7 Hamiltonian Dynamics

We have been treating $L$ as a function of $q$, $\dot{q}$, and $t$ where the variation in generalized velocities $\delta \dot{q}$ is not an independent function but just the time derivative of the variation of the generalized coordinate $\frac{d}{dt} \delta q$ and hence these are related in the same way that $q$ and $\dot{q}$ are related. In this way we have obtained $n$ second order differential equations for $q^a$, the Euler-Lagrange equations

$$\frac{\partial L(q, \dot{q}; t)}{\partial q^a} - \frac{d}{dt} \left( \frac{\partial L(q, \dot{q}; t)}{\partial \dot{q}^a} \right) = 0. \quad (7.1)$$

Any set of $n$ second order differential equations can be converted into $2n$ first order differential equations by means of Legendre transforming the function $L$. That is we can treat $q^a$ and $v^a = \dot{q}^a$ as independent variables, that is the variations $\delta q^a$ and $\delta v^a$ are independent. Rather than $v^a$ it is customary to use the momentum $p_a$ conjugate to $q^a$, that is $p_a \equiv \partial L/\partial \dot{q}^a$ as the other independent variable. Then we can eliminate $\dot{q}$ from our equations in favor of $p$ by viewing this formula

$$p_a = \frac{\partial L}{\partial \dot{q}^a} \quad (7.2)$$

as implicitly yielding

$$\dot{q}^a = \dot{q}^a(q^b, p_b; t). \quad (7.3)$$

We can use our definition of the Hamiltonian to determine the new first order dynamical equations of motion

$$H(q^a, p_a; t) = \sum_{a=1}^{n} p_a \dot{q}^a(q^b, p_b; t) - L(q^a, \dot{q}^a(q^b, p_b; t); t), \quad (7.4)$$

where now $H = H(q^a, p_a; t)$ is viewed as a function of the independent variables $(q, p; t)$ and where $\dot{q}$ occurs we use $\dot{q}^a(q^b, p_b; t)$. This is called a Legendre transformation. It changes the variables from $(q, \dot{q}; t)$ to $(q, p; t)$. 

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To find the equations of motion consider the change in $H$ caused by an increment of time $dt$

$$dH = \sum_{a=1}^{n} \left( \frac{\partial H}{\partial q^a} dq^a + \frac{\partial H}{\partial p_a} dp_a \right) + \frac{\partial H}{\partial t} dt,$$  \hspace{1cm} (7.5)

where $dq$ and $dp$ are independent. On the other hand we can use the definition of $H = \sum_{a=1}^{n} p_a \dot{q}^a - L$ to calculate its change

$$dH = \sum_{a=1}^{n} \left( dp_a \dot{q}^a + p_a dq^a - \frac{\partial L}{\partial q^a} dq^a - \frac{\partial L}{\partial \dot{q}^a} d\dot{q}^a \right) - \frac{\partial L}{\partial t} dt.$$ \hspace{1cm} (7.6)

But we have that $p_a = \frac{\partial L}{\partial \dot{q}^a}$ and the Euler-Lagrange equations

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^a} \right) = 0,$$ \hspace{1cm} (7.7)

are just

$$\dot{p}_a = \frac{\partial L}{\partial q^a}.$$ \hspace{1cm} (7.8)

So this dynamical input implies

$$dH = \sum_{a=1}^{n} (dp_a \dot{q}^a + p_a dq^a - \dot{p}_a dq^a - p_a dq^a) - \frac{\partial L}{\partial t} dt$$

$$= \sum_{a=1}^{n} (\dot{q}^a dp_a - \dot{p}_a dq^a) - \frac{\partial L}{\partial t} dt.$$ \hspace{1cm} (7.9)

Now this must be the same result as equation (7.5) for the change in $H, dH$, hence we find

$$\dot{q}^a = \frac{\partial H}{\partial p_a}$$

$$-\dot{p}_a = \frac{\partial H}{\partial q^a}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}.$$ \hspace{1cm} (7.10)
Substituting this into equation (7.5) yields
\[ dH = \sum_{a=1}^{n} (-\dot{p}_a dq^a + q^a dp_a) + \frac{\partial H}{\partial t} dt, \]  
(7.11)
which implies
\[ \frac{dH}{dt} = \sum_{a=1}^{n} (-\dot{p}_a q^a + \dot{q}^a \dot{p}_a) + \frac{\partial H}{\partial t} \]
\[ = \frac{\partial H}{\partial t} . \]  
(7.12)

The time derivatives of \( q^a \) and \( p_a \) are the 2n first order differential equations of motion for the \((q^a, p_a)\) system and are known as Hamilton’s equations
\[ \dot{q}^a = \frac{\partial H}{\partial p_a} \]
\[ -\dot{p}_a = \frac{\partial H}{\partial q^a} . \]  
(7.13)

\( q^a \) and \( p_a \) are called canonically conjugate variables. Recall the Euler-Lagrange equations of motion are \( n \) second order differential equations for \( q^a \). Note if \( \partial H/\partial t = 0 \) then \( H = \text{constant} \). Further, if \( U \) is velocity independent and \( x = x(q) \) with not explicit \( t \) dependence then
\[ H = E. \]  
(7.14)

Consider the example of a single particle in a conservative force field with potential energy \( U(x, y, z) \)
\[ L = T - U = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z). \]  
(7.15)
The Euler-Lagrange equations of motion are
\[ \frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = 0, \]  
(7.16)
which yields Newton’s 2nd law

\[ m \ddot{x}_i = -\partial_i U. \]  \hspace{1cm} (7.17)

Now Legendre transform to the Hamiltonian

\[ H(\bar{x}, \bar{p}; t) = \bar{p} \cdot \dot{\bar{x}} - L(\bar{x}, \dot{\bar{x}}(\bar{x}, \bar{p}; t); t), \]  \hspace{1cm} (7.18)

where we eliminate \( \dot{\bar{x}} \) in terms of \( \bar{x}, \bar{p} \) and \( t, \dot{\bar{x}} = \dot{x}(\bar{x}, \bar{p}; t) \) by the definition of \( \bar{p} \)

\[ p_i = \frac{\partial L}{\partial \dot{x}_i} = m \dot{x}_i, \]  \hspace{1cm} (7.19)

so that

\[ \dot{\bar{x}} = \frac{1}{m} \bar{p}. \]  \hspace{1cm} (7.20)

Hence

\[ H(\bar{x}, \bar{p}; t) = \frac{1}{m} \bar{p} \cdot \bar{p} - T + U \]

\[ = \frac{\bar{p}^2}{m} - \frac{1}{2} m \frac{1}{m^2} \bar{p} \cdot \bar{p} + U \]

\[ = \frac{\bar{p}^2}{2m} + U(\bar{x}). \]  \hspace{1cm} (7.21)

The dynamical equations of motion are Hamilton’s equations

\[ \dot{x}_i = \frac{\partial H}{\partial p_i} = \frac{1}{m} p_i \]

\[ -\dot{p}_i = \frac{\partial H}{\partial x_i} = \partial_i U. \]  \hspace{1cm} (7.22)

Thus we have two first order differential equations. Note we obtain the usual second order differential equation Newton’s law (Euler-Lagrange equation of motion) by substituting the \( p_i = m \dot{x}_i \) Hamilton equation into the \( \dot{p}_i = -\partial_i U \)
Hamilton equation, implying \( m\ddot{x}_i = -\partial_i U \). We can then solve this for \( x_i \) and \( p_i = m\dot{x}_i \).

In general it is easier to obtain the equations of motion by Lagrangian methods. However since \( q \) and \( p \) are independent this sometimes leads to simplification of the analysis of the problem. In particular if \( q^a \) does not appear in \( H \) then

\[
-p_a = \frac{\partial H}{\partial q^a} = 0,
\]

and \( p_a = \text{constant} \equiv \pi_a \). \( q^a \) is then called cyclic. So

\[
H = H(q^1, \ldots, q^n, p_1, \ldots, \pi_a, \ldots, p_n; t)
\]

depends on \( (2n - 2) \) variables now; we have reduced the number of degrees of freedom. \( q^a \) is said to be ignorable. Indeed \( \dot{q}^a = \partial H/\partial \pi_a \equiv \omega^a \) and hence \( q^a(t) = \int^t \omega^a dt \). This could lead to a practical simplification of the problem. If \( q^a \) is cyclic in \( H \) it is also cyclic in \( L \), that is \( \partial L/\partial q^a = 0 \) \( (H = p\dot{q} - L \text{ so that } \partial H/\partial q = p\partial \dot{q}/\partial q - \partial L/\partial q - \partial L/\partial \dot{q}\partial \dot{q}/\partial \dot{q} = -\partial L/\partial \dot{q} = 0.) \) So we see that \( p_a = \partial L/\partial \dot{q}^a \) = constant for cyclic variables.

Consider the two body central force problem with Lagrangian

\[
L = \frac{1}{2} m_1 \ddot{r}_1^2 + \frac{1}{2} m_2 \ddot{r}_2^2 - U(|\vec{r}_1 - \vec{r}_2|).
\]

Using new center of mass coordinates

\[
\begin{align*}
\vec{R} &= \frac{m_1}{m_1 + m_2} \vec{r}_1 + \frac{m_2}{m_1 + m_2} \vec{r}_2, \\
\vec{r} &= \vec{r}_1 - \vec{r}_2,
\end{align*}
\]

and the inverse relations

\[
\begin{align*}
\vec{r}_1 &= \vec{R} + \frac{m_2}{M} \vec{r} \\
\vec{r}_2 &= \vec{R} - \frac{m_1}{M} \vec{r}
\end{align*}
\]

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where the total mass of the two bodies is \( M = m_1 + m_2 \) and the reduced mass is defined as

\[
\mu \equiv \frac{m_1 m_2}{m_1 + m_2}.
\] (7.28)

Hence the Lagrangian becomes

\[
L = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 - U(|\vec{r}|).
\] (7.29)

If we choose our inertial frame to be at the center of mass, then the location of the center of mass is \( \vec{R} = 0 \) in that frame. Or more generally \( \vec{R} \) is cyclic. So \( \vec{P} = \partial L / \partial \dot{\vec{R}} = M \dot{\vec{R}} = \text{constant} \). Thus we find

\[
\vec{R} = \frac{\vec{P}}{M} t + \vec{R}_0.
\] (7.30)

So without loss of generality we can ignore \( \vec{R} \). Hence the Lagrangian for the “relative particle” is

\[
L = \frac{1}{2} \mu \dot{\vec{r}}^2 - U(|\vec{r}|).
\] (7.31)

Next we can work in cylindrical coordinates \((r, \theta, z)\) in which \( L \) becomes

\[
L = \frac{1}{2} \mu \left( \dot{r}^2 + r^2 \dot{\theta}^2 + z^2 \right) - U(\sqrt{r^2 + z^2}).
\] (7.32)

We see that \( \theta \) is cyclic so that

\[
\frac{\partial L}{\partial \dot{\theta}} = 0,
\] (7.33)

which implies that

\[
p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{constant} = l.
\] (7.34)

The Euler-Lagrange equations become

\[
\frac{\partial L}{\partial r} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = 0 = \mu r \dot{\theta}^2 - \frac{\partial U}{\partial r} - \mu \ddot{r}
\]
\[
\frac{\partial L}{\partial z} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}} \right) = 0 = -\frac{\partial U}{\partial z} - \mu \ddot{z}
\]
and
\[
l = \mu r^2 \dot{\theta}.
\]  
(7.35)

Since the two particles move in the same plane we can orient the coordinate axes so that there is no force in the z-direction and hence \(z(t)\) remains at its initial value taken to be zero (along with its z component of velocity) so \(z(t) = 0\). This implies that the potential energy is a function of \(r\) only \(U = U(r)\). The radial Euler-Lagrange equation becomes
\[
\mu \dddot{r} = \frac{l^2}{\mu r^3} - \frac{dU}{dr},
\]  
(7.36)

where the angular momentum equation for \(\dot{\theta} = l/\mu r^2\) has been used. For the \(1/r\) gravitational potential energy, the orbits will be conic sections as we have seen and will see again.

Finally we can state our variational dynamical principle in terms of \(q\) and \(p\) as independent variations
\[
\delta \int_{t_1}^{t_2} dt \left( \sum_{a=1}^{n} p_a \dot{q}^a - H(q, p; t) \right) = 0,
\]  
(7.37)

where \(\delta p_a\) and \(\delta q^a\) are independent but \(\delta \dot{q}^a = \frac{d}{dt} \delta q^a\) is not independent. Hence
\[
0 = \int_{t_1}^{t_2} dt \sum_{a=1}^{n} \left( \delta p_a \dot{q}^a + \frac{d}{dt} (p_a \delta q^a) - \dot{p}_a \delta q^a - \frac{\partial H}{\partial q^a} \delta q^a - \frac{\partial H}{\partial p_a} \delta p_a \right)
\]
\[
= \int_{t_1}^{t_2} dt \sum_{a=1}^{n} \left[ \left( \dot{q}^a - \frac{\partial H}{\partial q^a} \right) \delta p_a - \left( \dot{p}_a + \frac{\partial H}{\partial p_a} \right) \delta q^a \right].
\]  
(7.38)

Since \(\delta q^a\) and \(\delta p_a\) are independent we obtain Hamilton’s equations
\[
\dot{q}^a = \frac{\partial H}{\partial p_a},
\]
\[
-\dot{p}_a = \frac{\partial H}{\partial q^a}.
\]  
(7.39)

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