in $\mathcal{H}^*$ are denoted by a <u>bra</u> notation $\langle\phi|$ and are called bras or <u>bra vectors</u>. The label inside $\langle\ |$ denotes the vector of $\mathcal{H}^*$ under consideration.

Dirac then uses a bra-ket (bracket) notation to represent the mapping of each linear functional $\phi(|\psi\rangle)$.

That is he writes $\phi(|\psi\rangle)$ as

$$\phi(|\psi\rangle) = \langle\phi|\psi\rangle.$$

---

4.1) The inner product in $\mathcal{H}$ defines a bounded linear functional.

For $|\phi\rangle \in \mathcal{H}$

$$\left(\phi(|\psi\rangle) =\right) \; \langle\phi|\psi\rangle \equiv \left(|\phi\rangle, |\psi\rangle\right)$$

is a linear functional with norm

$\| \langle\phi| \| = \| |\phi\rangle \|$. Thus for every ket in $\mathcal{H}$, $|\phi\rangle \in \mathcal{H}$, their corresponds a bra in $\mathcal{H}^*$, $\langle\phi| \in \mathcal{H}^*$, defined by the inner product

$$\langle\phi|\psi\rangle = \left(|\phi\rangle, |\psi\rangle\right).$$

---

Since we have just shown that to every ket-vector there corresponds a bra-vector, we can ask whether there is a ket-vector corresponding to every bra-vector. A theorem due to Riesz guarantees the correspondence for bounded linear functionals

4.1.) <u>Riesz's Theorem</u>: Every bounded linear functional $(\phi =) \langle \phi |$ can be expressed in the form

$$(\phi(|\psi\rangle) =)\ \langle \phi | \psi \rangle = (|\phi\rangle, |\psi\rangle)$$

where $|\phi\rangle$ is an element of $\mathcal{H}$ uniquely determined by $\langle \phi |$ and $(|\phi\rangle, |\psi\rangle)$ is the inner product in $\mathcal{H}$.

---

For a proof see

1) Akhiezer and Glazman: "Theory of Linear Operators in Hilbert Space Vol. 1", page 33.

2) Riesz and Sz.-Nagy: "Functional Analysis", page 61.

3) Stakgold: "Boundary Value Problems of Mathematical Physics Vol. 1", page 136.

4) Jauch: "Foundations of Quantum Mechanics", page 31.

4.1) The idea of the proof is to expand any vector $|\psi\rangle$ in terms of an orthonormal basis of vectors $|e_n\rangle$

$$|\psi\rangle = \sum_n{}' c_n |e_n\rangle$$

Then any functional $\langle\phi|$ can be written as

$$\langle\phi|\psi\rangle = \sum_n{}' c_n \langle\phi|e_n\rangle .$$

The ket-vector in $\mathcal{H}$, $|\phi\rangle$, corresponding to the $\langle\phi|$ is

$$|\phi\rangle \equiv \sum_n{}' \langle\phi|e_n\rangle^* |e_n\rangle .$$

Since

$$(|\phi\rangle, |\psi\rangle) = \sum_{m,n}{}' \left(\langle\phi|e_n\rangle^* |e_n\rangle, c_m |e_m\rangle\right)$$

$$= \sum_{m,n}{}' \langle\phi|e_n\rangle c_m \underbrace{(|e_n\rangle, |e_m\rangle)}_{=\delta_{mn}}$$

$$= \sum_n{}' c_n \langle\phi|e_n\rangle$$

$$= \langle\phi|\psi\rangle .$$

4.1) Thus $\mathcal{H}$ and $\mathcal{H}^*$ are isomorphic.

In fact using Riesz's Theorem the scalar product in $\mathcal{H}^*$ can be defined, it is just that of corresponding kets in $\mathcal{H}$. Also then $\mathcal{H}^*$ can be shown to be a Hilbert space.

Since every functional has the form

$$\langle\phi|\psi\rangle = (|\phi\rangle, |\psi\rangle)$$

we will use the Dirac bra-ket notation in order to denote the inner product from now on. Hence recall the properties of the inner product using Dirac notation

1) $\langle\psi|\phi\rangle^* = \langle\phi|\psi\rangle$

2) $\langle\phi|\psi_1+\psi_2\rangle = \langle\phi|\psi_1\rangle + \langle\phi|\psi_2\rangle$

3) $\langle\phi|\lambda\psi\rangle = \lambda\langle\phi|\psi\rangle$

4) $\langle\psi|\psi\rangle \geq 0$ with equality only if $|\psi\rangle = 0$.

4.1) Properties 2) & 4) express the linearity of the functional in the ket-vector

$$\langle \phi | \psi_1 + \lambda \psi_2 \rangle = \langle \phi | \psi_1 \rangle + \lambda \langle \phi | \psi_2 \rangle$$

Property 1) implies, as previously, the functional is anti-linear in the bra-vector

$$\langle \phi_1 + \lambda \phi_2 | \psi \rangle = \langle \phi_1 | \psi \rangle + \lambda^* \langle \phi_2 | \psi \rangle .$$

The inner-product associates the bra-vector $\langle \phi_1 | + \lambda^* \langle \phi_2 |$ in $\mathcal{H}^*$ with the ket-vector $|\phi_1\rangle + \lambda |\phi_2\rangle$ in $\mathcal{H}$. (i.e. the bra corresponding to $|\lambda\phi\rangle = \lambda|\phi\rangle$ is $\langle \lambda\phi| = \lambda^* \langle \phi|$. The bra-vectors are like the complex conjugate of the wavefunctions, to each $\psi(\vec{r})$ there corresponds a $\psi^*(\vec{r})$ and vice versa. Corresponding to $\lambda\psi(\vec{r})$ is $(\lambda\psi(\vec{r}))^* = \lambda^* \psi^*(\vec{r})$.)

4.1) Technical Aside: It is useful to introduce non-normalizable vectors, therefore not belonging to ($\mathcal{H}$) $L^2$, in terms of which we may expand the normalizable vectors. For example plane-waves in 1-dimension are non-normalizable basis vectors

$$\psi(x) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar}\; e^{\frac{i}{\hbar}px}\,\tilde{\psi}(p)$$

$$\tilde{\psi}(p) = \int_{-\infty}^{+\infty} dx\; e^{-\frac{i}{\hbar}px}\,\psi(x) \quad .$$

This can be interpreted as expanding our state vectors $\psi$ in terms of basis vectors $\{u_p(x)\}$ with $p$ a continuous index labeling their components

i.e. $$u_p(x) \equiv e^{\frac{i}{\hbar}px} \quad .$$

As we know (Fourier Thm.) these basis vectors are complete

$$\int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar}\, u_p^*(y)\, u_p(x) = \delta(x-y)$$

and are "normalized" according to "continuum normalization conditions"

4.1) $$\int_{-\infty}^{+\infty} dx\; u_p^*(x)\, u_{p'}(x) = \delta(p-p').$$

The abstract vector corresponding to the plane-wave function $u_p(x)$ we denote $|u_p\rangle$, it is not an element of the Hilbert space of physical vectors, $|u_p\rangle \notin \mathcal{H}$.

However

$$\langle u_p|\psi\rangle = \int_{-\infty}^{+\infty} dx\; u_p^*(x)\, \psi(x)$$
$$= \tilde{\psi}(p) \in \mathbb{C}.$$

The Fourier transform defines a linear functional, thus $\langle u_p|$ is an element of the dual space of linear functionals but $|u_p\rangle \notin \mathcal{H}$. In order to maintain the symmetry of states in $\mathcal{H}$ and the dual space to $\mathcal{H}$, we <u>extend</u> our Hilbert space $\mathcal{H}$ to include <u>generalized</u> <u>kets</u> with infinite (continuum) norms but whose scalar product with every ket of $\mathcal{H}$ is finite. This extended space is denoted $\hat{\mathcal{H}}$ and is isomorphic to the extension of the space of bounded linear functionals $\hat{\mathcal{H}}^*$.

4.1) Thus to each bra vector $\langle\psi|$ of $\hat{\mathcal{H}}^*$ there corresponds a ket vector $|\psi\rangle \in \mathcal{H}^*$ and vice-versa.

---

From now on we will use Dirac notation when considering vector spaces.

In particular two vectors $|\psi\rangle$ and $|\phi\rangle$ are said to be orthogonal if

$$\langle\phi|\psi\rangle = 0.$$

A set of vectors $|\psi_1\rangle, |\psi_2\rangle, \ldots$ is independent if there is no set of complex numbers $\lambda_1, \lambda_2, \ldots$ for which the equation

$$\lambda_1 |\psi_1\rangle + \lambda_2 |\psi_2\rangle + \cdots = 0$$

is satisfied except $\lambda_1 = \lambda_2 = \cdots = 0$.

Note: Any set of orthogonal vectors $\{|\psi_k\rangle\}$ is independent since

$$0 = \sum_k \lambda_k |\psi_k\rangle \quad \Rightarrow$$

4.1)

$$0 = \langle \psi_j | \sum_k{}' \lambda_k \psi_k \rangle = \sum_k{}' \lambda_k \underbrace{\langle \psi_j | \psi_k \rangle}_{= \langle \psi_j | \psi_j \rangle \delta_{jk}}$$

$$= \lambda_j \langle \psi_j | \psi_j \rangle$$

$$\Rightarrow \lambda_j = 0 \quad \forall j .$$

---

A vector space has <u>dimensionality</u> $N$ if the maximum number of independent vectors is $N$. Any set of $N$-independent vectors $\{|\psi_k\rangle\}$ in such a space is <u>complete</u>, because given any vector $|\psi\rangle$, there must be some (unique) non-trivial linear relation

$$0 = \lambda |\psi\rangle + \sum_k{}' \lambda_k |\psi_k\rangle$$

with $\lambda \neq 0$ (otherwise the dimension is $(N+1)$).

---

<u>Theorem</u>: For a $N$-dimensional space, no set of $M < N$ vectors is complete.

4.1) Proof: Let $\{|\psi_k\rangle\}$ be a complete independent set of M vectors, and let $\{|\phi_\ell\rangle\}$ be a complete independent set of N vectors.

Then 1) $|\phi_1\rangle$ must be a linear combination of $|\psi_1\rangle, \ldots, |\psi_M\rangle$

2) let $|\psi_k\rangle$ be any vector in this linear combination, then

$$|\psi_1\rangle, \ldots, |\psi_{k-1}\rangle, |\phi_1\rangle, |\psi_{k+1}\rangle, \ldots, |\psi_M\rangle$$

must be a complete independent set.

3) Repeat this process with $|\phi_2\rangle$, then

$$|\psi_1\rangle, \ldots, |\psi_{k-1}\rangle, |\phi_1\rangle, |\psi_{k+1}\rangle, \ldots, |\psi_{\ell-1}\rangle, |\phi_2\rangle, |\psi_{\ell+1}\rangle, \ldots, |\psi_M\rangle$$

is a complete independent set.

4) Continue this procedure until all the $|\psi_k\rangle$ vectors are eliminated.

4.1) But then $|\phi_1\rangle, \ldots, |\phi_M\rangle$ is a complete set. Hence $|\phi_1\rangle, \ldots, |\phi_N\rangle$ cannot be independent for $N > M$, this is a contradiction.

---

Infinite dimensionality means that there is no maximum number of independent vectors.

The above theorem shows that no finite set of vectors can be complete in an infinite dimensional space.

---

Discrete Orthonormal Basis in $\mathcal{H}$:

If $\{|\phi_k\rangle\}$ forms a complete independent set of vectors, then any arbitrary ket vector $|\psi\rangle$ can be expanded as

$$|\psi\rangle = \sum_k{}' c_k |\phi_k\rangle.$$

The $\{c_k\}$ are the components of $|\psi\rangle$ in the $\{|\phi_k\rangle\}$ basis. The uniqueness of $\{c_k\}$

4.1) follows from the independence of $\{|\phi_k\rangle\}$.

Moreover, if the $\{|\phi_k\rangle\}$ basis is an <u>orthonormal basis</u>, that is $\langle\phi_k|\phi_\ell\rangle = \delta_{k\ell}$, then

$$\langle\phi_\ell|\psi\rangle = \langle\phi_\ell|\sum_k' c_k\phi_k\rangle$$
$$= \sum_k' c_k \langle\phi_\ell|\phi_k\rangle = \sum_k' c_k\delta_{k\ell}$$
$$= c_\ell .$$

Hence for any $|\psi\rangle$

$$|\psi\rangle = \sum_k' c_k|\phi_k\rangle$$
$$= \sum_k' \langle\phi_k|\psi\rangle\ |\phi_k\rangle$$
$$= \sum_k' |\phi_k\rangle\langle\phi_k|\psi\rangle$$

Since $|\psi\rangle$ is arbitrary, we have

$$\sum_k' |\phi_k\rangle\langle\phi_k| = \mathbb{1} \qquad \left(\begin{array}{c}\mathbb{1} \text{ is the} \\ \text{identity operator} \\ \text{in } \mathcal{H}\end{array}\right)$$

4.1) The operator $|\phi_k\rangle\langle\phi_k| \equiv P_k$ is the projection operator onto the $|\phi_k\rangle$ vector,

$$P_k|\psi\rangle \equiv \langle\phi_k|\psi\rangle\,|\phi_k\rangle\,. \quad \text{Hence}$$

$\sum_k' |\phi_k\rangle\langle\phi_k| = 1$ is the completeness or closure identity for the basis vectors $\{|\phi_k\rangle\}$.

---

If $|\psi\rangle = \sum_k' c_k|\phi_k\rangle$ and $|\psi'\rangle = \sum_k' c_k'|\phi_k\rangle$,

then $\langle\psi'| = \langle \sum_k' c_k' \phi_k| = \sum_k' c_k'^* \langle\phi_k|$,

recall due to the anti-linearity of the inner product (functional), so bra and ket vectors are anti-linearly related — they are conjugates of each other

So $\langle\psi'|\psi\rangle = \langle \sum_k' c_k'\phi_k | \sum_l' c_l\phi_l\rangle$

$$= \sum_k'\sum_l' c_k'^* c_l \underbrace{\langle\phi_k|\phi_l\rangle}_{=\delta_{kl}}$$

$\boxed{\langle\psi'|\psi\rangle = \sum_k' c_k'^* c_k}$ for inner products

and the norm of $|\psi\rangle$ is given by

- 4.1)
$$\|\psi\|^2 = \langle\psi|\psi\rangle = \sum_k |c_k|^2.$$

Thm: If $\{|\phi_k\rangle\}$ forms an independent set, then

$$|\psi_1\rangle = \lambda_{11}|\phi_1\rangle$$

$$|\psi_2\rangle = \lambda_{21}|\phi_1\rangle + \lambda_{22}|\phi_2\rangle$$

$$|\psi_3\rangle = \lambda_{31}|\phi_1\rangle + \lambda_{32}|\phi_2\rangle + \lambda_{33}|\phi_3\rangle$$

$$\vdots$$

can be chosen as an orthonormal set by the Gramm-Schmidt process.

Proof: This is true for 1-vector by letting

$$\lambda_{11} = \frac{1}{\sqrt{\langle\phi_1|\phi_1\rangle}}.$$

Assume it is true for $n$-vectors, then choose

$$|\psi_{n+1}\rangle = c\left[\rho_1|\psi_1\rangle + \cdots + \rho_n|\psi_n\rangle + |\phi_{n+1}\rangle\right]$$

where $c = \lambda_{(n+1)(n+1)}$ and for $0 \le k \le n$

i.e. $$|\psi_1\rangle = \frac{|\phi_1\rangle}{\sqrt{\langle\phi_1|\phi_1\rangle}} = \frac{|\phi_1\rangle}{\||\phi_1\rangle\|}$$

$$|\psi_2\rangle = \frac{|\phi_2\rangle - \langle\psi_1|\phi_2\rangle|\psi_1\rangle}{\||\phi_2\rangle - \langle\psi_1|\phi_2\rangle|\psi_1\rangle\|}$$

$$|\psi_3\rangle = \frac{|\phi_3\rangle - \langle\psi_1|\phi_3\rangle|\psi_1\rangle - \langle\psi_2|\phi_3\rangle|\psi_2\rangle}{\| \quad '' \quad \|}$$

$\vdots$

etc.

$$0 = \langle \psi_k | \psi_{n+1} \rangle = c\rho_k + c \langle \psi_k | \phi_{n+1} \rangle .$$

Hence $\boxed{\rho_k = -\langle \psi_k | \phi_{n+1} \rangle .}$ In addition

$$1 = \langle \psi_{n+1} | \psi_{n+1} \rangle$$

$$= |c|^2 \Big\{ \sum_{k=1}^{n} |\rho_k|^2 + \langle \phi_{n+1} | \phi_{n+1} \rangle + \sum_{k=1}^{n} \rho_k \langle \psi_k | \phi_{n+1} \rangle + \sum_{k=1}^{n} \rho_k^* \langle \phi_{n+1} | \psi_k \rangle \Big\}$$

Fixes the constant $c$. This completes the proof of the Gramm-Schmidt process.

---

In particular, for any subspace $S$ of dimension $M$ in a larger dimensional space, we can find an orthonormal set in which the first $M$ vectors are in $S$.

4.1) Examples of Discrete Bases:

1) $\mathbb{R}^3$ has basis vectors $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \{\hat{i}, \hat{j}, \hat{k}\}$ so that $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$. In Dirac notation we write $\hat{e}_1 = |e_1\rangle$, $\hat{e}_2 = |e_2\rangle$, $\hat{e}_3 = |e_3\rangle$ and their orthonormality as

$$\langle e_i | e_j \rangle = \delta_{ij} \; (= \hat{e}_i \cdot \hat{e}_j).$$

Any vector $|v\rangle \in \mathbb{R}^3$ has the unique expansion in terms of $\{|e_i\rangle\}$

$$(\vec{v} =) \; |v\rangle = \sum_{i=1}^{3} v_i |e_i\rangle \text{ where}$$

$v_i = \langle e_i | v \rangle$. Hence, the dot product of $\vec{v}$ and $\vec{u}\,(= |u\rangle)$ is

$$(\vec{v} \cdot \vec{u} =) \; \langle v | u \rangle = \sum_{i=1}^{3} v_i u_i .$$

---

- 4.1) 2) In wavefunction space $L^2(\mathbb{R}^3)$ we have come across several bases. In particular, we had the energy eigenfunction basis (ex. Hermite-polynomials)

$$H\psi_n(\vec{r}) = E_n\psi_n(\vec{r}) \quad , n=0,1,2,\ldots$$

Then any wavefunction

$$\psi(\vec{r}) = \sum_n{}' c_n\psi_n(\vec{r}).$$ The

inner product was given by

$$\langle\psi|\psi_n\rangle \equiv \int d^3r\, \psi^*(\vec{r})\psi_n(\vec{r})$$

and for orthonormal $\{|\psi_n\rangle\}$ we have

$$\langle\psi_n|\psi_m\rangle = \int d^3r\, \psi_n^*(\vec{r})\psi_m(\vec{r}) = \delta_{mn},$$

so $c_n = \langle\psi_n|\psi\rangle = \int d^3r\, \psi_n^*(\vec{r})\psi(\vec{r})$.

For another wavefunction $\phi(\vec{r})$ where

$\phi(\vec{r}) = \sum_n{}' d_n\psi_n(\vec{r})$ we have

$$\langle\phi|\psi\rangle = \sum_n{}' d_n^* c_n .$$

Continuous Orthonormal Basis in $\mathcal{H}$:

When we include vectors labelled by a continuous index in our space, we have bases sets of such vectors for $\mathcal{H}$. Their properties are analogous to the discrete index bases of $\mathcal{H}$.

If $\{|\phi_\alpha\rangle\}$ is a complete orthonormal basis depending on a continuous parameter $\alpha$, then arbitrary ket vectors $|\psi\rangle, |\phi\rangle$ can be written as

$$|\psi\rangle = \int d\alpha\, \psi(\alpha)\, |\phi_\alpha\rangle$$

$$|\phi\rangle = \int d\alpha\, \phi(\alpha)\, |\phi_\alpha\rangle .$$

Since

$$\langle \phi_\alpha | \phi_\beta \rangle = \delta(\alpha-\beta)$$, due to the orthonormal continuum normalization, we have

$$\psi(\alpha) = \langle \phi_\alpha | \psi \rangle = \int d\beta\, \psi(\beta) \underbrace{\langle \phi_\alpha | \phi_\beta \rangle}_{=\delta(\alpha-\beta)} .$$

Likewise the inner product between

$|\psi\rangle$ and $|\phi\rangle$ becomes

$$\langle\psi|\phi\rangle = \int d\alpha\, d\beta\, \psi^*(\alpha)\, \phi(\beta) \langle\phi_\alpha|\phi_\beta\rangle$$
$$= \int d\alpha\, d\beta\, \psi^*(\alpha)\, \phi(\beta)\, \delta(\alpha-\beta)$$
$$= \int d\alpha\, \psi^*(\alpha)\, \phi(\alpha) \; .$$

and the norm of a element of $\underline{\underline{\mathcal{H}}}$ becomes

$$\langle\psi|\psi\rangle = \int d\alpha\, \psi^*(\alpha)\psi(\alpha) \; .$$

---

<u>examples</u>: i) Clearly our plane-wave states have been such a set of basis vectors. Any wavefunction $\psi(\vec{r})$ can be expanded in terms of them

$$\psi(\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, \tilde{\psi}(\vec{k})$$

and defining $u_{\vec{k}}(\vec{r}) \equiv e^{i\vec{k}\cdot\vec{r}}$ as the continuum indexed basis vectors we have

$$\langle u_{\vec{k}} | u_{\vec{l}}\rangle = \delta^3(\vec{k}-\vec{l})\,(2\pi)^3$$
$$= \int d^3r\, u^*_{\vec{k}}(\vec{r})\, u_{\vec{l}}(\vec{r}) \, .$$

2) Another example of continuum labelled basis vectors are given by the position eigenfunctions $\{\varphi_{\vec{r}_0}\}$

$$\varphi_{\vec{r}_0}(\vec{r}) = \delta^3(\vec{r}-\vec{r}_0) .$$

They are complete, recall, since

$$\int d^3r_0 \, \varphi^*_{\vec{r}_0}(\vec{r})\,\varphi_{\vec{r}_0}(\vec{r}\,') = \delta^3(\vec{r}-\vec{r}\,'),$$

and continuum orthonormal

$$\int d^3r \, \varphi^*_{\vec{r}_0}(\vec{r})\,\varphi_{\vec{r}_0'}(\vec{r}) = \delta^3(\vec{r}_0-\vec{r}_0\,').$$

Any wavefunction $\psi(\vec{r})$ can be expanded in terms of them

$$\psi(\vec{r}) = \int d^3r_0 \, f(\vec{r}_0)\,\varphi_{\vec{r}_0}(\vec{r}) .$$

Here we have

$$\langle \varphi_{\vec{r}_0} | \psi \rangle = \int d^3r \, \varphi^*_{\vec{r}_0}(\vec{r})\,\psi(\vec{r}) = \psi(\vec{r}_0)$$

but also

$$= \int d^3r \int d^3r_0' \, f(\vec{r}_0\,')\,\varphi^*_{\vec{r}_0}(\vec{r})\,\varphi_{\vec{r}_0'}(\vec{r})$$

$$= \int d^3r_0' \, f(\vec{r}_0\,')\,\delta^3(\vec{r}_0-\vec{r}_0\,')$$

$$= f(\vec{r}_0) .$$