III.) Finally, the Schrödinger equation for the relative motion wavefunction can be studied (denote $\underline{\nabla_r}^2$ by $\underline{\nabla}^2$ now)

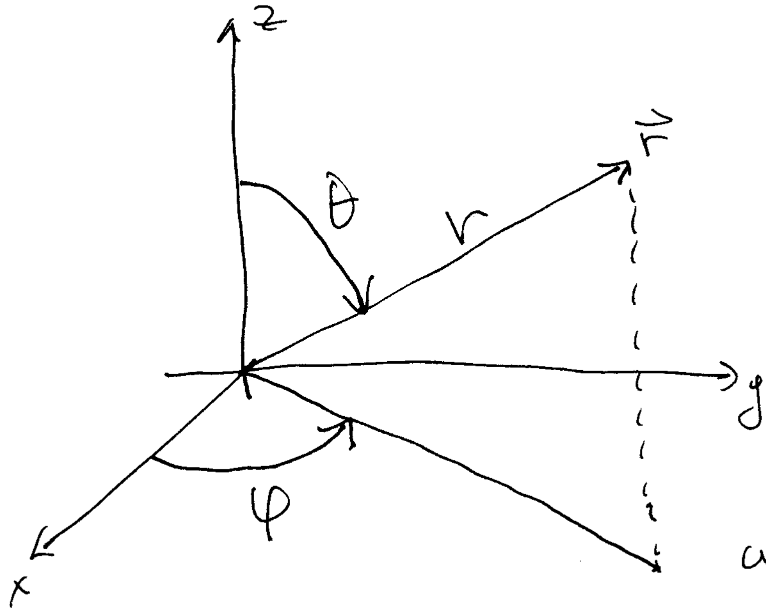
$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(|\vec{r}|)\right]\psi(\vec{r}) = E\,\psi(\vec{r})$$

This is just the Schrödinger equation for a particle of mass m moving in the central potential $V = V(|\vec{r}|)$.

A.) Spherical Polar Coordinates and Separation of Variables

Since the interaction of the (relative) particle with the potential only depends on the magnitude $|\vec{r}|$ of the position coordinate we can imagine that the motion in the other directions with $|\vec{r}|$ fixed should separate from the $|\vec{r}|$ radial motion — This is readily seen by using spherical polar coordinates:

iii. A.)



$x = r\sin\theta\cos\varphi$
$y = r\sin\theta\sin\varphi$
$z = r\cos\theta$

$0 \le \theta \le \pi$
$0 \le \varphi \le 2\pi$
and $r^2 = x^2 + y^2 + z^2$.

The Laplacian ∇^2 is spherical polar coordinates is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

In spherical polar coordinates

$$V = V(|\vec{r}|) = V(r).$$

Hence we try to solve Schrödinger's eq. by separation of variables

III.A.) <u>Ansatz</u>: $\psi(\vec{r}) = R(r)\, Y(\theta,\varphi)$

Plug into Sch. Eq.:

$$\left(-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)\right)Y(\theta,\varphi)$$

$$+\frac{R(r)}{r^2}\left(-\frac{\hbar^2}{2m}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y(\theta,\varphi)\right)$$

$$+\left[V(r)R(r)\right]Y(\theta,\varphi) = E\,R(r)\,Y(\theta,\varphi)$$

So multiply by r^2 and $\div\, RY$

$\Rightarrow$

$$\boxed{\underbrace{\frac{1}{R(r)}\left[\frac{-\hbar^2}{2m}\frac{d}{dr}\left(r^2\frac{d}{dr}R(r)\right) + r^2\left(V(r)-E\right)R(r)\right]}_{\text{function of } r \text{ only } \left(=\frac{-\hbar^2}{2m}\lambda\right)\ (=\text{separation constant})} = \underbrace{-\frac{1}{Y(\theta,\varphi)\sin^2\theta}\left(\frac{-\hbar^2}{2m}\right)\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}Y(\theta,\varphi)\right) + \frac{\partial^2}{\partial\varphi^2}Y(\theta,\varphi)\right]}_{\text{function of } (\theta,\varphi) \text{ only}}}$$

$$= -\frac{\hbar^2}{2m}\lambda \quad (=\text{separation constant})$$

III A) Thus we have separated the equations

1) The Radial Equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}R(r)\right) + \left\{\frac{2m}{\hbar^2}(E-V(r)) - \frac{\lambda}{r^2}\right\}R(r) = 0$$

2) The angular equation:

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}Y(\theta,\varphi)\right) + \lambda\sin^2\theta\, Y(\theta,\varphi) + \frac{\partial^2}{\partial\varphi^2}Y(\theta,\varphi) = 0$$

Since the φ-derivative is isolated in the $\measuredangle$ equation, we can try separation of angular variables

Ansatz:

$$Y(\theta,\varphi) = A\,\Theta(\cos\theta)\,\Phi(\varphi)$$

where A = constant which will be used to later normalize $Y(\theta,\varphi)$.

III. A) Plugging into the χ equation and dividing by $Y(\theta,\varphi)$ $\Rightarrow$

$$\underbrace{\frac{1}{\Theta(\cos\theta)}\left[\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\Theta(\cos\theta)\right)+\lambda\sin^2\theta\,\Theta(\cos\theta)\right]}_{\text{function of }\theta\text{ only}\ \equiv\ \nu\ =\ \text{(separation constant)}} = \underbrace{-\frac{1}{\Phi(\varphi)}\frac{d^2}{d\varphi^2}\Phi(\varphi)}_{\text{function of }\varphi\text{ only}\ \equiv\ \nu\ (=\text{separation constant})}$$

Thus we have separated the angular equation

1) Polar angular Equation:

$$\boxed{\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\Theta(\cos\theta)\right)+\left(\lambda-\frac{\nu}{\sin^2\theta}\right)\Theta(\cos\theta)=0}$$

2) Azimuthal angular equation:

$$\boxed{\frac{d^2}{d\varphi^2}\Phi(\varphi)+\nu\,\Phi(\varphi)=0}$$

-III.A) We can immediately solve the azimuthal eq.

$$\Phi(\varphi) = \begin{cases} A e^{i\sqrt{\nu}\varphi} + B e^{-i\sqrt{\nu}\varphi} & , \nu \neq 0 \\ A' + B'\varphi & , \nu = 0. \end{cases}$$

We must now impose our continuity conditions on the wavefunction, $\Phi(\varphi)$ and $\frac{d\Phi(\varphi)}{d\varphi}$ are continuous throughout the domain $0 \leq \varphi \leq 2\pi$. In particular after rotation by 2π (requiring single valuedness) we are back to the same point in space, So $\Phi(0) = \Phi(2\pi)$.

For $\nu \neq 0 \Rightarrow \sqrt{\nu} = m$, an integer

For $\nu = 0 \Rightarrow B' = 0$. Thus all

cases and solutions can be written as

$$\boxed{\Phi_m(\varphi) = A e^{im\varphi} \quad , m = 0, \pm 1, \pm 2, \ldots}$$

III A) The polar angle equation with $\nu = m^2$ now becomes

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta}\Theta(\cos\theta)\right) + \left(\lambda - \frac{m^2}{\sin^2\theta}\right)\Theta(\cos\theta) = 0$$

Define $\boxed{\xi \equiv \cos\theta}$ so that as $0 \le \theta \le \pi$
ξ ranges $-1 \le \xi \le +1$.

So $\frac{d}{d\theta} = \frac{d\xi}{d\theta}\frac{d}{d\xi} = -\sin\theta \frac{d}{d\xi}$ and

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta}\Theta(\cos\theta)\right)$$

$$= \frac{1}{\sin\theta}\left(-\sin\theta\frac{d}{d\xi}\right)\left(-\sin^2\theta \frac{d}{d\xi}\Theta(\xi)\right)$$

$$= \frac{d}{d\xi}\left[\sin^2\theta \frac{d}{d\xi}\Theta(\xi)\right]$$

$$= \frac{d}{d\xi}\left[(1-\xi^2)\frac{d\Theta(\xi)}{d\xi}\right].$$

$\Rightarrow$ The Polar angle eq. becomes

$$\boxed{\frac{d}{d\xi}\left[(1-\xi^2)\frac{d\Theta(\xi)}{d\xi}\right] + \left(\lambda - \frac{m^2}{1-\xi^2}\right)\Theta(\xi) = 0}$$

Sketch of Solution:

III A) 1) Notice as $\xi \to \pm 1$, the last term $\frac{-m}{1-\xi^2}\Theta$ is singular! We can eliminate this singularity by defining

$$\boxed{\Theta(\xi) \equiv (1-\xi^2)^{\frac{|m|}{2}} \mathcal{P}(\xi)}\ .$$

Plugging this into the Polar eq. we find (eventually) an equation for $\mathcal{P}(\xi)$:

$$(1-\xi^2)\frac{d^2}{d\xi^2}\mathcal{P}(\xi) - 2(|m|+1)\xi\frac{d}{d\xi}\mathcal{P}(\xi)$$

$$+ \underbrace{(\lambda - m^2 - |m|)}_{=(\lambda - |m|(|m|+1))}\mathcal{P}(\xi) = 0$$

2) This equation is satisfied by

$$\boxed{\mathcal{P}(\xi) \equiv \frac{d^{|m|}}{d\xi^{|m|}} P(\xi)}$$

with $P(\xi)$ obeying the differential equation for $m=0$!!:

$$\boxed{(1-\xi^2)\frac{d^2P(\xi)}{d\xi^2} - 2\xi\frac{dP(\xi)}{d\xi} + \lambda P(\xi) = 0}$$

III A) Thus we find for the range $-1 \leq \xi \leq 1$

$$\boxed{\begin{gathered}\Theta(\xi) = (1-\xi^2)^{\frac{|m|}{2}} \frac{d^{|m|}}{d\xi^{|m|}} P(\xi) \\ \text{where } P(\xi) \text{ obeys the equation} \\ (1-\xi^2)\frac{d^2}{d\xi^2}P(\xi) - 2\xi\frac{d}{d\xi}P(\xi) + \lambda P(\xi) = 0 \\ \text{That is} \\ \frac{d}{d\xi}\left[(1-\xi^2)\frac{d}{d\xi}P(\xi)\right] + \lambda P(\xi) = 0\end{gathered}}$$

This is just <u>Legendre's Equation</u> for $P(\xi)$

It has well behaved solutions in the region $-1 \leq \xi \leq +1$, including the end points $\xi = \pm 1$, only if the separation constant λ has the discrete values

$$\boxed{\lambda = l(l+1) \quad , \quad l = 0, 1, 2, \ldots}$$

The resulting solutions $P_l(\xi)$ are then polynomials in ξ, called the Legendre polynomials

- III A)

$$\boxed{P_\ell(\xi) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{d\xi^\ell}(\xi^2-1)^\ell}$$

Sketch of Solution:

1) Ansatz: Power Series Solution

$$P(\xi) = \sum_{L=0}^{\infty}{}' a_L \xi^L$$

Substitute into Legendre eq. $\Rightarrow$ recursion relation for a_L

1) $$2a_2 = -\lambda a_0$$

2) $$6a_3 = (2-\lambda)a_1$$

3) $L \geq 2$: $$(L+1)(L+2)\, a_{L+2} = [L(L+1) - \lambda]\, a_L$$

2) Suppose $\lambda \neq \ell(\ell+1)$, $\ell = 0, 1, 2, \ldots$

$$\frac{a_{L+2}}{a_L} = \frac{L(L+1)-\lambda}{(L+1)(L+2)} \xrightarrow{L\to\infty} 1$$

So as $\xi \to 1$, the series will diverge like

$$\frac{1}{(1-\xi^2)} = \sum_{L=0}^{\infty}{}' (-1)^L (\xi^2)^L$$ since this

III A) $\frac{a_{L+2}}{a_L} \to 1$ for all L. This implies the $\Theta(\xi)$ diverges like

$$\Theta(\xi) \overset{\xi\to\pm1}{\sim} \frac{1}{(1-\xi^2)^{1+\frac{|m|}{2}}} .$$

But the wavefunction ψ is to be finite over the whole domain $0 \le \theta \le \pi$, that is $-1 \le \xi \pm 1$.

Thus λ must be some integer

$$\boxed{\lambda = l(l+1) \qquad l = 0, 1, 2, \ldots}$$

Then the power series for $P(\xi)$ terminates after $L = l$,

$$\frac{a_{L+2}}{a_L} = \frac{L(L+1) - l(l+1)}{(L+1)(L+2)} ,$$

so for $L \ge l$, $a_{L+2} = 0$. The $P(\xi)$ are polynomials of order l, the Legendre polynomials.

Since $\Theta(\xi)$ is proportional to the

-III A) $|m|^{th}$ derivative of $P(\xi)$, it vanishes unless $\underline{|m| \leq l}$. Hence each

value of l allows $2l+1$ values of m, running from $-l$ to $+l$.

So with $\lambda = l(l+1)$, the recursion relation becomes

$$a_{L+2} = \frac{-(l-L)(L+l+1)}{(L+1)(L+2)} a_L$$

and all even powers are related to a_0 and all odd powers are related to a_1. Further if l is even then the even series terminates and we have $a_1 = 0$, $a_0 \neq 0$ and $P_l(\xi) = P_l(-\xi)$

If l = odd then the odd series terminates and we put $a_0 = 0$, $a_1 \neq 0$ and $P_l(-\xi) = -P_l(\xi)$.

So consider the even and odd series. We can solve the above recursion formula to yield:

III A.) a) $l = 2n$; $N = 1, 2, \ldots, n$; $n = 1, 2, \ldots$

$$a_{2N} = (-1)^N \frac{(2n+2N)!}{(2N)!\,(n+N)!\,(n-N)!}\, a_0$$

b) $l = 2n+1$; $n = 0, 1, 2, \ldots$; $N = 1, 2, \ldots, n$

$$a_{2N+1} = (-1)^N \frac{(2n+2N+2)!}{(n+N+1)!\,(n-N)!\,(2N+1)!}\, a_1$$

Thus we obtain the Legendre Polynomials $P_l(\xi)$:

a) For $l = 2n$, $n = 0, 1, 2, \ldots$

$$P_l(\xi) = a_0 \sum_{\substack{N=0 \\ (l=2n)}}^{n} (-1)^N \frac{(2n+2N)!}{(2N)!\,(n+N)!\,(n-N)!}\, \xi^{2N}$$

b) For $l = 2n+1$, $n = 0, 1, 2, \ldots$

$$P_l(\xi) = a_1 \sum_{\substack{N=0 \\ (l=2n+1)}}^{n} (-1)^N \frac{(2n+2N+2)!}{(2N+1)!\,(n+N+1)!\,(n-N)!}\, \xi^{2N+1}$$

III A) **By Convention:**

$$\boxed{a_0 \equiv \frac{(-1)^n}{2^{2n}} \quad ; \quad a_1 \equiv \frac{(-1)^n}{2^{2n+1}}}$$

Hence the first few Legendre Poly. are

$P_0(\xi) = 1$

$P_1(\xi) = \xi$

$P_2(\xi) = \frac{1}{2}(3\xi^2 - 1)$

$P_3(\xi) = \frac{1}{2}(5\xi^3 - 3\xi)$

$\vdots$

Properties of Legendre Polynomials:

1) $P_\ell(\xi)$ is real
2) $P_\ell(-\xi) = (-1)^\ell P_\ell(\xi)$
3) $P_\ell(\xi)$ is a polynomial in ξ of degree ℓ.
4) $P_\ell(\pm 1) = (\pm 1)^\ell$

III A) 5) The Rodrigues Formula:

$$P_\ell(\xi) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{d\xi^\ell} (\xi^2 - 1)^\ell$$

6) Orthogonality:

$$\int_{-1}^{+1} d\xi \, P_\ell(\xi) P_{\ell'}(\xi) = \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

7) Generating Function:

$$Z(s) = \sum_{\ell=0}^{\infty} s^\ell P_\ell(\xi) = \frac{1}{\sqrt{1 - 2\xi s + s^2}}$$

So

$$\ell! \, P_\ell(\xi) = \frac{\partial^\ell}{\partial s^\ell} Z(s) \Big|_{s=0}.$$

Thus using the Legendre polynomials, we have solutions to the polar angular equation. These are most easily expressed in terms of the

III A) Associated Legendre Functions

$P_l^m(\xi)$ for $l=0,1,2,\ldots$ and $m=-l,-l+1,\ldots,0,+1,+2,\ldots,+l$

$$\boxed{P_l^m(\xi) \equiv (1-\xi^2)^{\frac{|m|}{2}} \frac{d^{|m|}}{d\xi^{|m|}} P_l(\xi)}$$

Hence the polar angle eigenfunctions $\Theta(\xi)$ are given by the associated Legendre functions

$$\boxed{\Theta_{lm}(\xi) = P_l^m(\xi)}.$$

The properties of $P_l^m(\xi)$ follow from those of $P_l(\xi)$

1) $P_l^m(\xi)$ is real
2) $P_l^m(-\xi) = (-1)^{l+|m|} P_l^m(\xi)$
3) $P_l^m(\xi) = P_l^{-m}(\xi)$
4) $P_l^0(\xi) = P_l(\xi)$

III A) 5) Rodrigues Formula

$$P_\ell^m(\xi) = \frac{(-1)^{|m|}(\ell+|m|)!}{2^\ell \ell!\,(\ell-|m|)!}(1-\xi^2)^{-\frac{|m|}{2}} \frac{d^{\ell-|m|}}{d\xi^{\ell-|m|}}(\xi^2-1)^\ell$$

6) Generating Function: Differentiate the generating function for $P_\ell(\xi)$ $|m|$ times wrt ξ and multiply by $(1-\xi^2)^{\frac{|m|}{2}}$ yields

$$\sum_{\ell=|m|}^{\infty} P_\ell^m(\xi)\, s^\ell = \frac{(2|m|)!\,(1-\xi^2)^{\frac{|m|}{2}}\, s^{|m|}}{2^{|m|}(|m|)!\,(1-2\xi s+s^2)^{|m|+\frac{1}{2}}}$$

7) Orthogonality:

$$\int_{-1}^{+1} d\xi\, P_\ell^m(\xi) P_{\ell'}^m(\xi) = \frac{2}{2\ell+1}\frac{(\ell+|m|)!}{(\ell-|m|)!}\delta_{\ell\ell'}$$

8) Low Order Associated Legendre Functions:

$P_\ell^0(\xi) = P_\ell(\xi)$	$P_2^1(\xi) = 3\xi\sqrt{1-\xi^2}$
$P_1^1(\xi) = \sqrt{1-\xi^2}$	$P_2^2(\xi) = 3(1-\xi^2)$
	$\vdots$

III A) Putting this all back together, the angular eigenfunctions are given by $Y(\theta,\varphi) = A\,\Theta(\cos\theta)\,\Phi(\varphi)$ and are labelled by the (l,m) integers

$$\boxed{\begin{gathered} Y_l^m(\theta,\varphi) = A_{lm}\, P_l^m(\cos\theta)\, e^{\pm im\varphi} \\ \text{with } l = 0,1,2,\ldots \text{ and } m = -l, -l+1, \ldots, 0, 1, 2, \ldots, +l \end{gathered}}$$

Recall from page -196- the (polar & azimuthal) angular eigenvalue equations:

1) $$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta}\right] Y_l^m(\theta,\varphi)$$
$$= -l(l+1)\, Y_l^m(\theta,\varphi)$$

2) $$-i\frac{\partial}{\partial\varphi} Y_l^m(\theta,\varphi) = m\, Y_l^m(\theta,\varphi)$$
$$\Rightarrow \quad \left(\frac{\partial^2}{\partial\varphi^2} + m^2\right) Y_l^m(\theta,\varphi) = 0.$$

III A) Choosing the normalization constant A_{lm} as

$$A_{lm} \equiv (-1)^{\frac{|m|+m}{2}} \left[\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!} \right]^{1/2}$$

The $\underline{Y_l^m(\theta,\varphi)}$ are called $\underline{\text{Spherical Harmonics}}$

Their properties are given by those of P_l^m and $e^{im\varphi}$:

1) Orthonormality

$$\int_{4\pi} d\Omega \, Y_l^{m*}(\theta,\varphi) \, Y_{l'}^{m'}(\theta,\varphi) = \delta_{ll'} \delta_{mm'}$$

where the solid angle integration is given by

$$\int_{4\pi} d\Omega \equiv \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta$$

$$\left(\underset{\xi = \cos\theta}{=} \int_0^{2\pi} d\varphi \int_{-1}^{+1} d\xi \right)$$

III A) 2) Low Order Spherical Harmonics:

$$Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^{\pm 1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$$

$$Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_2^{\pm 2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi}$$

$$Y_2^{\pm 1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\varphi}$$

$$Y_2^0(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$$

$\vdots$

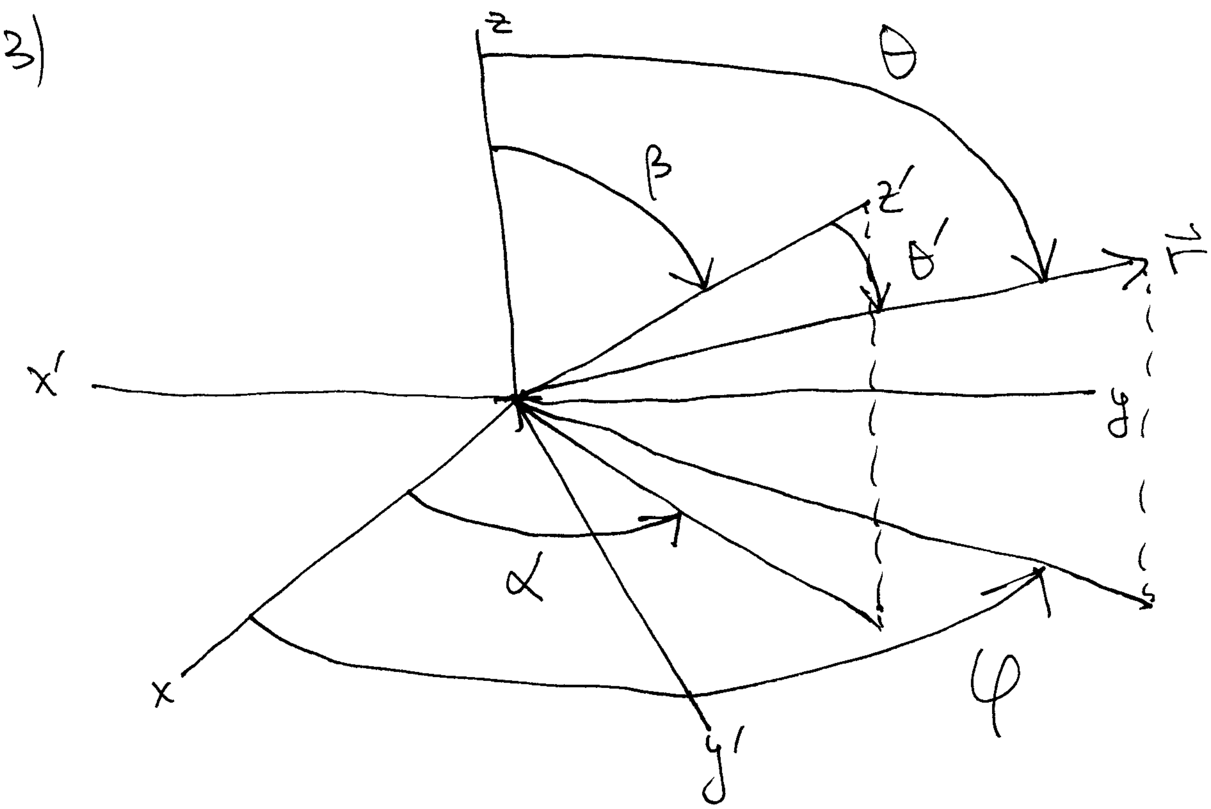
---

3) (Advanced Property) Addition Theorem For Spherical Harmonics

$$P_\ell(\cos\theta') = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{+\ell} Y_\ell^{m*}(\beta,\alpha)\, Y_\ell^m(\theta,\varphi)$$

with the angles $\alpha, \beta, \theta, \varphi, \theta'$ given by

III A) 3)



Finally, the energy eigenfunctions for a particle of mass m moving in a central potential $V = V(r)$ is

$$\psi_{lm}(\vec{r}) = R(r)\, Y_l^m(\theta, \varphi)$$

where $R(r)$ obeys the radical equation:

III. A)

$$\boxed{\left(\frac{-\hbar^2}{2m}\right)\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}R(r)\right) + \left[V(r) + \frac{1}{2mr^2}\hbar^2 l(l+1)\right]R(r) = E\,R(r)}$$

III.B) <u>The Radial Equation</u>:

This equation simplifies by a change of variables: Let

$$\boxed{R(r) = \frac{1}{r}u(r)}$$

Then $\frac{dR}{dr} = \left[r\frac{du}{dr} - u\right]\frac{1}{r^2}$

$$\frac{d}{dr}\left[r^2\frac{d}{dr}R\right] = r\frac{d^2u}{dr^2}$$

Hence the Radial Equation becomes

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu}$$

III B.) This radial equation looks like a one-dimensional Schrödinger equation with the effective potential

$$V_{eff} = V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}$$

containing an extra piece called the centrifugal term $\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}$. It tends to throw the particle outward – away from the origin just as the "centrifugal" (pseudo-)force in classical mechanics!

Now recall the normalization of the wavefunction

$$1 = \int d^3r \, |\psi_{lm}(\vec{r})|^2$$

$$= \int_{r=0}^{\infty} dr \, r^2 |R|^2 \underbrace{\int d\Omega \, |Y_{lm}|^2}_{4\pi \quad = 1}$$

$$= \int_{r=0}^{\infty} dr \, r^2 |R|^2 = \int_{r=0}^{\infty} dr \, |u(r)|^2$$

IIIB) Hence our normalization condition reduces to

$$\boxed{1 = \int_{r=0}^{\infty} dr\, |u(r)|^2}$$

Next, the potential must be specified.

c) The Hydrogen Atom

The Hydrogen atom is a two body bound state composed of a proton of mass $m_1 = 1.7 \times 10^{-27}$ Kg ($m_1 c^2 = 938$ MeV) and an electron of mass $m_2 = 9.1 \times 10^{-31}$ Kg ($m_2 c^2 = 0.511$ MeV). The proton carries positive charge $e > 0$ and the electron has opposite charge $-e$. The 2 particles are bound by the Coulomb potential (in SI units)

$$V(|\vec{r}_1 - \vec{r}_2|) = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_1 - \vec{r}_2|}$$

Which in spherical polar coordinates is given by

$$V(r) = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r}$$