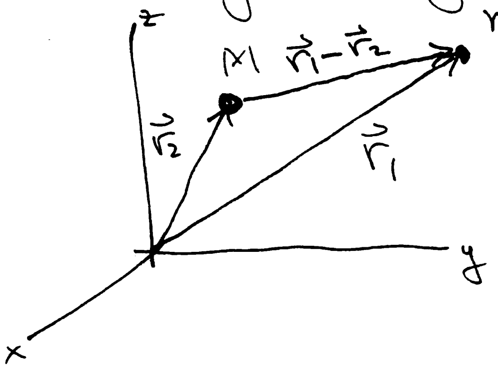
Newton's Law of Universal Gravitation

Newton discovered that the gravitational force acting on a body of mass m and caused by another body of mass M is attractive and given by



$$\vec{F}_m = -G \frac{mM}{|\vec{r}_1 - \vec{r}_2|^2} \frac{(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|}$$

force on m due to M

The negative sign means the force is attractive.

G = Newton's Constant = $6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2}$.

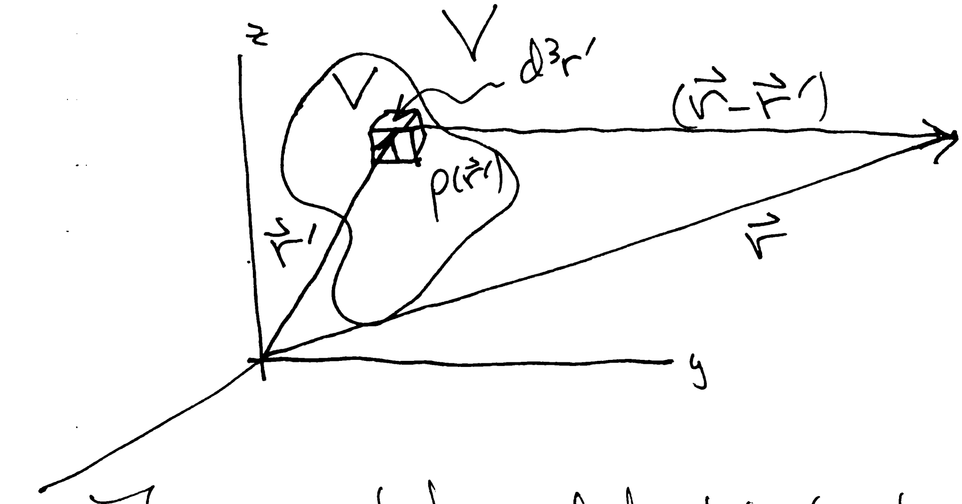
Further the force obeys the strong form of the 3rd law: hence the force acting on M due to m is

$$\vec{F}_M = -\vec{F}_m = -G \frac{mM}{|\vec{r}_1 - \vec{r}_2|^2} \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|}$$

and the force is central i.e. it acts along the line joining the particles.

As written this is a force between our elementary point particles. We may also consider a collection of point particles macroscopically as a continuous distibution of matter and for that matter, as we shall see, we can treat point particles themselves as a matter distribution. The total force of a matter distribution acting on a mass m is just the vector sum (integral) of the individual infinitesimal "point" mass forces. Hence the force on m at position $\vec{r}$ due to a mass distribution of mass density $\rho(\vec{r}')$ contained in volume V (each point labelled by $\vec{r}'$ in V) is given by

$$\vec{F}_m = -Gm \int \frac{\rho(\vec{r}')(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} d^3r'$$



The gravitational field (vectorfield) $\vec{g}$ is defined as the vectorfield representing the force per unit mass exerted on a test particle of mass m at each point in space for vanishingly small m (so that m does not affect the mass distribution and hence alter the gravitational field i.e. we want to view $\vec{g}(\vec{r})$ as a field intrinsic to the mass distribution ρ, not dependent upon the test mass)

$$\vec{g}(\vec{r}) = \lim_{m\to 0} \frac{1}{m} \vec{F}_m(\vec{r}) .$$

For a point source M located at $\vec{r}\,'$ we then have

$$\vec{g}(\vec{r}) = -G \frac{M}{|\vec{r}-\vec{r}\,'|^3}(\vec{r}-\vec{r}\,')$$

while for a mass distribution of density ρ we obtain

$$\vec{g}(\vec{r}) = -G \int_V \frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, d^3r' ,$$

Note: $\vec{g}$ has units of acceleration.

Further we have that $\vec{F}_m(\vec{r})$ and hence $\vec{g}(\vec{r})$ are <u>conservative</u> fields

To see this consider the curl of $\vec{g}$

$$\vec{\nabla}_{\vec{r}} \times \vec{g}(\vec{r}) = -G \int_V d^3r' \, \rho(\vec{r}\,') \, \vec{\nabla}_{\vec{r}} \times \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

Now

$$\vec{\nabla}_{\vec{r}} \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = \frac{1}{|\vec{r}-\vec{r}\,'|^3} \vec{\nabla}_{\vec{r}} \times (\vec{r}-\vec{r}\,')$$

$$+ \left(\vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}\,'|^3} \right) \times (\vec{r}-\vec{r}\,')$$

First consider in Cartesian coordinates the i^{th} component of the curl term on the RHS

$$\left[\vec{\nabla}_{\vec{r}} \times (\vec{r}-\vec{r}\,') \right]_i = \epsilon_{ijk} \, \partial_j (x-x')_k = \epsilon_{ijk} \underbrace{\partial_j x_k}_{=\delta_{jk}}$$

$$= \epsilon_{ijj} = 0$$

Also the gradient term has the i^{th} component

$$\left(\vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}\,'|^n} \right)_i = \partial_i \left(\sum_{j=1}^{3} (x_j - x_j')^2 \right)^{-\frac{n}{2}}$$

$$= -\frac{n}{2} \left(\sum_{k=1}^{3} (x_k - x_k')^2 \right)^{-\frac{n}{2}-1} \sum_{j=1}^{3} 2(x_j - x_j') \delta_{ij}$$

$$= \frac{-n(x_i - x_i')}{\left(\sum_{k=1}^{3} (x_k - x_k')^2 \right)^{\frac{n+2}{2}}}$$

$$= -n \frac{(x_i - x_i')}{|\vec{r}-\vec{r}\,'|^{n+2}}$$

hence
$$\vec{\nabla}_{\vec{r}}\left(\frac{1}{|\vec{r}-\vec{r}\,'|^n}\right) = -n\,\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^{n+2}}$$

So
$$\vec{\nabla}_{\vec{r}}\left(\frac{1}{|\vec{r}-\vec{r}\,'|^3}\right) = \frac{-3(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^5}$$

Putting this together, we have
$$\left[\vec{\nabla}_{\vec{r}}\left(\frac{1}{|\vec{r}-\vec{r}\,'|^3}\right)\right]\times(\vec{r}-\vec{r}\,') = -3\,\frac{\overbrace{(\vec{r}-\vec{r}\,')\times(\vec{r}-\vec{r}\,')}^{=0}}{|\vec{r}-\vec{r}\,'|^5}$$
$$=0$$

So
$$\boxed{\vec{\nabla}_{\vec{r}}\times\vec{g}(\vec{r}) = 0}$$

$\vec{g}(\vec{r})$ is a conservative field
(and hence so is $\vec{F}_m(\vec{r})$)
(clearly $\frac{\partial}{\partial t}\vec{g}(\vec{r})=0$)

Since $\vec{\nabla}\times\vec{g}=0 \Rightarrow$

$$\boxed{\begin{array}{l}\vec{g}(\vec{r}) = -\vec{\nabla}_{\vec{r}}\,\phi(\vec{r}) \text{ where} \\ \phi(\vec{r}) \text{ is the gravitational potential}\end{array}}$$

Using the gradient formula above we have that

$$\vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}'|} = -\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

So

$$\phi(\vec{r}) = -G\int_V d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \quad \text{for}$$

a mass distribution $\rho(\vec{r}')$ while for a point mass M at $\vec{r}'$ we have

$$\phi(\vec{r}) = -G\frac{M}{|\vec{r}-\vec{r}'|} .$$

In both cases taking the gradient of ϕ we recover the form of $\vec{g}(\vec{r})$

$$\vec{g}(\vec{r}) = -\vec{\nabla}_{\vec{r}}\, \phi(\vec{r}) .$$

Note: ϕ is defined only up to an additive constant since

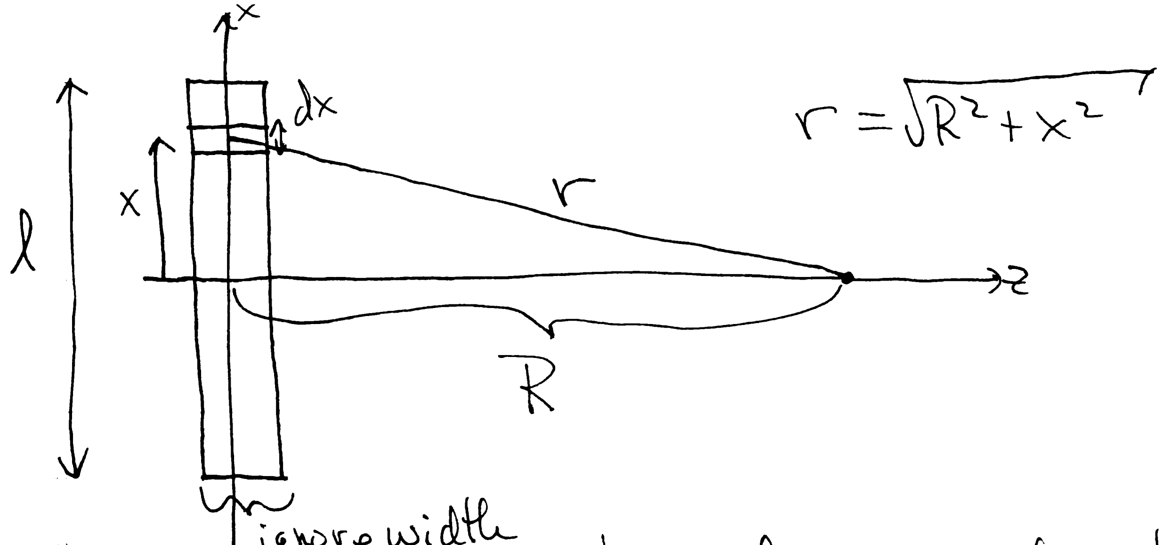
$$\phi' = \phi + C$$

yields the same $\vec{g}$;

$$\vec{g}' = -\vec{\nabla}\phi' = -\vec{\nabla}\phi - \cancel{\vec{\nabla} C}^{\,0} = \vec{g} .$$

In writing the above formulas we assumed that $\rho(\vec{r})$ was such (finite extent) that we could consistently choose the constant to be zero. That is ρ is such that $\phi(\vec{r}) \xrightarrow[\vec{r}\to\infty]{} 0$.

Example: Calculate the gravitational potential due to a thin rod of length l and mass M at a distance R from the center of the rod and in a direction $\perp$ to the rod. (M-T Problem 5.7)



hence linear mass density

The gravitational potential at R due to the mass in length dx of rod with density per unit length of $\lambda = \frac{M}{l}$ is

$$d\phi = -G\frac{\lambda dx}{r}$$

Since ϕ is a scalar function we simply add up the $d\phi$ contributions to obtain the total ϕ. So integrating over the length of the rod, we find the gravitational potential at R

$$\phi = -G\frac{M}{l}\int_{-l/2}^{+l/2}\frac{dx}{\sqrt{R^2+x^2}}$$

$$= -\frac{GM}{l}\ln\left[\frac{\frac{l}{2}+\sqrt{R^2+\frac{l^2}{4}}}{-\frac{l}{2}+\sqrt{R^2+l^2/4}}\right]$$

(Using eq. E.6 in M-T: $\int\frac{dx}{\sqrt{x^2+a^2}} = \ln\{x+\sqrt{x^2+a^2}\}$)

Hence

$$\boxed{\phi = -\frac{GM}{l}\ln\left[\frac{\sqrt{4R^2+l^2}+l}{\sqrt{4R^2+l^2}-l}\right]}$$

Suppose we are very far away from the rod compared to its length; $R >> l$. We may expect that the rod will look like a point particle approximately of mass M at the origin; hence

$$\phi \underset{R>>l}{=} -\frac{GM}{R}.$$

Let's check our intuition by expanding the log and $\sqrt{\ }$ in powers of $\frac{l}{R} \ll 1$

First
$$\phi = -\frac{GM}{l} \ln\left[\frac{\sqrt{1+\frac{l^2}{4R^2}} + \frac{l}{2R}}{\sqrt{1+\frac{l^2}{4R^2}} - \frac{l}{2R}}\right]$$

But
$$\sqrt{1+\frac{l^2}{4R^2}} \approx 1 + \frac{l^2}{8R^2}$$

So
$$\phi \approx -\frac{GM}{l} \ln\left[\frac{1+\frac{l^2}{8R^2}+\frac{l}{2R}}{1+\frac{l^2}{8R^2}-\frac{l}{2R}}\right]$$

but $\frac{l^2}{R^2} \ll \frac{l}{R} \ll 1$ so ignore quadratic terms
$$\phi \approx -\frac{GM}{l} \ln\left[\frac{1+\frac{l}{2R}}{1-l/2R}\right]$$

but $\frac{1}{1-\frac{l}{2R}} \approx 1+\frac{l}{2R}$ So
$$\phi \approx -\frac{GM}{l} \ln\left[\left(1+\frac{l}{2R}\right)\left(1+\frac{l}{2R}\right)\right]$$
$$\approx -\frac{GM}{l} \ln\left(1+\frac{l}{R}\right)$$ ignoring quadratic $\frac{l^2}{R^2}$ terms

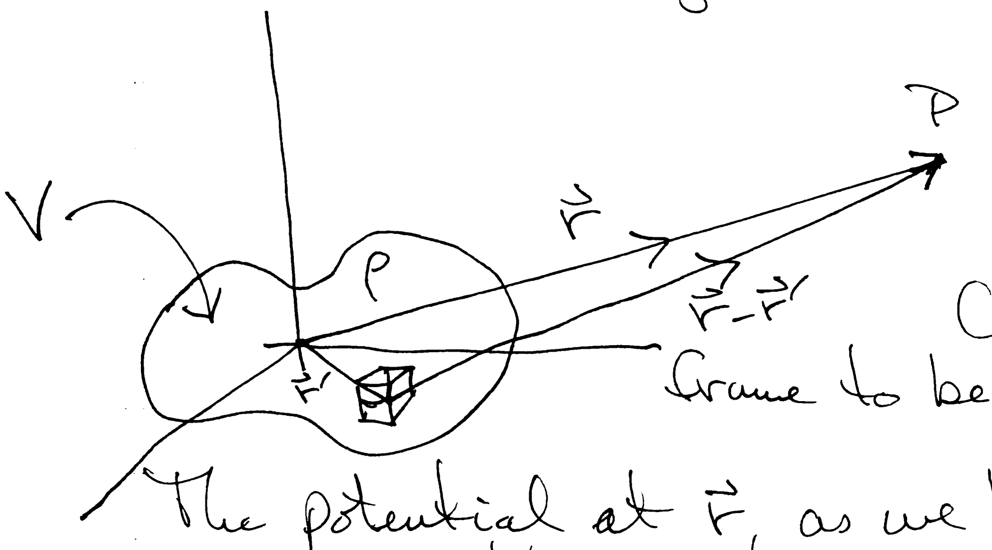
Finally
$$\ln(1+\epsilon) \approx \epsilon$$

Hence we find what we expected
$$\phi \approx -\frac{GM}{R} \text{ for } R \gg l$$
This is just the monopole term in a multipole

expansion for ϕ. As we've seen the monopole term approximates the mass distribution by a point mass equal to the total mass at the origin.

Indeed we can consider an expansion for the gravitational potential when we are observing the field at distances $\vec{r}$ much larger than the physical extent of the mass distribution.

Suppose a volume V of mass density ρ is such that the radius R of the smallest sphere which encloses V is very much less than the distance to the point of observation $R \ll |\vec{r}|$



Choose the inertial frame to be inside V.

The potential at $\vec{r}$, as we have seen, is exactly

$$\phi(\vec{r}) = -G \int_V d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

Consider the denominator for $\frac{|\vec{r}\,'|}{|\vec{r}|} \ll 1$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{\sqrt{(\vec{r}-\vec{r}\,')\circ(\vec{r}-\vec{r}\,')}}$$

$$= \frac{1}{\sqrt{r^2 - 2\vec{r}\cdot\vec{r}\,' + \vec{r}\,'^2}}$$

$$= \frac{1}{\sqrt{r^2}\sqrt{1-\left(\frac{2\vec{r}\cdot\vec{r}\,'}{r^2} - \frac{r'^2}{r^2}\right)}} \qquad \left(\begin{array}{l}\text{call}\\ \epsilon = \left(\frac{2\vec{r}\cdot\vec{r}\,'}{r^2} - \frac{r'^2}{r^2}\right)\end{array}\right)$$

$$= \frac{1}{r}\,\frac{1}{\sqrt{1-\epsilon}}$$

But $\frac{1}{\sqrt{1-\epsilon}} = 1 + \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 + \cdots$

$$= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n!\,2^n}\,\epsilon^n$$

So

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\left[1 + \frac{1}{2}\left(\frac{2\vec{r}\cdot\vec{r}\,'}{r^2} - \frac{r'^2}{r^2}\right) + \frac{3}{8}\left(\frac{2\vec{r}\cdot\vec{r}\,'}{r^2} - \frac{r'^2}{r^2}\right)^2 + \cdots\right]$$

Grouping equal powers of $\frac{|\vec{r}\,'|}{|\vec{r}|} \ll 1$

$$= \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}\,'}{r^3} + \frac{1}{2}\left(\frac{3(\vec{r}\cdot\vec{r}\,')^2}{r^5} - \frac{r'^2}{r^{\cancel{2}\,3}}\right) + \cdots$$

Thus we find

$$\phi(\vec{r}) = -G\int_V d^3r'\,\rho(\vec{r}\,')\Big[\frac{1}{r} + \frac{\vec{r}}{r^3}\cdot\vec{r}\,' + \frac{1}{2}\Big(\frac{3(\vec{r}\cdot\vec{r}\,')^2}{r^5} - \frac{r'^2}{r^3}\Big) + \cdots\Big]$$

Next we can bring all factors of $\vec{r}$ outside the $\vec{r}\,'$ integration. To do this in the 3rd term consider it in a Cartesian coordinate system $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
$\vec{r}\,' = x'\hat{i} + y'\hat{j} + z'\hat{k}$

So

$$3(\vec{r}\cdot\vec{r}\,')^2 - r'^2 r^2$$

$$= 3(x_i x_i')(x_j x_j') - r'^2 x_i x_i$$

$$= x_i x_j\left[3x_i' x_j' - \delta_{ij} r'^2\right]$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j\left[3x_i' x_j' - \delta_{ij} r'^2\right]$$

Hence we find

$$\boxed{\phi(\vec{r}) = -G\left\{\frac{1}{r}\int_V \rho(\vec{r}')\,d^3r' + \frac{\vec{r}}{r^3}\cdot\int_V d^3r'\rho(\vec{r}')\vec{r}' + \frac{1}{2}\frac{x_i x_j}{r^5}\int_V d^3r'\rho(\vec{r}')\left(3x_i'x_j' - \delta_{ij}r'^2\right) + \cdots\right\}}$$

Each term in this multipole expansion can be interpreted as

1) $M \equiv \int_V \rho(\vec{r}')\,d^3r'$ is the total mass enclosed in volume V. Its contribution to the gravitational potential is $-\frac{GM}{r}$, as if the total mass of the distribution were concentrated in a point mass at the origin. This is called the <u>monopole</u> term.

2) Dipole:
$$\vec{d} \equiv \int_V d^3r'\rho(\vec{r}')\vec{r}'$$

The mass distribution acts like a dipole distribution centered on the origin and contributes $-G\frac{\vec{d}\cdot\vec{r}}{r^3}$ to the potential. The dipole moment

$\vec{d}$ is the first moment of the density distribution.

3) Quadrupole Moment: the 3rd term contributes

$$-G\,\frac{1}{2}\,\frac{x_i Q_{ij} x_j}{r^5} \quad \text{to the}$$

potential where

$$Q_{ij} = \int_V d^3r'\,\rho(\vec{r}\,')\,(3x_i'x_j' - \delta_{ij}r'^2) \quad \text{is}$$

the quadrupole moment of the mass distribution & is related to the Inertial tensor as we will see later.

Note that Q_{ij} has 9-matrix Elements since $i=1,2,3$ and $j=1,2,3$. Only 5 of these are independent. First $Q_{ij}=Q_{ji}$ is a symmetric matrix. Second

$$Q_{ii} = \int_V d^3r'\,\rho(\vec{r}\,')\,(3\underbrace{x_i'x_i'}_{=\vec{r}\,'^2} - \underbrace{\delta_{ii}}_{=3}r'^2)$$

$$= 0. \text{ The trace of } Q_{ij} \text{ is zero.}$$

So

$Q_{ij} = Q_{ji}$ are 3 equations

$\mathrm{Tr}\,Q = Q_{ii} = 0$ is $\underline{1 \text{ equation}}$

4 equations

Hence $9-4=5$ independent matrix elements.

Q_{ij} is a symmetric, traceless 3×3 matrix.

Note: all the multipoles depend on the location of the origin except the monopole since it is just the total mass of the system.

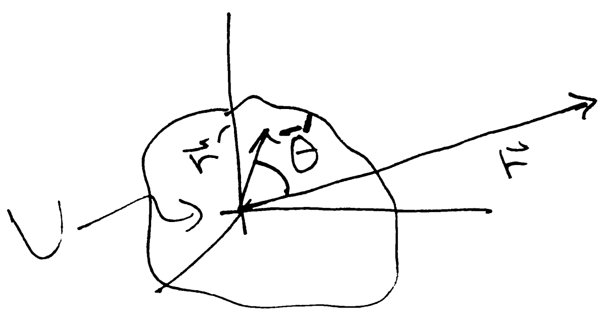
Hence we obtain the multipole expansion of the gravitational potential

$$\phi(\vec{r}) = -\frac{GM}{r} - G\frac{\vec{d}\cdot\vec{r}}{r^3} - G\frac{\frac{1}{2}x^T Q x}{r^5} + \cdots$$

The general form of the multipole expansion can be obtained by expanding $\frac{1}{|\vec{r}-\vec{r}'|}$ in terms of Legendre polynomials

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\underbrace{\cos\theta'}_{=\hat{r}'\cdot\hat{r}})$$

Angle between $\vec{r}$ & $\vec{r}'$ is θ'



(Note: we must express $\bar{\theta}'$ in terms of θ' and θ in order to do the integral and have a result that only depends on θ.)

So

$$\boxed{\begin{array}{l}\phi(\vec{r}) = -G \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int d^3r' \rho(\vec{r}') \, r'^n P_n(\cos\bar{\theta}') \\ \text{where } \bar{\theta}' = \sphericalangle \text{ between } \vec{r} \,\&\, \vec{r}',\end{array}}$$

and

$P_0(\cos\bar{\theta}') = 1$ $\qquad$ $P_2(\cos\bar{\theta}') = \frac{1}{2}(3\cos^2\bar{\theta}' - 1)$

-insert- -1- to -4- $\quad$ $P_1(\cos\bar{\theta}') = \cos\bar{\theta}'$ $\qquad$ P_3 ... etc.

As we know $\phi(\vec{r})$, in the case of azimuthal symmetry of the mass distribution, can itself be written as an expansion in terms of Legendre polynomials with θ the polar coordinate of $\vec{r}$ in that case. This is a result of the fact that outside the mass distribution $\phi(\vec{r})$ obeys Laplace's equation - as we shall see - $\nabla^2\phi = 0$.

Towards this end consider in particular the mass density $\rho(\vec{r})$ of a point mass M at position $\vec{r}_0$.

We can approximate the $\rho(\vec{r})$ by sharper and sharper ~~peaked~~ mass distributions

So we have 2-forms of expansions of ϕ

First our general Taylor (Maclurin) series multipole expansion in Cartesian Coord.

$$\phi(\vec{r}) = -\frac{GM}{r} - G\frac{\vec{d}\cdot\vec{r}}{r^3} - G\frac{\frac{1}{2}x^T Q x}{r^5} - \cdots$$

$$= -G\sum_{l=0}^{\infty}\left(\int d^3r'\rho(\vec{r}')\frac{(-1)^l}{l!}\sum_{i_1\cdots i_l=1}^{3} x'_{i_1}\cdots x'_{i_l}\times\right)\times\frac{\partial^l}{\partial x_{i_1}\cdots\partial x_{i_l}}\left(\frac{1}{r}\right)$$

Now In the case of azimuthal symmetry we have the general expansion in spherical polar coordinates

$\cancel{\phi(r,\theta)}$

$$\phi(r,\theta) = -G\sum_{l=0}^{\infty}\left[A_l r^l + \frac{B_l}{r^{l+1}}\right]P_l(\cos\theta)$$

If ρ has compact support $\Rightarrow \phi \xrightarrow{r\to\infty} 0 \Rightarrow A_l = 0$

So

$$\phi(r,\theta) = -\frac{G}{r}\sum_{l=0}^{\infty}\frac{B_l}{r^l}P_l(\cos\theta)$$

For azimuthal symmetric mass distribution $\rho = \rho(r, \theta)$ then

$$\vec{d} = d\hat{k} \Rightarrow \vec{d} \cdot \vec{r} = r d \cos\theta = r d P_1(\cos\theta)$$

and

$$Q_{xy} = Q_{xz} = Q_{yz} = 0$$

while $Q_{xx} = Q_{yy} = -\frac{1}{2} Q_{zz}$

So $\frac{1}{2} x^T Q x = \frac{1}{4} Q_{zz} [3z^2 - r^2]$

$$= \frac{1}{4} Q_{zz} r^2 (3\cos^2\theta - 1)$$

$$= \frac{1}{2} Q_{zz} r^2 P_2(\cos\theta) \equiv Q r^2 P_2$$

So the multipole in spherical coord- becomes

$$\boxed{\phi(r,\theta) = -\frac{GM}{r} - \frac{Gd}{r^2} P_1(\cos\theta) - \frac{GQ}{r^3} P_2(\cos\theta) - \cdots}$$

Or more generally we can expand the potential in terms of spherical harmonics $Y_\ell^m(\theta,\varphi)$ so that

$$\phi(r,\theta,\varphi) = -G \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \left[A_{\ell m} r^\ell + \frac{B_{\ell m}}{r^{\ell+1}}\right] Y_\ell^m(\theta,\varphi)$$

If ρ has compact support $\Rightarrow \phi \xrightarrow{r\to\infty} 0 \Rightarrow A_{\ell m} = 0$

$$\Rightarrow \quad \phi(r,\theta,\varphi) = \frac{-G}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \frac{B_{\ell m}}{r^\ell} Y_\ell^m(\theta,\varphi)$$

where

$$Y_\ell^m(\theta,\varphi) = (-1)^{\frac{|m|+m}{2}} \left[\frac{2\ell+1}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}\right]^{1/2} \times P_\ell^m(\cos\theta)\, e^{+im\varphi}.$$

1) Orthonormality:

$$\int_{4\pi} d\Omega\, Y_\ell^{m*}(\theta,\varphi)\, Y_{\ell'}^{m'}(\theta,\varphi) = \delta_{\ell\ell'}\,\delta_{mm'}$$

2) Addition Theorem (Advanced property)

$$P_\ell(\cos\bar{\theta}') = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{+\ell} Y_\ell^{m*}(\theta',\varphi')\, Y_\ell^m(\theta,\varphi)$$

So in general we find

$$\phi(\vec r) = -G\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int d^3r'\,\rho(\vec r')\,r'^{l}\,P_l(\cos\bar\theta')$$

$$= -G\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\frac{4\pi}{2l+1}\sum_{m=-l}^{+l} Y_l^m(\theta,\varphi)\left[\int d^3r'\,\rho(\vec r')\,r'^{l}\times\right.$$
$$\left.\times\, Y_l^{m*}(\theta',\varphi')\right]$$

That is

$$B_{lm} = \frac{4\pi}{2l+1}\int d^3r'\,\rho(\vec r')\,r'^{l}\,Y_l^{m*}(\theta',\varphi') .$$

Now if $\rho(\vec r)$ is azimuthally symmetric so that

$\rho = \rho(r,\theta)$ only $\Rightarrow$ only $m=0$ contributes

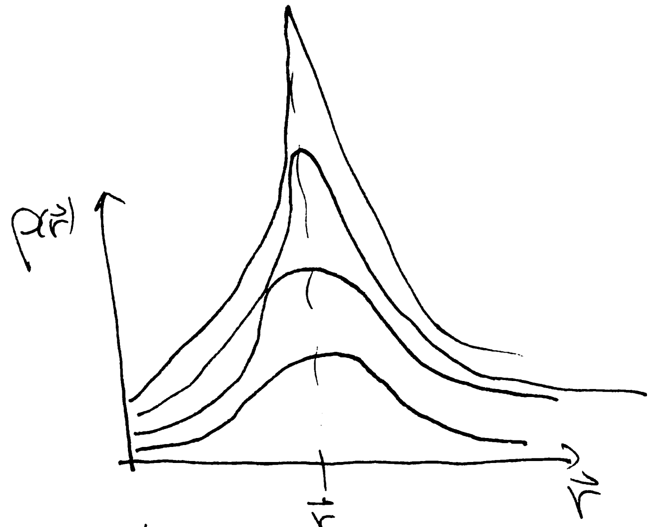
& $\displaystyle\int_{\varphi=0}^{2\pi} d\varphi'\, Y_l^{m*}(\theta',\varphi') = \sqrt{\pi}\,\sqrt{2l+1}\,P_l(\cos\theta')\,\delta_{m0}$

$$\Rightarrow B_{lm} = \frac{4\pi^{3/2}}{\sqrt{2l+1}}\int_0^{\infty} dr'\int_{-1}^{+1} d(\cos\theta')\,\rho(r',\theta')\,r'^{l+2}\,P_l(\cos\theta')\,\delta_{m0}$$

and

$$\phi(\vec r) = -G\sum_{l=0}^{\infty}\frac{B_{l0}}{r^{l+1}}\,Y_l^0(\theta,\varphi)$$

$$= -\frac{G}{r}\sum_{l=0}^{\infty}\frac{P_l(\cos\theta)}{r^l}\underbrace{\left(B_{l0}\sqrt{\frac{2l+1}{4\pi}}\right)}_{\equiv B_l}$$



In the limit we expect these to have infinite height but no spatial extent — yet we must have the total mass of the particle

$$M = \int d^3r\, \rho(\vec{r}) .$$

The Dirac delta function, $\delta^3(\vec{r})$, is precisely this limiting function. It is not an ordinary function but is called a distribution or generalized function.

$\delta^3(\vec{r})$ has the properties

1) $\delta^3(\vec{r}) = 0$ for $\vec{r} \neq 0$

2) $\int_V d^3r\, \delta^3(\vec{r}) = \begin{cases} 1 & \text{if } V \text{ contains } \vec{r}=0 \\ 0 & \text{if } V \text{ does not contain } \vec{r}=0 . \end{cases}$

From this definition of δ we have that

$$\int d^3r' \, \delta^3(\vec{r}'-\vec{r}) \, f(\vec{r}')$$

$$= \int d^3(\vec{r}'-\vec{r}) \, \delta^3(\vec{r}'-\vec{r}) \, f(\vec{r}'-\vec{r}+\vec{r})$$

$$= \int d^3R \, \delta^3(\vec{R}) \, f(\vec{R}+\vec{r})$$

$$= \int d^3R \, \delta^3(\vec{R}) \left[f(\vec{r}) + \vec{R}\cdot\vec{\nabla} f(\vec{r}) + \cdots \right]$$

$$= f(\vec{r}) \underbrace{\int d^3R \, \delta^3(\vec{R})}_{=1} + \left(\underbrace{\int d^3R \, \delta(\vec{R}) \vec{R}}_{=0} \right) \cdot \vec{\nabla} f(\vec{r}) + \underbrace{\cdots}_{=0}$$

$$= f(\vec{r})$$

Thus $\delta^3(\vec{r}'-\vec{r})$ evaluates $f(\vec{r}')$ at $\vec{r}'=\vec{r}$: $f(\vec{r})$

$$\boxed{\int d^3r' \, \delta^3(\vec{r}'-\vec{r}) \, f(\vec{r}') = f(\vec{r})}\,.$$

So we can represent the point particle ~~mass~~ density by

$$\rho(\vec{r}) = M \, \delta^3(\vec{r}-\vec{r}_0)$$

for a mass M at position $\vec{r}_0$.

Hence the gravitational potential for a point mass M at position $\vec{r}_0$ is recovered from our density formula as:

$$\phi(\vec{r}) = -G\int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= -G\int d^3r' \frac{M\delta^3(\vec{r}'-\vec{r}_0)}{|\vec{r}-\vec{r}'|}$$

$$\phi(\vec{r}) = -\frac{GM}{|\vec{r}-\vec{r}_0|}$$ ✓ as we found earlier according to Newton's Law of gravitation.

Further note that

$$\nabla^2_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}'|} = -4\pi\delta^3(\vec{r}-\vec{r}')$$

Proof: for $\vec{r} \neq \vec{r}'$

$$\nabla^2_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}'|} = \vec{\nabla}_{\vec{r}} \cdot \vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}'|} = -\vec{\nabla}_{\vec{r}} \cdot \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$= -\vec{\nabla}_{\vec{r}}\left(\frac{1}{|\vec{r}-\vec{r}'|^3}\right) \cdot (\vec{r}-\vec{r}')$$

$$- \frac{1}{|\vec{r}-\vec{r}'|^3} \underbrace{\vec{\nabla}_{\vec{r}} \cdot (\vec{r}-\vec{r}')}_{=3}$$

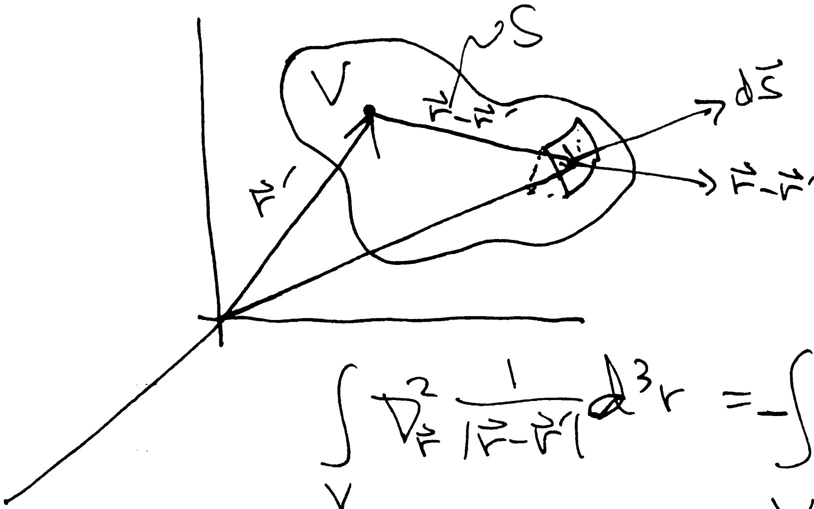
Recalling that $\vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}'|^3} = \frac{-3(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^5}$

$\Rightarrow$

$$\nabla^2_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}'|} = \frac{+3(\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^5} - \frac{3}{|\vec{r}-\vec{r}'|^3}$$

$$= 0. \quad (\text{for } \vec{r} \neq \vec{r}')$$

Note the above 2 terms are $+\infty, -\infty$ as $\vec{r} \to \vec{r}'$ and hence indeterminate. However we may use Gauss's divergence theorem to integrate this expression over a volume V with surface S



$$\int_V \nabla^2_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}'|} d^3r = -\int_V \vec{\nabla}_{\vec{r}} \cdot \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} d^3r$$

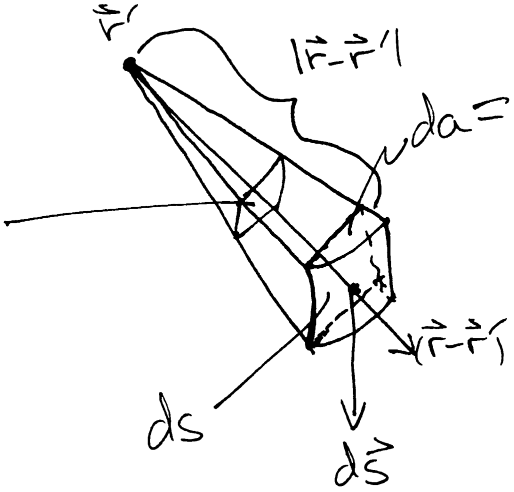
$$\stackrel{G.T.}{=} -\oint_S \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \cdot d\vec{S}$$

Now $\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|} \cdot d\vec{S}$ = element of surface area projected onto the plane $\perp$ to $(\vec{r}-\vec{r}')$

$\equiv da$

But $\frac{da}{|\vec{r}-\vec{r}\,'|^2} = d\Omega$ = element of solid angle subtended by $d\vec{S}$ from the $\vec{r}\,'$ position

i.e.



$\Rightarrow$ solid $\measuredangle = d\Omega = \frac{da}{|\vec{r}-\vec{r}\,'|^2}$

Now if $\vec{r}\,' \in V$ then this integral is 4π since da will cover the surface of the sphere of radius $|\vec{r}-\vec{r}\,'|$ as $d\vec{S}$ covers V's surface.

If $\vec{r}\,' \notin V$ then each $d\vec{S}$ will be paired with an oppositely directed $d\vec{S}$ with the same $d\Omega$. hence the integral $= 0$.

So for our case $\vec{r}\,' \in V$ we have

$$\boxed{\int_V \nabla^2_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}\,'|} d^3r = -4\pi}$$

To see this directly recall that $\nabla^2_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}\,'|} = 0$ except at $\vec{r} = \vec{r}\,'$. Hence the volume integral vanishes except for a small sphere about $\vec{r}\,'$. Let $\vec{R}$ be the radius vector of a small sphere about $\vec{r}\,'$: $\vec{r}-\vec{r}\,' = \vec{R}$

then $\quad d\vec{S} = R^2 d\Omega\, \hat{n} \quad$ and

$$\oint_S \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \cdot d\vec{S} = \oint_S \hat{R} \cdot \hat{n}\, d\Omega \quad \text{but } \hat{n} = \hat{R}$$

for the small sphere and $= \oint d\Omega = 4\pi$ as long as $\vec{r}\,' \in V$ independent of R (let $R \to 0$)

So we have 1) $\nabla^2_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}\,'|} = 0$ for $\vec{r} \neq \vec{r}\,'$

and 2) $\int_V d^3r\, \nabla^2_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}\,'|} = -4\pi$ for $\vec{r}\,' \in V$

$$\Rightarrow \quad \boxed{\begin{aligned} \nabla^2_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}\,'|} &= -4\pi\, \delta^3(\vec{r}-\vec{r}\,') \\ &= -\vec{\nabla}_{\vec{r}} \cdot \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \end{aligned}}$$

This is a useful result since the gravitational field for a mass density ρ (including the point mass) is

$$\vec{g}(\vec{r}) = -G \int_V \frac{\rho(\vec{r}\,')\,(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, d^3r'$$

Consider the divergence of $\vec{g}$

$$\vec{\nabla}\cdot\vec{g}(\vec{r}) = -G \int_V d^3r'\,\rho(\vec{r}\,')\,\underbrace{\vec{\nabla}_{\vec{r}}\cdot\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}}_{=4\pi\delta^3(\vec{r}-\vec{r}\,')}$$

$$= -G \int_V d^3r'\,\rho(\vec{r}\,')\,4\pi\,\delta^3(\vec{r}-\vec{r}\,')$$

$$\boxed{\vec{\nabla}\cdot\vec{g}(\vec{r}) = -4\pi G\,\rho(\vec{r})}$$ Along with

$$\boxed{\vec{\nabla}\times\vec{g}(\vec{r}) = 0}$$ These are the (Newtonian) gravitational field equations.

Further we can integrate the divergence field equation over a volume V bounded by surface S

$$\int_V d^3r\, \vec{\nabla}_{\vec{r}} \cdot \vec{g}(\vec{r}) = -4\pi G \int_V \rho(\vec{r})\, d^3r$$

$$= -4\pi G M$$ where M is the total mass contained in Volume V.

By Gauss's Theorem however

$$\int_V d^3r\, \vec{\nabla}_{\vec{r}} \cdot \vec{g}(\vec{r}) \overset{G.T.}{=} \oint_S \vec{g}(\vec{r}) \cdot d\vec{S}$$ the flux of $\vec{g}$ thru S

hence

$$\boxed{\oint_S \vec{g}(\vec{r}) \cdot d\vec{S} = -4\pi G M}$$

Note that $\vec{\nabla} \times \vec{g} = 0 \Rightarrow \vec{g} = -\vec{\nabla}\phi$

So that $\vec{\nabla} \cdot \vec{g} = -\vec{\nabla} \cdot \vec{\nabla}\phi = -\nabla^2\phi$

Thus the field eq.'s become Poisson's eq. for the gravitational potential

$$\boxed{\nabla^2 \phi(\vec{r}) = 4\pi G \rho(\vec{r})}$$

(Newton gave solution $\phi(\vec r) = -G\int d^3r' \frac{\rho(\vec r')}{|\vec r - \vec r'|}$, circle complete)

For points $\vec r$ outside the mass distribution $\rho(\vec r) = 0$ $\Rightarrow$ $\boxed{\nabla^2\phi = 0}$ Laplace's eq.

Finally, the work per unit mass, dw, done by an outside force in order to displace a body a distance $d\vec r$ in a gravitational field is

$$dw = -\vec g \cdot d\vec r = \vec\nabla\phi \cdot d\vec r$$
$$= \partial_i\phi\, dx_i = d\phi$$

, it is just the difference in gravitational potential between the points.

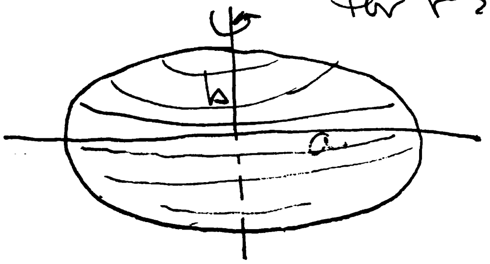
Hence the gravitational potential energy U is just

$$U = m\phi .$$

(If the ϕ and P.E. are 0 at $\vec r \to \infty$, then $U, \phi \le 0$ and increase when work is done <u>on</u> a body.)

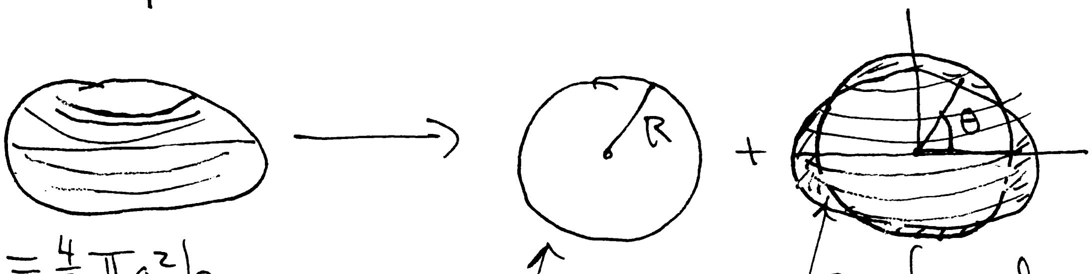
~~gravitational field~~

Ex The earth (sun) has the shape, approximately, of an oblate ellipsoid of revolution as shown below. Find the gravitational potential for $r \gg a$ to first order in η.



$b = a(1-\eta)$ $\quad$ ($\eta \approx 0.0034$ for earth)

1)



$V = \frac{4}{3}\pi a^2 b$

$M = \rho V$

sphere of same volume of ellipsoid.

surface density of $\pm$ mass $\sigma(\theta)$

$V = \frac{4}{3}\pi R^3 \Rightarrow R^3 = a^2 b = a^3(1-\eta)$
$= b^3 \frac{1}{(1-\eta)^2}$

$\Rightarrow$ to 1st order in η

$R \approx a(1 - \frac{1}{3}\eta)$

$R \approx b(1 + \frac{2}{3}\eta)$

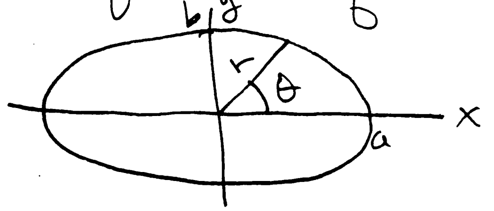
So $\phi(\vec{r}) = \phi_{sphere}(\vec{r}) + \phi_{shell}(\vec{r})$

$$\phi_{sphere}(\vec{r}) = -\frac{GM}{r}$$

$$\phi_{shell}(\vec{r}) = -G \int_{surface} \frac{\sigma(\theta')\,da'}{|\vec{r} - \vec{r}'|}$$

So first we must find $\sigma(\theta)$

Equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$x = r\cos\theta$

$y = r\sin\theta$

$$\Rightarrow \quad r^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) = 1 = r^2\,\frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2b^2}$$

$$\Rightarrow \quad r^2 = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta} = \frac{a^2b^2}{a^2 - (a^2-b^2)\cos^2\theta}$$

$$\boxed{r^2 = \frac{b^2}{1-\left(1-\frac{b^2}{a^2}\right)\cos^2\theta}} \qquad e^2 = \left(1-\frac{b^2}{a^2}\right)$$

if $b = a(1-\eta)$; $b^2 = a^2(1-\eta)^2 \Rightarrow$

$$r^2 = \frac{b^2}{1-(2\eta-\eta^2)\cos^2\theta}$$

if $\eta \ll 1$ $\quad r^2 \simeq \frac{b^2}{1-2\eta\cos^2\theta} \simeq b^2(1+2\eta\cos^2\theta)$

$\Rightarrow$

$$\boxed{r \simeq b\sqrt{1+2\eta\cos^2\theta} \simeq b(1+\eta\cos^2\theta)}$$

So the shell of mass will have thickness

$$\Delta t = r - R = b(1+\eta\cos^2\theta) - b(1+\tfrac{2}{3}\eta)$$
$$= b\left[\eta\cos^2\theta - \tfrac{2}{3}\eta\right]$$

$$\Delta t = -\tfrac{2}{3}\eta b\left[1 - \tfrac{3}{2}\cos^2\theta\right] = -\tfrac{2}{3}\eta b\left[1-\tfrac{3}{2}+\tfrac{3}{2}\sin^2\theta\right]$$

$$\boxed{\Delta t = \tfrac{1}{3}\eta b\left[1-3\sin^2\theta\right]}$$

if $\theta' = \frac{\pi}{2} - \theta$ = colatitude

$$\sigma(\theta') = \frac{1}{3}\eta b\rho\{1 - 3\cos^2\theta'\} = -\frac{2}{3}\eta b\rho P_2(\cos\theta')$$

Hence the surface mass density of the shell is

$$\boxed{\sigma(\theta) = \rho\Delta t = \frac{1}{3}\eta b\rho[1 - 3\sin^2\theta]}$$

Note: as expected σ is positive when $r > R$ and negative when $r < R$ and the total mass of the shell

$$M_{shell} = \int_{shell} \sigma(\theta)\, da$$

symmetry

(to first order in η we replace shell again with sphere since η is already present)

$$= \frac{1}{3}\eta b\rho R^2 \int_{\varphi=0}^{2\pi} d\varphi\, 2\int_{\theta=0}^{\frac{\pi}{2}} d\theta \cos\theta [1 - 3\sin^2\theta]$$

$$= \frac{4\pi}{3}\eta b\rho R^2 \int_0^{+1} d\xi [1 - 3\xi^2]$$

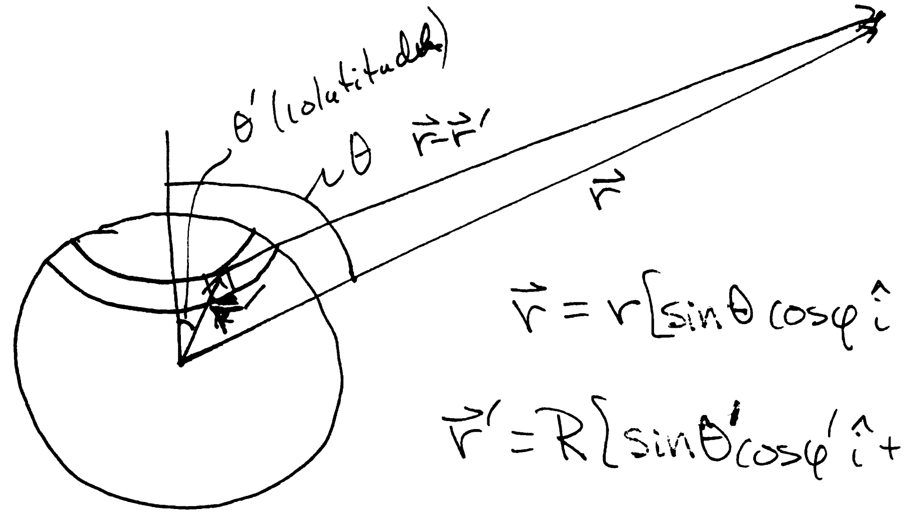
where $\xi = \sin\theta$, $d\xi = \cos\theta d\theta$

So

$$\boxed{M_{shell} = \frac{4\pi}{3}\eta b\rho R^2 \left[\, \xi\Big|_0^{+1} - \xi^3\Big|_0^{+1} \right] = \frac{4\pi}{3}\eta b\rho R^2 [1 - 1] = 0}$$

Now we need to find Φ_{shell} to first order in η again we can consider the shell a spherical shell since $\sigma \sim \eta$ already

So we first note that the mass distribution is azimuthally symmetric hence $\Phi_{shell}(\vec{r})$ is independent of φ, as we must obtain.



$$\vec{r} = r[\sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k}]$$

$$\vec{r}\,' = R[\sin\theta'\cos\varphi'\,\hat{i} + \sin\theta'\sin\varphi'\,\hat{j} + \cos\theta'\,\hat{k}]$$

Now recall for $r >> a$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}\,'}{r^3} + \frac{1}{2}\left[\frac{3(\vec{r}\cdot\vec{r}\,')^2}{r^5} - \frac{r'^2}{r^2}\right] + \cdots$$

So (use colatitude now $\theta' \to \frac{\pi}{2} - \theta'$)

$$\phi_{shell}(\vec{r}) = -G\int_{\varphi'=0}^{2\pi} d\varphi' \int_{\theta'=0}^{\pi} d\theta' R^2 \sin\theta'\, \sigma\!\left(\tfrac{\pi}{2}-\theta'\right)\left[\frac{1}{r} + \frac{\vec{r}\cdot\vec{r}\,'}{r^3} + \frac{1}{2}\left(\frac{3(\vec{r}\cdot\vec{r}\,')^2}{r^5} - \frac{r'^2}{r^3}\right) + \cdots\right]$$

$$= -\,GR^2\,\tfrac{1}{3}\eta b\rho \int_{\varphi'=0}^{2\pi} d\varphi' \int_{(\cos\theta')=-1}^{+1} d(\cos\theta')\,[1-3\cos^2\theta'] \times$$

$$\times\left[\frac{1}{r} + \frac{R}{r^2}(s\theta c\varphi s\theta' c\varphi' + s\theta s\varphi s\theta' s\varphi' + c\theta c\theta') \right.$$

$$\left. + \frac{1}{2}\frac{R^2}{r^3}\left(3(s\theta c\varphi s\theta' c\varphi' + s\theta s\varphi s\theta' s\varphi' + c\theta c\theta')^2 - 1\right) + \cdots\right]$$

Again the $M_{shell}=0$ so the first term $\left(\frac{1}{r}\right)$ vanishes (monopole); next we have

$$\int_0^{2\pi} d\varphi' \begin{cases}\cos\varphi' \\ \sin\varphi'\end{cases} = \begin{cases}\sin\varphi' \\ -\cos\varphi'\end{cases}\Bigg|_0^{2\pi} = 0$$

and $\displaystyle\int_{-1}^{+1} d\xi\, \xi\,(1-3\xi^2)$ ($\xi = \cos\theta'$ now)

$$= \frac{1}{2}\xi^2\Big|_{-1}^{+1} - \frac{3}{4}\xi^4\Big|_{-1}^{+1} = \frac{1}{2}(1-1) - \frac{3}{4}(1-1) = 0 - 0$$

$$= 0 .$$

So the $\frac{1}{r^2}$ dipole term also vanishes.

Finally

$$\Phi_{shell}(\vec{r}) = -\frac{1}{2} G R^2 \eta b \rho \frac{R^2}{r^3} \int_{\varphi'=0}^{2\pi} d\varphi' \int_{\xi=-1}^{+1} d\xi\,(1-3\xi^2) \times$$

$$\times \Big[\big[s\theta c\varphi\, \underbrace{s\theta'}_{} c\varphi' + s\theta s\varphi \underbrace{s\theta'}_{\sqrt{1-\xi^2}} s\varphi' + c\theta \xi \big]^2 - 1 \Big]$$

Since the last term has no φ' dependence all cross terms with it $=0$ while the cross term $2c\varphi' s\varphi' = \sin 2\varphi'$ also integrate to 0. <u>So all cross terms vanish</u>. As well the last term (-1) integrates to 0.

hence

$$\phi_{shell}(\vec{r}) = -\frac{1}{2} G \frac{R^4 \eta b \rho}{r^3} \int_{\varphi'=0}^{2\pi} d\varphi' \int_{\xi=-1}^{+1} d\xi \, (1-3\xi^2) \times$$

$$\times \left[s^2\theta c^2\varphi (1-\xi^2) c^2\varphi' + s^2\theta s^2\varphi (1-\xi^2) s^2\varphi' + c^2\theta \xi^2 \right]$$

But $\int_0^{2\pi} d\varphi' \begin{Bmatrix} \cos^2\varphi' \\ \sin^2\varphi' \end{Bmatrix} = \pi \quad ; \int_0^{2\pi} d\varphi' \, 1 = 2\pi$

$\Rightarrow$

$$\phi_{shell}(\vec{r}) = -\frac{\pi}{2} G \frac{R^4 \eta b \rho}{r^3} \int_{\xi=-1}^{+1} d\xi \left[(1-4\xi^2+3\xi^4) \left[s^2\theta c^2\varphi + s^2\theta s^2\varphi \right] + 2c^2\theta (\xi^2 - 3\xi^4) \right]$$

$$= -\frac{\pi}{2} G \frac{R^4 \eta b \rho}{r^3} \int_{\xi=-1}^{+1} d\xi \left[\sin^2\theta \, (1-4\xi^2+3\xi^4) + 2\cos^2\theta \, (\underbrace{\xi^2}_{} - \underbrace{3\xi^4}_{}) \right]$$

$$= -\frac{\pi}{2} G \frac{R^4 \eta b \rho}{r^3} \left[\sin^2\theta \left(\overbrace{\xi \Big|_{-1}^{+1}}^{2} - \overbrace{\frac{4}{3}\xi^3 \Big|_{-1}^{+1}}^{8/3} + \overbrace{\frac{3}{5}\xi^5 \Big|_{-1}^{+1}}^{6/5} \right) + 2\cos^2\theta \left(\underbrace{\frac{1}{3}\xi^3 \Big|_{-1}^{+1}}_{2/3} - \underbrace{\frac{3}{5}\xi^5 \Big|_{-1}^{+1}}_{6/5} \right) \right]$$

$$= -\frac{\pi}{2} G \frac{R^4 \eta b \rho}{r^3} \left[\sin^2\theta \left(\frac{30-40+18}{15} \right) + \cos^2\theta \left(\frac{20-36}{15} \right) \right]$$

Finally:

$$\phi_{shell}(\vec{r}) = -\frac{\pi}{2} G \frac{R^4 \eta b \rho}{r^3}\left[\frac{8}{15}\left(\sin^2\theta - 2\cos^2\theta\right)\right]$$

$$= -\frac{4\pi}{15} G \eta \frac{R^3 a^2 \rho}{r^3}\left[1 - 3\cos^2\theta\right]$$

where $\eta R = \eta a = \eta b$ to this 1st order in η.

$$\boxed{\phi_{shell}(\vec{r}) = -\frac{1}{5} G M \eta \frac{a^2}{r^3}\left[1 - 3\cos^2\theta\right]}$$

So for $r \gg a$

$$\boxed{\phi(\vec{r}) = -\frac{GM}{r} - \frac{1}{5} G M \eta \frac{a^2}{r^3}\left[1 - 3\cos^2\theta\right]}$$

As expected for the case of azimuthal symmetry and $\nabla^2\phi = 0$

$$\Rightarrow \quad \phi = \phi(r,\theta) = \sum_{n=0}^{\infty}\left(a_n r^n + \frac{b_n}{r^{n+1}}\right) P_n(\cos\theta)$$

and we have found

$$\phi(\vec{r}) = \frac{-GM}{r} P_0(\cos\theta) + \frac{2}{5}\frac{GM\eta a^2}{r^3} P_2(\cos\theta)$$
