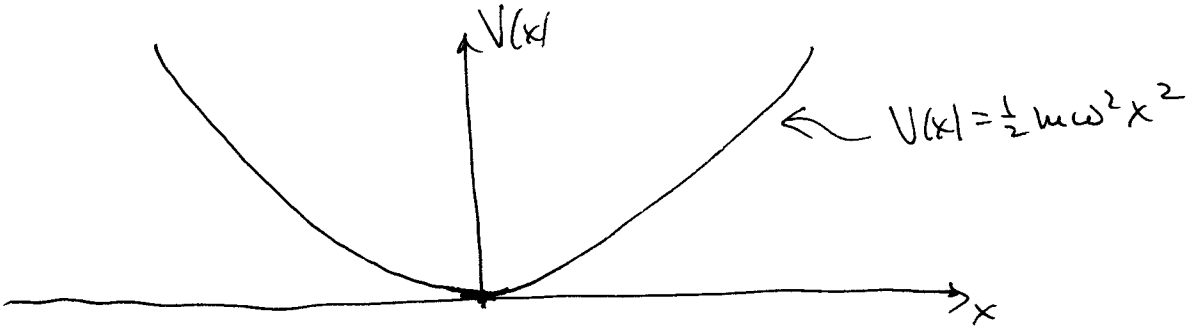
B) The Simple Harmonic Oscillator

Models system undergoing small oscillations.

$$V(x) = \frac{1}{2} m\omega^2 x^2$$



where m is the mass of the particle and ω a constant with units of $\frac{1}{t}$ frequency.
(classically $\omega = \sqrt{k/m}$, k = spring constant and ω is the freq. of classical oscillators)

Once again $V = V(x)$ is indept of time
So we have stationary state solutions to the time-dep. Sch-eq.

$$\psi(x,t) = \psi(x) e^{-iEt/\hbar}$$

And $\psi(x)$ obeys the time-independent Sch. Eq.

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2 \right] \psi(x) = E \psi(x)$$

B) Since V is not define in regions of space or has no singularities the only condition we have on V is ψ is finite as $x \to \pm\infty$. (specifically impose) (of course ψ is continuous & $d\psi/dx$ is continuous.)

There are 2 methods of solution: 1) Algebraic 2) Analytic.

1) Algebraic Method: we will reduce the problem to a first order DE for the ground state. Then find all excited states by acting on the ground state with a "raising" operator.

Schrödinger's Equation:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2\right]\psi = E\psi$$

$$\left[\underbrace{\frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2}_{H}\right]\psi(x) = E\psi(x)$$

This can be factored

-11B) Consider the complex combinations:

$$a \equiv \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}\,\underline{X} + i\frac{1}{\sqrt{m\hbar\omega}}\,\underline{P}\right]$$

$$= \frac{i}{\sqrt{\hbar\omega}}\,\frac{1}{\sqrt{2m}}\left[\frac{\hbar}{i}\frac{d}{dx} - im\omega x\right]$$

$$\left(= \frac{i}{\sqrt{\hbar\omega}}\,a_- \quad \text{in Griffiths}\right)$$

$$a^\dagger \equiv \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}\,\underline{X} - i\frac{1}{\sqrt{m\hbar\omega}}\,\underline{P}\right]$$

$$= \frac{-i}{\sqrt{\hbar\omega}}\,\frac{1}{\sqrt{2m}}\left[\frac{\hbar}{i}\frac{d}{dx} + im\omega x\right]$$

$$\left(= \frac{-i}{\sqrt{\hbar\omega}}\,a_+ \quad \text{in Griffiths}\right)$$

where $\underline{X} = \underline{X}^\dagger$, $\underline{P} = \underline{P}^\dagger$ has been used.

...B) Now inverting

$$\underline{X} = \frac{1}{\sqrt{2}}\sqrt{\frac{\hbar}{m\omega}}\,(a^\dagger + a) \quad (= \underline{X}^\dagger)$$

$$\underline{P} = \frac{i}{\sqrt{2}}\sqrt{m\hbar\omega}\,(a^\dagger - a) \quad (= \underline{P}^\dagger)$$

Since $[\underline{X}, \underline{P}] = i\hbar \Rightarrow$

$$[a, a^\dagger] = \frac{1}{2}\left[\sqrt{\frac{m\omega}{\hbar}}\underline{X} + i\sqrt{\frac{1}{m\hbar\omega}}\underline{P},\ \sqrt{\frac{m\omega}{\hbar}}\underline{X} - i\frac{1}{\sqrt{m\hbar\omega}}\underline{P}\right]$$

$$= \frac{-i}{2}\frac{1}{\hbar}[\underline{X}, \underline{P}] + \frac{i}{2}\frac{1}{\hbar}\underbrace{[\underline{P}, \underline{X}]}_{= -[\underline{X}, \underline{P}]}$$

$$= \frac{-i}{\hbar}[\underline{X}, \underline{P}]$$

$$= 1$$

So $\boxed{[a, a^\dagger] = 1 \quad \text{and} \quad [a, a] = 0 = [a^\dagger, a^\dagger]}$

Further we may expand $a^\dagger a$

(8)

$$a^\dagger a = \frac{1}{2}\left[\sqrt{\frac{m\omega}{\hbar}}\hat{X} - i\frac{1}{\sqrt{m\hbar\omega}}\hat{P}\right]\left[\sqrt{\frac{m\omega}{\hbar}}\hat{X} + i\frac{1}{\sqrt{m\hbar\omega}}\hat{P}\right]$$

$$= \frac{1}{2}\left[\frac{m\omega}{\hbar}\hat{X}^2 + \frac{1}{m\hbar\omega}\hat{P}^2 + \underbrace{\frac{i}{\hbar}\hat{X}\hat{P} - \frac{i}{\hbar}\hat{P}\hat{X}}\right]$$

$$\frac{i}{\hbar}\underbrace{[\hat{X},\hat{P}]}_{=i\hbar} = -1$$

$\Rightarrow$

$$\boxed{a^\dagger a = \frac{1}{\hbar\omega}\left[\frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2 - \frac{1}{2}\hbar\omega\right]}$$

Note

$$aa^\dagger = a^\dagger a + aa^\dagger - a^\dagger a$$

$$= a^\dagger a + [a, a^\dagger] = a^\dagger a + 1$$

$$\boxed{aa^\dagger = \frac{1}{\hbar\omega}\left[\frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2 + \frac{1}{2}\hbar\omega\right]}$$

-I-B) So we have "factored" the Hamiltonian (two ways)

$$H = \hbar\omega\left[a^\dagger a + \tfrac{1}{2}\right]$$

$$= \hbar\omega\left[aa^\dagger - \tfrac{1}{2}\right]$$

So the Schrödinger equation maybe written 2 ways

$$H\psi = E\psi$$

$$\hbar\omega\left[a^\dagger a + \tfrac{1}{2}\right]\psi = E\psi$$

$$\hbar\omega\left[a\,a^\dagger - \tfrac{1}{2}\right]\psi = E\psi$$

That is letting $\boxed{\epsilon \equiv \frac{E}{\hbar\omega}}$

1) $\boxed{a^\dagger a\,\psi = (\epsilon - \tfrac{1}{2})\psi}$

2) $\boxed{a\,a^\dagger\psi = (\epsilon + \tfrac{1}{2})\psi}$

Schrödinger's Eq.

B) Now multiply (2) by $a^\dagger$

$$a^\dagger a(a^\dagger \psi) = (\epsilon + \tfrac{1}{2})(a^\dagger \psi)$$

This is true if either

a) $a^\dagger \psi = 0$

or

b) $a^\dagger a[a^\dagger \psi] = (\hat{\epsilon} - \tfrac{1}{2})[a^\dagger \psi]$

where $\hat{\epsilon} = \epsilon + 1$. This is just equation (1) with energy $\hat{\epsilon} = \epsilon + 1$, that is $\hat{E} = \hbar\omega \hat{\epsilon} = \hbar\omega(\epsilon + 1) = \hbar\omega + E$.

So if ψ is a solution to Sch. Eq. with energy $E = \hbar\omega\epsilon$, then we have another solution $[a^\dagger \psi]$ with energy $\hat{E} = (E + \hbar\omega) = \hbar\omega\hat{\epsilon}$.

Now we still have case (a)

$$a^\dagger \psi = 0 = \frac{-i}{\sqrt{\hbar\omega}} \frac{1}{\sqrt{2m}} \left[\frac{\hbar}{i}\frac{d}{dx} + i m\omega x\right]\psi$$

This is just a 1st order DE for ψ

(B) $\Rightarrow$ $\boxed{\psi = N e^{+\frac{1}{2}\frac{m\omega^2}{\hbar\omega}x^2}}$

Now recall $\psi(x)$ must be finite as $x \to \pm\infty$, and $\psi(x)$ must be normalizable So $\psi \to 0$ as $x \to \pm\infty$. This solution is ruled out by these BC.

So only the case (b) applies

> Given $\psi(x)$ an eigenfunction with energy ϵ, $[a^\dagger\psi]$ is another eigenfunction with energy $(\epsilon+1)$.

Since this worked so well, let's try it the other way — multiply (1) by a

$$a\,a^\dagger[a\psi] = (\epsilon - \tfrac{1}{2})[a\psi]$$

This equation is satisfied if either

a) $a\psi = 0$

or b) $a\,a^\dagger[a\psi] = (\hat{\epsilon} + \tfrac{1}{2})[a\psi]$

-13) Where now $\hat{\hat{E}} = E - 1$.

So case (b) we see that given an energy eigenfunction ψ with energy E, then $[a\psi]$ is another energy eigenfunction with energy $E-1 \equiv \hat{E}$, i.e.

$\hat{\hat{E}} = \hbar\omega\hat{E} = E - \hbar\omega$, a lower energy.

But $H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 X^2$ is a non-negative operator, so its eigenvalues must be non-negative — indeed

$$\int_{-\infty}^{+\infty} dx\, \psi^* a^\dagger a \psi = \int_{-\infty}^{+\infty} dx\, [a\psi]^* [a\psi]$$

Since $(a^\dagger)^\dagger = a$ or just do an integration by parts

$$= \int_{-\infty}^{+\infty} dx\, \psi^* \left[\frac{-i}{\sqrt{\hbar\omega}} \frac{1}{\sqrt{2m}} \left(\frac{\hbar}{i}\frac{d}{dx} + im\omega x\right)\right] a\psi$$

$$= \frac{-i}{\sqrt{\hbar\omega}} \frac{1}{\sqrt{2m}} \frac{\hbar}{i} \cancel{\left(\psi^* a\psi\right)\Big|_{-\infty}^{+\infty}} \to 0$$

$$+ \int_{-\infty}^{+\infty} dx \left[\frac{-i}{\sqrt{\hbar\omega}} \frac{1}{\sqrt{2m}} \left(-\frac{\hbar}{i}\frac{d}{dx} + im\omega x\right)\right] \psi^* a\psi$$

-[B)

$$= \int_{-\infty}^{+\infty} dx \left[\frac{i}{\sqrt{\hbar\omega}} \frac{1}{i2m} \left(\frac{\hbar}{i} \frac{d}{dx} - im\omega x \right) \psi \right]^* a\psi$$

$$= \int_{-\infty}^{+\infty} dx \, [a\psi]^* [a\psi] = \int_{-\infty}^{+\infty} dx \, |a\psi|^2 \geq 0$$

But Sch. Eq. was $a^\dagger a \psi = (\epsilon - \frac{1}{2})\psi$

So $$\int_{-\infty}^{+\infty} dx \, \psi^* a^\dagger a \psi = (\epsilon - \tfrac{1}{2}) \int_{-\infty}^{+\infty} dx \, \psi^* \psi \geq 0$$

$\Rightarrow$

$\epsilon \geq \frac{1}{2}$	For all eigenfunctions ψ

So now if we start with some ϵ & ψ and continue to let a act on ψ

eigenfunction	eigenvalue
ψ	ϵ
$a\psi$	$\epsilon - 1$
$a^2\psi$	$\epsilon - 2$
$\vdots$	
$a^N\psi$	$\epsilon - N$

we lower the energy by 1 each time

II B) Eventually N will be large enough so that

$$(\epsilon - N - \tfrac{1}{2}) < 0.$$

This is forbidden, since H is a non-negative operator. The only way out of this contradiction is that there must exist some N such that one more application of a causes the function to vanish suppose

$$\psi_0(x) \equiv a^N \psi(x)$$

Then

$$\boxed{a\psi_0(x) = 0}$$

But this is just the case (a)

Hence, we can always generate (from ψ) a lower energy eigenfunction by the application of a, unless we have that ψ is the ground state ψ_0. Then

$$a\psi_0 = 0$$

iii) But this is just a 1st order DE for ψ_0

$$0 = a\psi_0 = \frac{i}{\sqrt{\hbar\omega}}\frac{1}{\sqrt{2m}}\left[\frac{\hbar}{i}\frac{d}{dx} - im\omega x\right]\psi_0$$

$$\Rightarrow \boxed{\psi_0(x) = N_0 e^{-\frac{1}{2}\frac{m\omega^2}{\hbar\omega}x^2}}$$

Since this 1st DE this solution is unique up to the normalization constant N_0 — the ground state is not degenerate — there is only one, unique lowest energy state.

Now since $a\psi_0 = 0$ we have from Schrödinger's equation —

$$a^\dagger a\psi_0 = (\epsilon_0 - \tfrac{1}{2})\psi_0$$

$$\| \quad \Rightarrow$$

$$0$$

$$\boxed{\epsilon_0 = \frac{1}{2} \Rightarrow E_0 = \frac{1}{2}\hbar\omega}$$

The ground state energy is not zero it is $E_0 = \frac{1}{2}\hbar\omega$ (also called the zero point energy)

II B) Since this is the lowest energy, all other energies can be reached by applying the "raising operator" $a^\dagger$ to ψ_0 repeatedly. This corresponds to raising the energy by one, hence we have the energy spectrum. The energies are labelled by an index $n = 0, 1, 2, \ldots$ with values

$$\epsilon_n = n + \tfrac{1}{2} \quad , \quad n = 0, 1, 2, \ldots$$

That is

$$E_n = (n + \tfrac{1}{2})\hbar\omega \quad , \quad n = 0, 1, 2, \ldots$$

The n^{th} excited state eigenfunction $\psi_n(x)$ corresponding to eigenvalue E_n is simply

$$\psi_n(x) = \frac{N_n}{N_0} (a^\dagger)^n \psi_0(x)$$

with Schrödinger's Eq.

$$a^\dagger a \psi_n = (\epsilon_n - \tfrac{1}{2})\psi_n = n\psi_n$$

$$a a^\dagger \psi_n = (\epsilon_n + \tfrac{1}{2})\psi_n = (n+1)\psi_n$$

-B) where $N_0, N_1, \ldots$ are the normalization constants. So we find

$$\psi_0 = N_0 e^{-\frac{1}{2}\frac{m\omega^2}{\hbar\omega}x^2}$$

$$\left(\frac{N_1}{N_0}a^\dagger\psi_0 =\right)\ \psi_1 = N_1\left[\frac{-i}{\sqrt{2m\hbar\omega}}\right]\left[\frac{\hbar}{i}\frac{d}{dx}+im\omega x\right]e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$$

$$= N_1\sqrt{\frac{2m\omega}{\hbar}}\,x\,e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$$

$$\left(\frac{N_2}{N_1}a^\dagger\psi_1 = \frac{N_2}{N_0}(a^\dagger)^2\psi_0 =\right)\ \psi_2 = N_2\sqrt{\frac{2m\omega}{\hbar}}\left(\frac{-i}{\sqrt{2m\hbar\omega}}\right)\left[\frac{\hbar}{i}\frac{d}{dx}+im\omega x\right]x e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$$

$$= N_2\left[\frac{2m\omega}{\hbar}x^2 - 1\right]e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$$

and so on.

Remarks about details:

1) If ψ corresponds to energy ϵ so that

$$\psi_0 = (a)^N\psi \quad \text{and} \quad a\psi_0 = 0$$

then $aa^\dagger\psi_0 = aa^\dagger a^N\psi = (\epsilon - N + \frac{1}{2})(a^N\psi)$

$$= (\epsilon - N + \tfrac{1}{2})\,\psi_0$$

i.e.

ψ	has eigenvalue	ϵ	(a is a "lowering" operator)
$a\psi$	" "	$\epsilon-1$	
$a^2\psi$	" "	$\epsilon-2$	
$\vdots$			
$a^N\psi$	" "	$\epsilon-N$	

IIB) 1) But $[a, a^\dagger] = 1$

So $a a^\dagger \psi_0 = (a a^\dagger - a^\dagger a + a^\dagger a)\psi_0$

$= [a, a^\dagger]\psi_0 + a^\dagger \underbrace{a \psi_0}_{=0}$

$= \psi_0$

$= (\epsilon - N + \tfrac{1}{2})\psi_0$

$\Rightarrow [\epsilon - N + \tfrac{1}{2} - 1]\psi_0 = 0$

$\Rightarrow \boxed{\epsilon = N + \tfrac{1}{2}}$, $N = 0, 1, 2, \ldots$

Hence the totality of eigenvalues are given by the integers $+\frac{1}{2}$ i.e. odd-half-integers — we have not missed any values in the energy spectrum.

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2) The ground state is non-degenerate, i.e. the solution to 1st order DE has one arbitrary constant — No the normalization of the state.

All excited states are non-degenerate

II.B. 2) Since the are built up from ψ_0 by action of raising operators a^+.

$a^+ a \psi_{n+1,i} = (\epsilon_{n+1} - \frac{1}{2}) \psi_{n+1,i}$, but $a\psi_{n+1,i} = c_i \psi_n$ and $a\psi_{n+1,i}$ exists.

$$\Rightarrow \underbrace{a^+ a \psi_{n+1,i}}_{} = c_i a^+ \psi_n$$

$$= (\epsilon_{n+1} - \tfrac{1}{2}) \psi_{n+1,i}$$

$$\Rightarrow \psi_{n+1,i} = \frac{c_i}{(\epsilon_{n+1} - \frac{1}{2})} a^+ \psi_n$$

all $\psi_{n+1,i} \propto a^+ \psi_n$ hence non-degenerate.

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3) Normalization:

$\int_{-\infty}^{+\infty} dx \, |\psi_n(x)|^2 \equiv 1$, This determines N_n $\frac{N_{n-1}}{N_0}$

But $\psi_n = \frac{N_n}{N_0} (a^+)^n \psi_0 = \frac{N_n}{N_0} a^+ \frac{N_0}{N_{n-1}} (a^+)^{n-1} \psi_0$

$$= \frac{N_n}{N_{n-1}} a^+ \psi_{n-1} \quad \text{or} \quad \boxed{a^+ \psi_n = \frac{N_n}{N_{n+1}} \psi_{n+1}}$$

$$a\psi_n = \frac{N_n}{N_{n-1}} \underbrace{a a^+ \psi_{n-1}}$$

$$= (\epsilon_{n-1} + \tfrac{1}{2}) \psi_{n-1}$$

$$\boxed{a\psi_n = \frac{N_n}{N_{n-1}} n \psi_{n-1}}$$

IIB.3) The point of this being that $\epsilon_{n+1} = \epsilon_n + 1$.

$a^\dagger \psi_n \propto \psi_{n+1}$

$a\psi_n \propto \psi_{n-1}$

$aa^\dagger \psi_n = (\epsilon_n + \frac{1}{2})\psi_n \Rightarrow$

(i.e. $a^\dagger a(a^\dagger \psi_n) = (\epsilon_n + \frac{1}{2})(a^\dagger \psi_n)$

$= [\epsilon_n + 1 - \frac{1}{2}](a^\dagger \psi_n)$)

(i.e. $a^\dagger a \psi_n = (\epsilon_n - \frac{1}{2})\psi_n$

$\Rightarrow aa^\dagger(a\psi_n) = (\epsilon_n - \frac{1}{2})(a\psi_n)$

$= [\underbrace{\epsilon_n - 1}_{= \epsilon_{n-1}} + \frac{1}{2}](a\psi_n)$)

The "length" of $a^\dagger \psi_n$ and $a\psi_n$ can be found from

$$\int_{-\infty}^{+\infty} dx\, |a^\dagger \psi_n|^2 = \int_{-\infty}^{+\infty} dx\, (a^\dagger \psi_n)^* (a^\dagger \psi_n)$$

$$= \int_{-\infty}^{+\infty} dx\, \psi_n^* \underbrace{(aa^\dagger \psi_n)}_{= (\epsilon_n + \frac{1}{2})\psi_n}$$

(Hermitian Conjugate $\leftrightarrow$ integrate by parts)

$$= (\epsilon_n + \tfrac{1}{2}) \underbrace{\int_{-\infty}^{+\infty} dx\, |\psi_n|^2}_{= 1}$$

$$= (\epsilon_n + \tfrac{1}{2}) = n + \tfrac{1}{2} + \tfrac{1}{2}$$

$$= n + 1$$

II B 3) So even though ψ_n is normalized we find

$$\boxed{\int_{-\infty}^{+\infty} dx\, |a^\dagger \psi_n|^2 = (n+1)}$$

Likewise

$$\int_{-\infty}^{+\infty} dx\, |a\psi_n|^2 = \int_{-\infty}^{+\infty} dx\, (a\psi_n)^* (a\psi_n)$$

Hermitian conjugate

$$= \int_{-\infty}^{+\infty} dx\, \psi_n^* \underbrace{(a^\dagger a \psi_n)}_{= (\epsilon_n - \frac{1}{2})\psi_n}$$

integrate by parts

$$= (\epsilon_n - \tfrac{1}{2}) \underbrace{\int_{-\infty}^{+\infty} dx\, |\psi_n|^2}_{=1} = (\epsilon_n - \tfrac{1}{2})$$

$$= n + \tfrac{1}{2} - \tfrac{1}{2} = n$$

So $\boxed{\int_{-\infty}^{+\infty} dx\, |a\psi_n|^2 = n}$

(From above
$a\psi_n = \frac{N_n}{N_{n-1}} n \psi_{n-1} \Rightarrow n = \left|\frac{N_n}{N_{n-1}}\right|^2 n^2 \Rightarrow N_n = \frac{1}{\sqrt{n}} N_{n-1}$
$\Rightarrow N_n = \frac{1}{\sqrt{n!}} N_0$)

II B 3) Thus we have that the normalized functions are related by

$$a^\dagger \psi_n = \sqrt{(n+1)}\, \psi_{n+1}$$
$$a\, \psi_n = \sqrt{n}\, \psi_{n-1}$$

(choose constant of proportionality positive real)

Now this allows us to determine N_n:

$$\psi_n = \frac{N_n}{N_0} (a^\dagger)^n \psi_0$$
$$= \frac{N_n}{N_0} (a^\dagger)^{n-1} \underbrace{a^\dagger \psi_0}_{=\sqrt{1}\,\psi_1} = \frac{N_n}{N_0} \sqrt{1}\, (a^\dagger)^{n-1} \psi_1$$
$$= \frac{N_n}{N_0} \sqrt{1}\, (a^\dagger)^{n-2} \underbrace{a^\dagger \psi_1}_{=\sqrt{2}\,\psi_2} = \frac{N_n}{N_0} \sqrt{1}\sqrt{2}\, (a^\dagger)^{n-2} \psi_2$$
$$= \frac{N_n}{N_0} \sqrt{1}\sqrt{2}\, (a^\dagger)^{n-3} \underbrace{a^\dagger \psi_2}_{=\sqrt{3}\,\psi_3} = \frac{N_n}{N_0} \sqrt{1\cdot 2\cdot 3}\, (a^\dagger)^{n-3} \psi_3$$

and so on

$$\psi_n = \frac{N_n}{N_0} \sqrt{1\cdot 2\cdot 3 \cdots n}\, \psi_n = \frac{N_n}{N_0} \sqrt{n!}\, \psi_n$$

$$\Rightarrow \boxed{N_n = \frac{1}{\sqrt{n!}} N_0}$$

IIB 3) Finally we must normalize ψ_0 by hand

$$1 \equiv \int_{-\infty}^{+\infty} dx\, |\psi_0|^2 = |N_0|^2 \int_{-\infty}^{+\infty} dx\, e^{-\frac{m\omega}{\hbar}x^2}$$

$$= |N_0|^2 \sqrt{\frac{\pi\hbar}{m\omega}} \equiv 1$$

$$\Rightarrow \boxed{N_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}}$$

with N_0 chosen as positive real.

$$\Rightarrow \boxed{N_n = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}}$$

Hence the normalized energy eigenfunctions are

$$\boxed{\psi_n(x) = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} (a^\dagger)^n e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}}$$

II. B) 4) Orthogonality: From our general theorems we know that

$$m \neq n : \int_{-\infty}^{+\infty} dx\, \psi_m^* \psi_n = 0 \quad \text{. They are orthogonal}$$

Let's recall the argument: Use Sch. eq.

$$a^+ a \psi_n = (\epsilon_n - \tfrac{1}{2})\psi_n = n\psi_n$$
$$\Rightarrow \psi_m^* a^+ a \psi_n = n \psi_m^* \psi_n$$

and c.c. $(a^+ a \psi_m)^* \psi_n = m \psi_m^* \psi_n$

The difference is then integrated

$$(n-m)\int_{-\infty}^{+\infty} dx\, \psi_m^* \psi_n = \int_{-\infty}^{+\infty} dx \left[\psi_m^* a^+ a \psi_n - (a^+ a \psi_m)^* \psi_n\right]$$

we can integrate the RHS second term by parts twice - discarding the $x = \pm\infty$ terms - this is just the Hermitian conjugation property -

$$\int_{-\infty}^{+\infty} dx\, [a^+ a \psi_m]^* \psi_n = \int_{-\infty}^{+\infty} dx \left(\underbrace{\frac{+i}{\sqrt{\hbar\omega}} \frac{1}{\sqrt{2m}} \left[\frac{-\hbar}{i}\frac{d}{dx} - i m\omega x\right]}_{\text{integrate by parts first}} \psi_m^* \right) \times \underbrace{\frac{-i}{\sqrt{\hbar\omega}} \frac{1}{\sqrt{2m}} \left[\frac{-\hbar}{i}\frac{d}{dx} + i m\omega x\right]}_{\text{integrate by parts second}} \psi_n$$

$= a^\dagger$

II.B.4) $= \int_{-\infty}^{+\infty} dx\, \psi_m^* \Big(\overbrace{\frac{-i}{\sqrt{\hbar\omega}} \frac{1}{\sqrt{2m}} \Big[+\frac{\hbar}{i}\frac{d}{dx} + i m\omega x \Big]}^{} \times$

$\times \underbrace{\frac{+i}{\sqrt{\hbar\omega}} \frac{1}{\sqrt{2m}} \Big[+\frac{\hbar}{i}\frac{d}{dx} - i m\omega x \Big]}_{} \psi_n \Big)$

$= a$

$= \int_{-\infty}^{+\infty} dx\, \psi_m^*\, a^\dagger a \psi_n$ This cancels the first term on the RHS

$\Rightarrow$

$$(n-m)\int_{-\infty}^{+\infty} dx\, \psi_m^* \psi_n = 0$$

Since $m \neq n \Rightarrow$ $\boxed{\int_{-\infty}^{+\infty} dx\, \psi_m^* \psi_n = 0}$

So we have <u>Orthonormal</u> eigenfunctions

$$\boxed{\int_{-\infty}^{+\infty} dx\, \psi_m^*(x)\, \psi_n(x) = \delta_{mn}}$$

5) Complete Set of functions – see below analytic.

II. B.)2) Analytic Method: Let's continue by going back to the Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2\right]\psi = E\psi$$

Let's introduce dimensionless variables:

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$K = \frac{E}{\frac{1}{2}\hbar\omega} = 2\epsilon \quad \text{from above}$$

So we have

$$\frac{d}{d\xi} = \sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx} \;;\; \frac{d^2}{d\xi^2} = \frac{\hbar}{m\omega}\frac{d^2}{dx^2}$$

and Sch. eq. becomes

$$\left[-\frac{\hbar^2}{2m}\frac{m\omega}{\hbar}\frac{d^2}{d\xi^2} + \frac{1}{2}m\omega^2\frac{\hbar}{m\omega}\xi^2\right]\psi\left(\sqrt{\frac{\hbar}{m\omega}}\xi\right) = \frac{1}{2}\hbar\omega K\,\psi\left(\sqrt{\frac{\hbar}{m\omega}}\xi\right)$$

So dividing the eq. by $\frac{1}{2}\hbar\omega$

II B) 2) $$\left(-\frac{d^2}{d\xi^2} + \xi^2\right)\psi\left(\sqrt{\frac{\hbar}{m\omega}}\,\xi\right) = K\,\psi\left(\sqrt{\frac{\hbar}{m\omega}}\,\xi\right)$$

Define $$\varphi(\xi) \equiv \psi(x) = \psi\left(\sqrt{\frac{\hbar}{m\omega}}\,\xi\right)$$

We finally have Schrödinger's Eq.:

$$\boxed{\frac{d^2}{d\xi^2}\varphi(\xi) = (\xi^2 - K)\,\varphi(\xi)}$$

Consider $|x| \to \infty$ that is $|\xi| \to \infty$, then $\xi^2 >> K$ and the eq. approaches

$$\frac{d^2}{d\xi^2}\varphi(\xi) \sim \xi^2\varphi(\xi)$$ ← for $\xi^2 >> 1$ (asymptotically equal)

This has solutions of the form

$$\varphi \sim A e^{-\xi^2/2} + B e^{+\xi^2/2}$$

i.e. $$\frac{d}{d\xi}\varphi \sim -\xi A e^{-\xi^2/2} + B\xi e^{+\xi^2/2}$$

$$\frac{d^2}{d\xi^2}\varphi \sim -A e^{-\xi^2/2} + \xi^2 A e^{-\xi^2/2} + B e^{-\xi^2/2} + \xi^2 B e^{+\xi^2/2} \sim \xi^2\varphi \checkmark$$

but $\xi^2 >> 1$ so ignore 1st & 3rd term

II B 2) Since φ must $\to 0$ as $\xi \to \pm\infty$ the $e^{+\xi^2/2}$ term is not normalizable $\Rightarrow$

$\boxed{B=0}$; So asymptotically

$$\varphi(\xi) \sim A e^{-\frac{1}{2}\xi^2}$$

So we factor out this large ξ behavior by defining

$$\boxed{\varphi(\xi) \equiv h(\xi) e^{-\frac{1}{2}\xi^2}}$$

Then plug this into Sch. Eq.

$$\frac{d\varphi}{d\xi} = \frac{dh}{d\xi} e^{-\frac{1}{2}\xi^2} - \xi h e^{-\frac{1}{2}\xi^2}$$

$$\frac{d^2\varphi}{d\xi^2} = \left[\frac{d^2h}{d\xi^2} - 2\xi\frac{dh}{d\xi} + (\xi^2-1)h\right] e^{-\frac{1}{2}\xi^2}$$

$\Rightarrow$ Sch. Eq.:

$$\left[\frac{d^2h}{d\xi^2} - 2\xi\frac{dh}{d\xi} + (\xi^2-1)h\right] e^{-\frac{1}{2}\xi^2}$$

$$= (\xi^2 - K) h e^{-\frac{1}{2}\xi^2}$$

II B 2) Hence

$$\boxed{\frac{d^2h}{d\xi^2} - 2\xi\frac{dh}{d\xi} + (K-1)h = 0}$$

Ansatz: Power Series Solution:

$$h(\xi) = a_0 + a_1\xi^1 + a_2\xi^2 + \cdots$$
$$= \sum_{n=0}^{\infty} a_n \xi^n$$

Differentiate term by term and plug into Sch. Eq.

$$\frac{dh}{d\xi} = \sum_{n=1}^{\infty} n a_n \xi^{n-1} = \sum_{(m=n)\ m=0}^{\infty} m a_m \xi^{m-1}$$

$$\frac{d^2h}{d\xi^2} = \sum_{n=2}^{\infty} n(n-1) a_n \xi^{n-2}$$

Re-write the same so they are over the same range

$$\frac{d^2h}{d\xi^2} = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} \xi^m$$

$(m = n-2)$

II B 2) So we have Sch. Eq.

$$\sum_{m=0}^{\infty} \left[(m+2)(m+1) a_{m+2} \xi^m - 2m a_m \xi^m + (K-1) a_m \xi^m \right] = 0$$

Set summand to zero term by term

$$\Rightarrow$$

$$(m+2)(m+1) a_{m+2} = (2m+1-K) a_m$$

That is

$$a_{m+2} = \frac{(2m+1-K)}{(m+1)(m+2)} a_m$$

~~OR~~ letting $m = n$

$$\boxed{a_{n+2} = \frac{(2n+1-K)}{(n+1)(n+2)} a_n}$$

This is a <u>recursion relation</u> which determines the power series coefficients a_n.

*B 2) The recursion relation relates every other coefficient, hence given a_0 it gives $a_2, a_4, \ldots$. Likewise given a_1 it gives $a_3, a_5, \ldots$.

So we write $h(\xi) = h_{even}(\xi) + h_{odd}(\xi)$

with $h_{even}(\xi) \equiv a_0 + a_2\xi^2 + a_4\xi^4 + \cdots$

$h_{odd}(\xi) \equiv a_1\xi + a_3\xi^3 + a_5\xi^5 + \cdots$.

This is as expected for a 2nd order DE: the general solution contains 2 arbitrary constants a_0 and a_1.

Now this is all equivalent to the Schr. eq. and as we found in the asymptotic analysis not all solutions are normalizable.

For large n the recursion relation is approximately

$$a_{n+2} \sim \frac{2n}{n^2} a_n$$

$$\sim \frac{2}{n} a_n$$

Now recall the series

$$e^{\xi^2} = \sum_{n=0}^{\infty} c_n \xi^n = \sum_{n=0}^{\infty} \frac{\xi^{2n}}{n!}$$

$$= \sum_{\substack{m=0 \\ \text{even} \\ (2n=m)}}^{\infty} \frac{\xi^m}{\left(\frac{m}{2}\right)!} = \cdots + \frac{\xi^m}{\left(\frac{m}{2}\right)!} + \frac{\xi^{m+2}}{\left(\frac{m+2}{2}\right)!} + \cdots$$

So the ratio of coefficients for large m is just

$$\frac{c_{m+2}}{c_m} = \frac{\frac{1}{\left(\frac{m+2}{2}\right)!}}{\frac{1}{\left(\frac{m}{2}\right)!}} = \frac{1}{\frac{m+2}{2}}$$

$$\sim \frac{2}{m}$$

exactly what we find for our recursion relation! (Plus finite powers of ξ^2)

Hence $h(\xi)$ behaves like e^{ξ^2} and so

$$\varphi(\xi) = h(\xi)\, e^{-\frac{1}{2}\xi^2}$$

$$\xi^2 \to \infty \sim e^{+\frac{1}{2}\xi^2}$$ again!

Which is not normalizable!!

II B 2) The only way out of this dilemma is if the power series terminates (once again our BC (normalizable & finite Ψ) imply "quantization")

There is a highest n call it N such that the recursion relation yields

$$a_{N+2} = 0 .$$

This truncates the rest of the series along with, if N is even demanding $a_1 = 0$ (odd series) from the beginning and vice versa if N is odd we require also $a_0 = 0$ from the beginning. Hence we have only finite even or odd polynomial solutions.

Looking at the recursion relation

$$a_{n+2} = \frac{(2n+1-k)}{(n+1)(n+2)} a_n$$

This vanishes for $n = N$ if $k = 2N+1$ for $N = 0, 1, 2, \ldots$

$$\boxed{K \equiv 2N+1}$$

$(K \to K_N)$

Recall $K_N = 2\epsilon_N = \dfrac{E_N}{\frac{1}{2}\hbar\omega}$

Hence the energy is quantized as we found previously

$$E_N = \tfrac{1}{2}\hbar\omega K_N = \left(N + \tfrac{1}{2}\right)\hbar\omega$$

for $N = 0, 1, 2, \ldots$

For these allowed values of K_N the recursion relation becomes

$$a_{n+2} = \frac{-2(N-n)}{(n+1)(n+2)}\, a_n$$

a) If $N = 0$ only $a_{n=0} \neq 0$, the relation $\Rightarrow a_2 = 0 = a_4 = \cdots$. In addition we set $a_1 = 0 \Rightarrow a_3 = a_5 = \cdots = 0$

So we find $N=0 : h_0(\xi) = a_0$

and $$\varphi_0(\xi) = h_0(\xi)\, e^{-\xi^2/2}$$

$$= a_0 e^{-\xi^2/2}$$

$$= \psi_0(x)$$

-B 2) Which, as previously, was the ground state wavefunction corresponding to the lowest energy bound state

$$E_0 = \frac{1}{2}\hbar\omega .$$

---

1) If $N=1$ we set $a_0 = 0$ by hand and this $\Rightarrow a_2 = a_4 = \cdots = 0$
But $a_1 \neq 0$, however the recursion relation $\Rightarrow a_3 = 0$ hence $a_5 = a_7 = \cdots = 0$
So for $N=1$

$$h_1(\xi) = a_1 \xi$$

and $$\psi_1(x) = \varphi_1(\xi) = a_1 \xi e^{-\frac{1}{2}\xi^2}$$

corresponding to the wavefunction for the 1st excited state with energy $E_1 = \frac{3}{2}\hbar\omega$.

---

2) If $N=2$ we set $a_1 = 0$ by hand $\Rightarrow$
$a_3 = a_5 = \cdots = 0$ ($h_{odd} = 0$) but $a_0 \neq 0$

The recursion relation $\Rightarrow$

$$a_2 = -2a_0 \,, \quad a_4 = 0 \Rightarrow a_6 = a_8 = \cdots = 0.$$

$\xi = \beta x$ 2) 2) Hence $h_2(\xi) = a_0(1-2\xi^2)$

and the second excited state wavefunction

$$\psi_2(x) = h_2(\xi) e^{-\frac{1}{2}\xi^2}$$

$$= a_0(1-2\xi^2) e^{-\frac{1}{2}\xi^2}$$

corresponding to energy $E_2 = \frac{5}{2}\hbar\omega$.

So in general $h_N(\xi)$ will be a polynomial of degree N in ξ, even if N is even, odd if N is odd.

Apart from the overall normalization factor a_0 or a_1. These are the Hermite polynomials denoted $H_n(\xi)$ (let's go back to calling N by lower case n)

± B 2) By convention the $H_n(\xi)$ are defined so that the highest power of ξ in $H_n(\xi)$ has $(+2)^n$ as the coefficient.

The first few Hermite polynomials are defined as

$$H_0(\xi) = 1 \qquad (\text{i.e } h_0 \text{ with } a_0 = 1)$$
$$H_1(\xi) = 2\xi \qquad (\text{i.e. } h_1 \text{ with } a_1 = 2)$$
$$H_2(\xi) = 4\xi^2 - 2 \qquad (\text{i.e. } h_2 \text{ with } a_0 = -2)$$
$$\vdots$$
$$H_3(\xi) = 8\xi^3 - 12\xi$$
$$H_4(\xi) = 16\xi^4 - 48\xi^2 + 12$$
$$\vdots$$

Hence the general eigenfunction is given by

$$\psi_n(x) = \varphi_n(\xi) = A_n H_n(\xi)\, e^{-\frac{1}{2}\xi^2}$$
$$= A_n H_n\left(\sqrt{\tfrac{m\omega}{\hbar}}\, x\right) e^{-\frac{m\omega}{2\hbar}x^2}$$

where A_n is just the a_0 or a_1 factor times the factor that takes us from

II-B) The 2^n of the Hermite polynomial to the coefficient of ξ^n that comes from the recursion relation for the highest power of ξ. It will be fixed by normalization.

These eigenfunctions $\psi_n(x)$ correspond to the energy eigenvalues $E_n = (n+\frac{1}{2})\hbar\omega$, for $n = 0, 1, 2, \ldots$.

Mathematicians have studied the Hermite polynomials extensively. We summarize some of their properties here.

Hermite Polynomials

1) $H_n(-\xi) = (-1)^n H_n(\xi)$; $H_n(\xi)$ is an even (odd) function of ξ for n even (odd).

2) $H_n(\xi)$ obeys the differential equation

$$\left[\frac{d^2}{d\xi^2} - 2\xi\frac{d}{d\xi} + 2n\right] H_n(\xi) = 0.$$

This is just the $h(\xi)$ equation (p. 116) with $K = \sqrt{2n+1}$, i.e. Schrödinger's Equation.

The Rodrigues Formula:

3) $$H_n(\xi) \equiv (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2}$$

$$\left(= (2\xi - \frac{d}{d\xi})^n \, 1\right)$$

n^{th} order polynomial in ξ.

Note that $$\frac{d^n}{d\xi^n} e^{-\xi^2} = \frac{d^n}{d\xi^n} e^{-\frac{1}{2}\xi^2} e^{-\frac{1}{2}\xi^2}$$

$$= \frac{d^{n-1}}{d\xi^{n-1}} e^{-\frac{1}{2}\xi^2} \left(\frac{d}{d\xi} - \xi\right) e^{-\frac{1}{2}\xi^2}$$

Continuing to differentiate

$$\frac{d^n}{d\xi^n} e^{-\xi^2} = e^{-\frac{1}{2}\xi^2} \left(\frac{d}{d\xi} - \xi\right)^n e^{-\frac{1}{2}\xi^2}$$

$$\Rightarrow \quad e^{+\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2} = e^{+\frac{1}{2}\xi^2} \left(\frac{d}{d\xi} - \xi\right)^n e^{-\frac{1}{2}\xi^2}$$

Hence we also have

$$H_n(\xi) = (-1)^n e^{+\frac{1}{2}\xi^2} \left(\frac{d}{d\xi} - \xi\right)^n e^{-\frac{1}{2}\xi^2}$$

II B) 3) Now recall that

$$a^\dagger = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}\,x - i\frac{1}{\sqrt{m\hbar\omega}}p\right]$$

$$a^\dagger = \left[\sqrt{\frac{m\omega}{2\hbar}}\,x - \sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\xi - \frac{d}{d\xi}\right]$$

and from our algebraic method the eigenfunctions were given by

$$\psi_n(x) = \frac{N_n}{N_0}(a^\dagger)^n \psi_0(x) \quad , \psi_0 = N_0 e^{-\frac{1}{2}\xi^2}$$

$$= N_n \frac{1}{2^{n/2}}\left[\xi - \frac{d}{d\xi}\right]^n e^{-\frac{1}{2}\xi^2}$$

$$\boxed{\psi_n(x) = N_n \frac{1}{2^{n/2}} H_n(\xi)\, e^{-\frac{1}{2}\xi^2}} = A_n H_n(\xi) e^{-\frac{1}{2}\xi^2}$$

we already found $N_n = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$ to yield orthonormal eigenfunctions

So

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_n(\xi)\, e^{-\frac{1}{2}\xi^2}$$

are orthonormal.

IIB 4) $H_n(\xi)$ can be obtained from the generating function $Z(s)$ by differentiation

$$Z(s) \equiv \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(\xi)$$

$$= e^{-s^2 + 2s\xi}$$

Thus $$H_n(\xi) = \frac{\partial^n}{\partial s^n} Z(s) \Big|_{s=0}$$

Consider the Gaussian

$$G(\xi) = e^{-\xi^2}$$

Then $$\frac{d}{d\xi} G(\xi) = -2\xi e^{-\xi^2} = (-1) H_1(\xi) e^{-\xi^2}$$

$$\frac{d^2}{d\xi^2} G(\xi) = [4\xi^2 - 2] e^{-\xi^2} = (-1)^2 H_2(\xi) e^{-\xi^2}$$

In general

$$\frac{d^n G(\xi)}{d\xi^n} = (-1)^n H_n(\xi) e^{-\xi^2}$$

4) Prove this by induction. it is true for $n=1,2$ as seen above. Assume it is true for $n-1$

$$\frac{d^{n-1}}{d\xi^{n-1}} G(\xi) = (-1)^{n-1} H_{n-1}(\xi) e^{-\xi^2}$$

Then just differentiate

$$\frac{d^n}{d\xi^n} G(\xi) = (-1)^n e^{-\xi^2} \left(2\xi - \frac{d}{d\xi}\right) H_{n-1}(\xi)$$

So we require $\boxed{H_n(\xi) = \left(2\xi - \frac{d}{d\xi}\right) H_{n-1}(\xi)}$

Then $\frac{d^n}{d\xi^n} G(\xi) = (-1)^n H_n(\xi) e^{-\xi^2}$.

Since $H_{n-1}(\xi)$ is a polynomial of degree $n-1$
$\Rightarrow H_n(\xi)$ is a polynomial of degree n

And indeed we have

$$\boxed{H_n(\xi) = (-1)^n e^{+\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2}}$$ as we began.

Note $H_n(\xi) = \left(2\xi - \frac{d}{d\xi}\right) H_{n-1} = \left(2\xi - \frac{d}{d\xi}\right)^2 H_{n-2}$

$$= \cdots = \left(2\xi - \frac{d}{d\xi}\right)^n H_0$$

HB 4) Thus, since $H_0 = 1$,

$$\boxed{H_n(\xi) = \left(2\xi - \frac{d}{d\xi}\right)^n 1}$$

Now $G_1(\xi + s) = e^{-(\xi+s)^2}$ apply Taylor expansion

$$= \sum_{n=0}^{\infty} \frac{s^n}{n!} \frac{d^n G_1(\xi)}{d\xi^n}$$

$$= \sum_{n=0}^{\infty} \frac{s^n}{n!} (-1)^n H_n(\xi) e^{-\xi^2}$$

Now multiply by e^{ξ^2} and let $s \to -s \Rightarrow$

$$e^{\xi^2} G_1(\xi - s) = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(\xi)$$

but $G_1(\xi - s) = e^{-(\xi-s)^2} = e^{-\xi^2 + 2s\xi - s^2}$

$\Rightarrow$

$$\boxed{e^{-s^2+2s\xi} = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(\xi)}$$

$$\equiv Z(s)$$

II B 5) Recursion Relations (use these to show Hn obey Sch. Eq. (2) above)

1) $\frac{dH_n(\xi)}{d\xi} = 2n H_{n-1}(\xi)$

2) $H_{n+1}(\xi) = 2\xi H_n(\xi) - \frac{dH_n(\xi)}{d\xi}$

These follow from $Z(s)$

1) $\frac{\partial Z(s)}{\partial \xi} = \sum_{n=0}^{\infty} \frac{s^n}{n!} \frac{dH_n(\xi)}{d\xi}$

$= \frac{\partial}{\partial \xi}\left(\underbrace{e^{-s^2+2s\xi}}_{=Z(s)}\right) = 2s Z(s)$

$= 2\sum_{n=0}^{\infty} \frac{s^{n+1}}{n!} H_n(\xi)$

$= 2\sum_{m=1}^{\infty} \frac{s^m}{(m-1)!} H_{m-1}(\xi)$

$(m = n+1)$

$= \sum_{m=1}^{\infty} \frac{s^m}{m!} 2m H_{m-1}(\xi)$

$\leftarrow 0$ okay gives nothing

let $m \to n$ $= \sum_{n=0}^{\infty} \frac{s^n}{n!} 2n H_{n-1}(\xi)$

-[B5] $\Rightarrow$ 1) $\dfrac{dH_n(z)}{dz} = 2nH_{n-1}(z)$

---

2) Likewise

$$\frac{\partial}{\partial s}Z(s) = \sum_{n=1}^{\infty} \frac{s^{n-1}}{(n-1)!} H_n(z) = \sum_{\substack{m=0 \\ (m=n-1)}}^{\infty} \frac{s^m}{m!} H_{m+1}(z)$$

$$= (-2s+2z)Z(s)$$

$$= [2z-2s] \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(z)$$

$$= 2z \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(z) - 2 \sum_{n=0}^{\infty} \frac{s^{n+1}}{(n+1)!}(n+1) H_n(z)$$

$$= 2z \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(z) - 2 \sum_{\substack{m=1 \\ (m=n+1)}}^{\infty} \frac{s^m}{m!} m H_{m-1}(z)$$

(extend to $m=0$ okay)

$\Rightarrow$

$$H_{n+1}(z) = 2z H_n(z) - \underbrace{2n H_{n-1}(z)}_{\text{from (1)} = \frac{dH_n(z)}{dz}}$$

$\Rightarrow$ 2) $H_{n+1}(z) = 2z H_n(z) - \dfrac{dH_n(z)}{dz}$

---

6) $H_n^*(\xi) = H_n(\xi)$; $H_n(\xi)$ are real

7) The Hermite Polynomials are orthogonal wrt $e^{-\xi^2}$

$$\int_{-\infty}^{+\infty} d\xi\, e^{-\xi^2} H_m(\xi) H_n(\xi) = \delta_{mn} 2^n n! \sqrt{\pi}$$

Proof: let $m \geq n$

$$\int_{-\infty}^{+\infty} d\xi\, e^{-\xi^2} H_m(\xi) H_n(\xi)$$

$$= \int_{-\infty}^{+\infty} d\xi\, \cancel{e^{-\xi^2}} (-1)^{m+n} \left[\cancel{e^{\xi^2}} \frac{d^m}{d\xi^m} e^{-\xi^2} \right] \left[e^{\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2} \right]$$

$$= (-1)^{m+n} \int_{-\infty}^{+\infty} d\xi \left(\frac{d^m}{d\xi^m} e^{-\xi^2} \right) \left(e^{\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2} \right)$$

Integrate the first term by parts m-times throwing away the terms at $\xi \to \pm\infty$ since $\xi^k e^{-\xi^2} \to 0$ as $\xi \to \pm\infty$

$$= (-1)^{2m+n} \int_{-\infty}^{+\infty} d\xi\, e^{-\xi^2} \frac{d^m}{d\xi^m} \underbrace{\left(e^{+\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2} \right)}$$

$= n^{th}$ degree polynomial
$= b_n \xi^n + b_{n-1} \xi^{n-1} + \cdots + b_0$

II B 7) but $m \geq n$, thus the m^{th} derivative vanishes unless $m = n$; Then only the $b_n m!$ term survives.

From $Z(s) = e^{-s^2 + 2s\xi}$ and $H_n(\xi) = \left.\frac{\partial^n}{\partial s^n} Z(s)\right|_{s=0}$

we see that the $b_n \xi^n$ term comes from the n^{th} s-derivative on $e^{2s\xi} \Rightarrow$

$$b_n = (-2)^n \quad ; \text{ i.e. } H_n(\xi) = 2^n \xi^n + O(\xi^{n-2})$$

So
$$\int_{-\infty}^{+\infty} d\xi\, e^{-\xi^2} H_m(\xi) H_n(\xi) = \delta_{mn} (-1)^n \int_{-\infty}^{+\infty} d\xi\, e^{-\xi^2} (-2)^n n!$$

but $\int_{-\infty}^{+\infty} d\xi\, e^{-\xi^2} = \sqrt{\pi}$

Hence we obtain our result

$$\boxed{\int_{-\infty}^{+\infty} d\xi\, e^{-\xi^2} H_m(\xi) H_n(\xi) = \delta_{mn} 2^n n! \sqrt{\pi}}$$

7) This agrees with the normalization we found from the algebraic method.

$$\psi_n(x) = \varphi_n(\xi) = A_n H_n(\xi) e^{-\frac{1}{2}\xi^2}$$

Hence

$$\int_{-\infty}^{+\infty} dx\, \psi_m^*(x)\psi_n(x) \equiv \delta_{mn}$$

$$= A_m^* A_n \int_{-\infty}^{+\infty} dx\, H_m(\xi) H_n(\xi) e^{-\xi^2}$$

recall $\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$

$$= A_m^* A_n \sqrt{\frac{\hbar}{m\omega}} \underbrace{\int_{-\infty}^{+\infty} d\xi\, e^{-\xi^2} H_m(\xi) H_n(\xi)}_{= \delta_{mn} 2^n n! \sqrt{\pi}}$$

$$= \delta_{mn} |A_n|^2 \sqrt{\frac{\hbar}{m\omega}}\, 2^n n! \sqrt{\pi}$$

$$\Rightarrow \boxed{A_n = \frac{1}{\sqrt{2^n n!}} \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}}$$

as we found on page -127-.

II B) Hence, To Summarize: The orthonormal energy eigenfunctions ψ_n, with eigenvalues E_n, for the simple harmonic oscillator in one-dimension which obeys Sch. eq.

$$\left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2\right]\psi_n(x) = E_n\psi_n(x)$$

are given by

$$\psi_n(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\, H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right) e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$$

with H_n the n^{th} order Hermite polynomial and

$$E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega \quad \text{for } n = 0, 1, 2, \ldots$$

Note: 1) The ground state, $n=0$, has energy $E_0 = \frac{1}{2}\hbar\omega$, the zero-point energy of the SHO.

-II.B 2) The wavefunctions are orthonormal

$$\int_{-\infty}^{+\infty} dx\, \psi_m^*(x)\, \psi_n(x) = \delta_{mn}$$

3) The wavefunctions form a complete set of states

$$\sum_{n=0}^{\infty} \psi_n^*(y)\, \psi_n(x) = \delta(x-y)$$

i.e. Any function $\psi(x)$ may be expanded in terms of the energy eigenstates

$$\boxed{\psi(x) = \sum_{n=0}^{\infty} c_n \psi_n(x)}$$

Since $\{\psi_n\}$ are orthonormal we have

$$\int_{-\infty}^{+\infty} dx\, \psi_m^*(x)\, \psi(x) = \sum_{n=0}^{\infty} c_n \underbrace{\int_{-\infty}^{+\infty} dx\, \psi_m^*(x)\, \psi_n(x)}_{=\delta_{mn}}$$

$$= \sum_{n=0}^{\infty} c_n \delta_{mn}$$

$$= c_m$$

Thus

$$\boxed{c_n = \int_{-\infty}^{+\infty} dx\, \psi_n^*(x)\, \psi(x)}$$