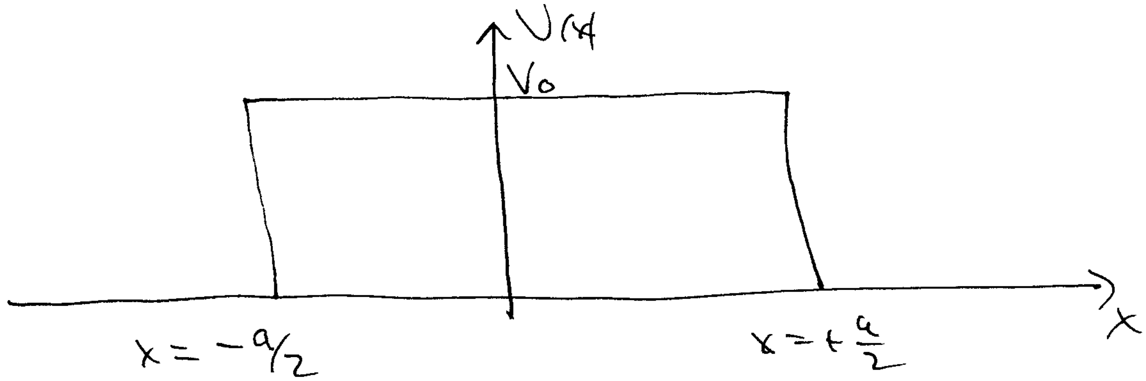
II. E) Finally let's consider scattering from a potential barrier

$$V(x) = \begin{cases} 0 , & x < -\frac{a}{2} \\ V_0 > 0 , & -\frac{a}{2} \le x \le \frac{a}{2} \\ 0 , & x > \frac{a}{2} \end{cases}$$



Of course there are no bound states we only have $E > 0$, scattering states.

Schrödinger's Equation is given by

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \qquad \text{for } |x| > \frac{a}{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi \text{ for } |x| \le \frac{a}{2} .$$

Now we still have 2 cases depending upon the incident particle's energy

$\underline{E > V_0}$ and $\underline{E < V_0}$.

II E) Case 1: $\boxed{E > V_0}$: Define

$$k \equiv \sqrt{\frac{2mE}{\hbar^2}} > 0 \quad ; \quad E = \frac{\hbar^2 k^2}{2m}$$

$$q \equiv \sqrt{\frac{2m(E-V_0)}{\hbar^2}} > 0 \quad \text{since } E > V_0.$$

(Note this is just the potential well case with $V_0 \to -V_0$)

The Sch. Eq. becomes

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \quad ; \quad |x| > \frac{a}{2}$$

$$\frac{d^2\psi}{dx^2} + q^2\psi = 0 \quad ; \quad |x| \leq \frac{a}{2}$$

And once again we have the general solution

$$\psi(x) = N \begin{cases} e^{ikx} + r\,e^{-ikx} & , \ x < -\frac{a}{2} \\ A e^{iqx} + B e^{-iqx} & , \ -\frac{a}{2} \leq x \leq \frac{a}{2} \\ t e^{ikx} + C e^{-ikx} & , \ x > \frac{a}{2}. \end{cases}$$

As in the potential well case

$$\psi_{incident} = N e^{ikx} \qquad \left(x < -\tfrac{a}{2}\right)$$

$$\psi_{ref.} = N r e^{-ikx} \qquad \left(x < -\tfrac{a}{2}\right)$$

-II E) While $\psi_{trans} = N t\, e^{ikx}$ $(x > \frac{a}{2})$.

So we choose $\boxed{C = 0}$ and the BC are the same as in the well case. So just recalling these results with $V_0 \to -V_0$ we have

$$\boxed{R = \frac{V_0^2 \sin^2\left(\sqrt{\frac{2m(E-V_0)}{\hbar^2}}\, a\right)}{4E(E-V_0) + V_0^2 \sin^2\left(\sqrt{\frac{2m(E-V_0)}{\hbar^2}}\, a\right)}}$$

and

$$\boxed{T = \frac{4E(E-V_0)}{4E(E-V_0) + V_0^2 \sin^2\left(\sqrt{\frac{2m(E-V_0)}{\hbar^2}}\, a\right)}}$$

As previously "resonance" scattering $(T=1)$ occurs at

$$\sqrt{\frac{2m(E-V_0)}{\hbar^2}}\, a = n\pi$$

<u>now</u> $n = 0, 1, 2, \ldots$ at which $T = 1$ and hence $R = 0$.

-II E) And as before the minimum of T occurs at

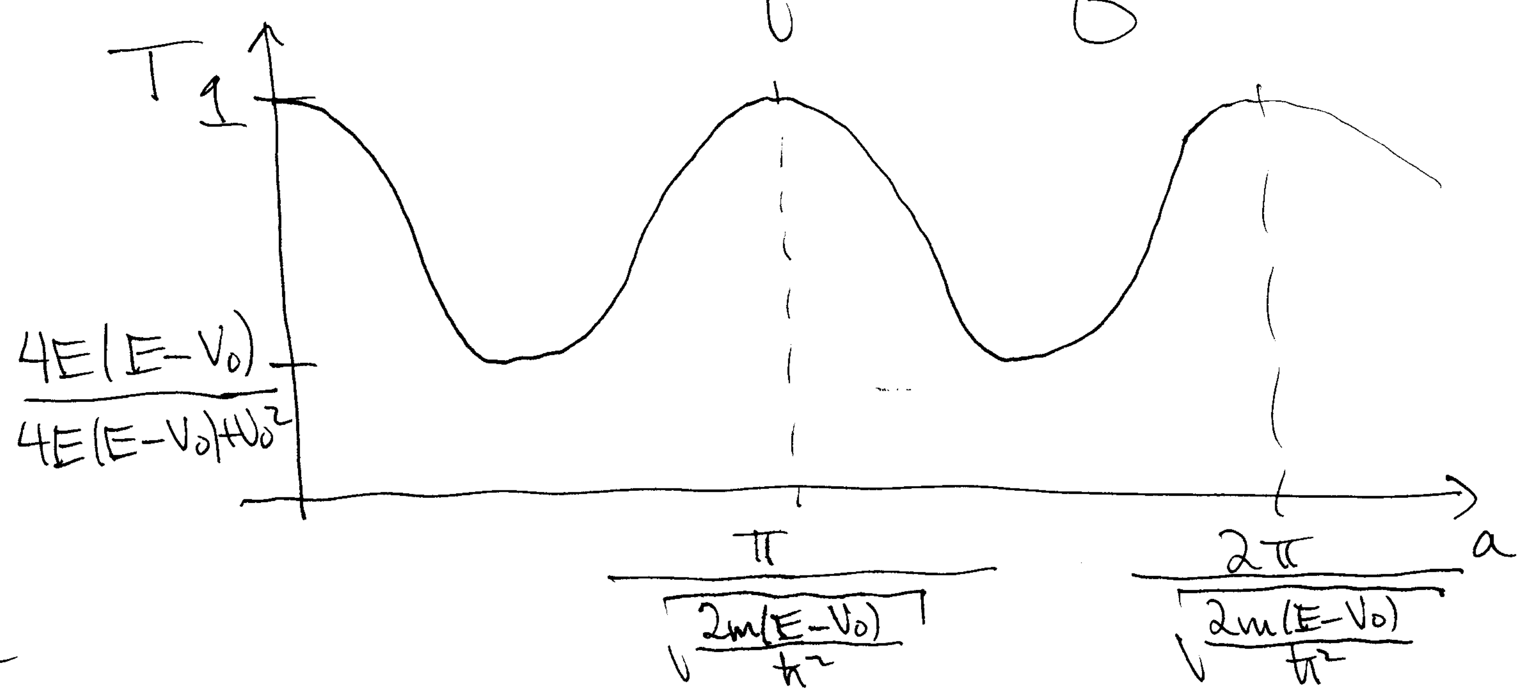
$$\sqrt{\frac{2m(E-V_0)}{\hbar^2}}\,a = \left(n+\tfrac{1}{2}\right)\pi \,, \quad n=0,1,2,\ldots$$

where

$$\boxed{T = \frac{4E(E-V_0)}{4E(E-V_0)+V_0^2} \qquad R = \frac{V_0^2}{4E(E-V_0)+V_0^2}}\,.$$

and

T plotted as a function of a



iii E) Case 2: $\boxed{E < V_0}$ $E > 0$

Barrier Penetration (Tunneling)

As before, define $k \equiv \sqrt{\frac{2mE}{\hbar^2}} > 0$

But now $\kappa \equiv \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} > 0$

The Schrödinger Equation now becomes

$$\frac{d^2}{dx^2}\psi + k^2\psi = 0 \quad , \quad |x| > \frac{a}{2}$$

$$\frac{d^2}{dx^2}\psi - \kappa^2\psi = 0 \quad , \quad -\frac{a}{2} \leq x \leq \frac{a}{2} \quad .$$

As previously we can find the solutions using the JBC on ψ and $\frac{d\psi}{dx}$. Equivalently we note that the previous Case 1 problem is converted into this Case 2 problem by the replacement

$$q \rightarrow -i\kappa \quad .$$

Using the fact that

$$\sin qa \rightarrow \sin(-i\kappa a) = -i\sinh \kappa a$$

we have

-IIE.) Case 2:) $\sin^2 qa \to -\sinh^2 \varkappa a$
and $q^2 \to -\varkappa^2$.

Thus the transmission coefficient in <u>case 2</u> is

$$\boxed{T = \frac{4\varkappa^2 k^2}{4\varkappa^2 k^2 + (\varkappa^2 + k^2)^2 \sinh^2 \varkappa a} = \frac{4E(V_0 - E)}{4E(V_0 - E) + V_0^2 \sinh^2\left(\sqrt{\frac{2m(V_0 - E)}{\hbar^2}}\, a\right)}}$$

The reflection coefficient is $R = 1 - T$.

Note: For $\varkappa a = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}\, a \gg 1$,

$$\sinh^2(\varkappa a) \to \frac{1}{4} e^{2\varkappa a} \gg 1$$

Hence

$$\boxed{T \approx \frac{16E(V_0 - E)}{V_0^2}\, e^{-2\sqrt{\frac{2m(V_0 - E)}{\hbar^2}}\, a}}$$

-II E.) Case 2: Remarks: Quantum Mechanically $T \neq 0$, there is a nonzero probability that the particle appears to the right of the barrier, $x > \frac{a}{2}$ even when its energy $0 < E < V_0$. The particle is said to have penetrated the barrier or tunneled through the barrier. Classically the particle would never appear in the region $x > \frac{a}{2}$ when its energy $E < V_0$.

There are many examples of this tunneling phenomenon in Nature; they include the tunnel diode, the Josephson effect, the α-decay of nuclei.

Let's determine T for two different types of particles.

1) Consider incident electrons

$$\frac{1}{\chi} = \frac{\hbar}{\sqrt{2m(V_0 - E)}} = \frac{1.96\,\text{Å}}{\sqrt{V_0 - E}}$$

with V_0, E typically in eV. Typical atomic physics values for V_0, E are

$$E = 1\text{eV}, \quad V_0 = 2\text{eV}.$$

-II E) This gives $\frac{1}{\kappa} = 1.96\,\text{Å}$. For $a = 1\,\text{Å}$, this yield

$$\kappa a = \frac{1}{1.96}$$

. Plugging these numbers into T we find

$$T \simeq 0.78$$

. This is quite a large probability that the e^- tunnels through the barrier.

2) Incident protons: for this case T will be much smaller since

$$\frac{m_{e^-}}{m_p} \simeq \frac{1}{1836}, \quad \text{hence}$$

$$\frac{1}{\kappa} = \frac{1.96\,\text{Å}}{\sqrt{1836}\,\sqrt{V_0 - E}} = \frac{4.6\times10^{-2}}{\sqrt{V_0 - E}}\,\text{Å}$$

with V_0, E in eV. For the above values

$$\kappa a = \frac{\sqrt{1836}}{1.96} >> 1, \quad \text{hence}$$

$$T \simeq 4\times10^{-19}, \quad \text{quite negligible.}$$