(Chapter 6) Calculus of Variations

Consider the problem of a bead of mass \( m \) constrained to slide along a wire curve in a constant gravitational field \( g \).

\[(0,0) = (x_1, y_1)\]

\[\bigg(\frac{v}{mg}, \frac{(x_2, y_2)}{g}\bigg)\]

Suppose we ask the question: What shape should the curved wire be so that the particle initially at rest at the origin arrives at \((x_2, y_2)\) in
The least amount of time (this is known as the brachistochrone problem).

Thus we desire to find the equation of the curve: \( h(x, y) = 0 \) or parametrically

\[ x = x(\alpha) \quad \text{where } \alpha \text{ is some parameter} \]
\[ y = y(\alpha) \quad \text{with } 0 \leq \alpha \leq \alpha_0 \text{ say} \]

or for that matter \( y = y(x) \) describes the curve.

In order to proceed we must relate the time to the position coordinates of the particle. This can be done by using the conservation of energy; that is the total energy of the particle is a constant

\[ E = T + U. \]

We choose the zero of potential energy to be at \( x = 0 \): \( U(x = 0) = 0 \), so that initially with \( \dot{x} = 0 \) we have

\[ E = U = 0 \text{ at the origin}. \]
In general \( U = -mgx \) and \( T = \frac{1}{2}mv^2 \)

Hence \( E = 0 = \frac{1}{2}mv^2 - mgx \)

\[ \Rightarrow v = \sqrt{2g x} \]

Now the time required to go from the origin to \((x_2, y_2)\) is just the integral of the distance along the curve (the "arc length") divided by the velocity.

More specifically

\[ t = \frac{ds}{v} = \frac{1}{\sqrt{2g x}} \]

where

\[ ds = \sqrt{dx^2 + dy^2} \]

\[ ds = \text{distance along curve travelled in time } dt \]

\[ = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx \]

where \( y = y(x) \) describes the curve.

So the time \( dt \) to go \( ds \) is just

\[ dt = \frac{ds}{v} = \frac{ds(x, y)}{\sqrt{2g x}} \]
Hence the total time elapsed is
\[ t = \int_{(0,0)}^{(x_2,y_2)} \frac{ds(x,y)}{N(x,y)} = \int_{x=0}^{x_2} \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{2g x}} \, dx \]

The question we asked is what curve, \( y = y(x) \), yields a minimum for the transit time \( t \).

This is a more complicated question than just finding the minimum of an ordinary function, for even a function you only need its values at a point; here, the time \( t \) depends on the whole functional form of \( y(x) \), i.e., many independent functions \( y(x) \) each with a different value at all points in \([0,x_2]\).

Since the value of the integral depends upon the form of \( y \), i.e., what it is at each \( x \), \( t = t[y] \) is called a functional if written
\[ t = t[y] \] (map set of sets into a set, function of \( n \) variables).
Hence we need to find a condition which will be necessary for extremizing a functional. In order to set up this mathematical requirement in more general terms, let \( J = J[y] \) be a functional of \( y \) where \( y = y(x) \) is a function of \( x \), given by

\[
J[y] = \int_{x_1}^{x_2} f(y(x), y'(x), x) \, dx
\]

with \( x_1, x_2 \) fixed and where \( f \) is a given function of \( y, y' = \frac{dy}{dx} \) and \( x \).

Then \( J \) is extremized by \( y \) if given any neighboring function \( y = y(x, \xi) \) no matter how close \( y(x, \xi) \) to \( y(x) \), \( J \) of this new function is larger than and \( J[y] \).

\[\text{i.e.}\]
\[y = y(x)\] shortest time path
\[
\text{neighboring path } y = y(x, \xi) \]
\[
\equiv y(x) + \alpha \eta(x) \]

with \( \eta(0) = 0 = \eta(x_2) \)

(\( x_2, y_{\xi} \))

Fixed endpoints.
\[
\text{then } \exists [y] < \exists [y + \Delta y] \text{ for } y(x) \text{ to be the extremum of } \exists.
\]

Thus we can parameterize our neighboring paths by parameter \( \alpha \) (is positive and negative) by letting

\[
y(\alpha, x) = y(x) + \alpha \frac{dy}{dx} \tag{185}
\]

where \( y(0, x) = y(x) \) and \( y(1, x) \) is an arbitrary \( C^1 \) function which vanishes at the end points of our integral (endpoints are said to be fixed).

So \( y(x_1) = y(x_2) = 0 \Rightarrow (y(\alpha, x) = y(x) + \alpha \frac{dy}{dx}) \)

at \( x = x_1, y(x_1) = 0 \Rightarrow y(\alpha, x_1) = y(x_1) \)

and \( x = x_2, y(x_2) = 0 \Rightarrow y(\alpha, x_2) = y(x_2) \)

(as we have drawn for our bead path above)

Now let us ask what \( J[y + \Delta y] > J[y] \)

implies
So we have

\[ J[y(\alpha, x)] > J[y(0, x)]. \]

Let's Taylor expand our function about \( \alpha = 0 \), (i.e. \( J[y(\alpha, x)] = J(\alpha) \))

Recall

\[ J[y(\alpha, x)] = J[y + \alpha \eta] \]

\[ = \int_{x_1}^{x_2} f(y + \alpha \eta, y' + \alpha \eta'; x) \, dx \]

Taylor expanded about \( \alpha = 0 \)

\[ = \int_{x_1}^{x_2} \left[ f(y, y'; x) + \alpha \eta y' (\frac{\partial f}{\partial y})(y, y'; x) \right. \]

\[ + \alpha \eta' y (\frac{\partial f}{\partial y'})(y, y'; x) \]

\[ \left. + O(\alpha^2) \right] \, dx \]

\[ = J[y] + \alpha \int_{x_1}^{x_2} \left[ \frac{\partial f}{\partial y} \eta + \frac{\partial f}{\partial y'} \eta' \right] \, dx + O(\alpha^2) \]
The last term can be simplified by integrating the \( \frac{dy}{dx} \) by parts

\[
\int_{x_1}^{x_2} \frac{df}{dy'} \frac{dy}{dx} \, dx = \int_{x_1}^{x_2} \frac{d}{dx} \left( \frac{df}{dy'} \right) \, dx
- \int_{x_1}^{x_2} \frac{df}{dy'} \frac{d}{dx} \left( \frac{df}{dy'} \right) \, dx
\]

\[
= \frac{df}{dy'} \eta(x_1) \bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{df}{dy'} \frac{d}{dx} \left( \frac{df}{dy'} \right) \, dx
\]

But \( \eta(x_1) = \eta(x_2) = 0 \); the endpoints are fixed

So the first term vanishes. Thus

\[
J[y + \alpha \eta] = J[y] + \alpha \int_{x_1}^{x_2} \left[ \frac{df}{dy'} - \frac{d}{dx} \frac{df}{dy'} \right] \eta(x) \, dx + o(\alpha^2)
\]

For small \( \alpha \) the variation in \( J \) (differential change in \( J \))

\[
\delta J[y] = J[y + \alpha \eta] - J[y]
\]

It changes sign when \( \alpha \) changes sign, so the only way for

\[
J[y] < J[y + \alpha \eta]
\]
for all $\alpha$ is for $8J = 0$; that is

$$\int_{x_1}^{x_2} \left[ \frac{df}{dy} - \frac{d}{dx} \frac{df}{dy} \right] y(x) \, dx = 0$$

(similarly if $J[y] > J[y + \alpha y]$ for all $\alpha$)

This the necessary condition for a maximum or a minimum, i.e. an extremum, of $J[y]$ is that $8J = 0$ (if stationary)

$$8J = \int_{x_1}^{x_2} \left[ \frac{df}{dy} - \frac{d}{dx} \frac{df}{dy} \right] y(x) \, dx = 0$$

But $y(x)$ was arbitrary hence the integrand must vanish $\Rightarrow$

$$\frac{df}{dy} - \frac{d}{dx} \frac{df}{dy} = 0$$

This is called the Euler-Lagrange equation for $f$ and

$$8y = \frac{d}{dy} - \frac{d}{dx} \frac{d}{dy}$$

is often
called the Euler-(Lagrange) derivative.

To summarize: The functional

\[ J[y] = \int_{x_1}^{x_2} f(y, y', y''; x) \, dx, \]

with \( f \) given, is extremized by \( y \), then \( y \) obeys

\[ \delta_y f = 0 = \frac{df}{dy} - \frac{d}{dx} \frac{df}{dy'}; \]

That is, \( y \) obeys the Euler-Lagrange equation.

Remarks: 1) This is a necessary condition for a max. or min. To determine which we need to find the \( O(dx^2) \) terms for sufficiency.

2) If \( f \) is a function of higher than first derivatives \( y''', y'''', \) etc., we proceed analogously to find

\[ 0 = \frac{df}{dy} - \frac{d}{dx} \frac{df}{dy'} + \frac{d^2}{dx^2} \frac{df}{dy''} + \cdots. \]
Let's apply these results to our bead on a wire problem: We want to minimize the time of transit from the origin to \((x_2, y_2)\),

\[
\mathcal{T}[y] = \int_{x=0}^{x_2} \frac{\sqrt{1+y'(x)^2}}{\sqrt{2g} \sqrt{x}} \, dx
\]

where \(y'(x) = \frac{dy}{dx}\). The necessary condition for an extremum (one can check it is indeed a minimum) is that

\[
\delta\mathcal{T}[y] = 0 \Rightarrow \text{Euler-Lagrange equation for } y(x)
\]

\[
\frac{2}{dy} \left[ \frac{\sqrt{1+y'(x)^2}}{\sqrt{2g} \sqrt{x}} \right] - \frac{d}{dx} \frac{d}{dy'} \left[ \frac{\sqrt{1+y'(x)^2}}{\sqrt{2g} \sqrt{x}} \right] = 0
\]

\[
= 0 \quad \text{since indep. of } y
\]

\[
\Rightarrow \quad \frac{d}{dx} \left( \frac{1}{\sqrt{2g} \sqrt{x}} \frac{1}{\sqrt{1+y'(x)^2}} \right) = 0
\]

\[
\Rightarrow \quad \frac{1}{\sqrt{2g} \sqrt{x}} \frac{1}{\sqrt{1+y'(x)^2}} y' = \text{constant in } x
\]

squaring

\[
\Rightarrow \quad \frac{y'(x)^2}{x(1+y'(x)^2)} = \text{constant} = \frac{1}{2a}
\]
Solving for \( y' \)

\[
y' = \frac{x}{2a} \left( 1 + y'^2 \right)
\]

\[
y' \left( 1 - \frac{x}{2a} \right) = \frac{x}{2a}
\]

\[
\frac{dy}{dx} = \frac{\sqrt{x}}{(2a - x)^{1/2}} = \frac{x}{(2ax - x^2)^{1/2}}
\]

or

\[
y = \int \frac{x \, dx}{\sqrt{2ax - x^2}}
\]

where the constant of integration will be determined later.

Let

\[
x = a (1 - \cos \theta)
\]

\[
dx = a \sin \theta \, d\theta
\]

\[
y = \int \alpha (1 - \cos \theta) \, d\theta
\]

\[
y = a [\theta - \sin \theta] + b
\]

with \( b = \) constant of integration
Now at $\theta = 0 \Rightarrow x = 0$ and so must $y = 0 \Rightarrow b = 0$.

Hence we find the parametric equation for the shortest time curve:

$$x = a(1 - \cos \theta)$$
$$y = a(\theta - \sin \theta)$$

This is called a cycloid curve. It passes through the origin and $a$ is determined so that $y = y_2$ at $x = x_2$.

i.e.

$$\cos \theta = \frac{a-x}{a}$$
$$a \sqrt{a^2 - (a-x)^2 + 2ax}$$

$$a-x = \sqrt{2ax - x^2}$$

$$\theta = \cos^{-1} \frac{a-x}{a}$$

$$\theta = \sin^{-1} \frac{\sqrt{2ax - x^2}}{a}$$

So

$$y = a \left[ \sin^{-1} \frac{\sqrt{2ax - x^2}}{a} - \frac{\sqrt{2ax - x^2}}{a} \right]$$

hence we have $y = y(x)$ and at $x = x_2; y = y_2$

$$y_2 = a \left[ \sin^{-1} \frac{\sqrt{2ax_2 - x^2}}{a} - \frac{\sqrt{2ax_2 - x^2}}{a} \right]$$

This is an equation for $a = a(x_2, y_2)$. 
So recall for an ordinary function of a single variable \( f(x) \) that

The minimizer of the function is \( x_0 \) (denote \( x_0 \))

\[ f(x_0) < f(x_0 + \alpha dx) \]

\[ f(x_0 + \alpha dx) = f(x_0) + \alpha dx \frac{df}{dx} \bigg|_{x=x_0} \]

So,

\[ df = f(x_0 + \alpha dx) - f(x_0) = \alpha dx \frac{df}{dx} \bigg|_{x=x_0} \]

(ie \( dx = \pm \))

Since \( \alpha \) is \( \pm \), it is necessary for \( df = 0 \) (but not sufficient) for a minimum — it extremizes \( f \) \( \Rightarrow \) \( \frac{df}{dx} \bigg|_{x=x_0} = 0 \) \( \Rightarrow \) equation for \( x_0 \)

Analogously

\[ J[y] = \int x^2 f(y, \frac{dy}{dx}; x) \, dx \text{ is a functional of } y \]

\( y \) \( \Delta \) minimizes \( J \) if \( J[y] < J[y + \alpha \frac{dy}{dx}] \)

So Taylor expand about \( x = 0 \)

\[ J[y + \alpha \frac{dy}{dx}] = J[y] + \alpha \frac{df}{dy} \bigg|_{x=0} \frac{dy}{dx} + \frac{1}{2} \alpha^2 \frac{d^2 f}{dy^2} \bigg|_{x=0} \frac{dy}{dx}^2 \]

\[ = J[y] + \alpha \frac{df}{dy} \bigg|_{x=0} \frac{dy}{dx} + \frac{1}{2} \alpha^2 \frac{d^2 f}{dy^2} \bigg|_{x=0} \frac{dy}{dx}^2 \]
Since $y$ is an odd function, $\int_{-a}^{a} y \, dx = 0$.

So $\int_{-a}^{a} \left( \frac{dy}{dx} \right)^2 \, dx = 0.$

Therefore, for $y$ to be a minimum, $\frac{dy}{dx} = 0$.

So $f(x) = \int_{-g(x)}^{g(x)} \, dx = \int_{-g(x)}^{g(x)} \, dy = 0.$

\[
\int_{g(x)}^{g(x)} \, dy = 0
\]

\[
\int_{g(x)}^{g(x)} \, dx = 0
\]

For $x = 0$, we have $f(0) = \int_{-g(0)}^{g(0)} \, dy = 0.$
The brachistochrone problem is particularly simple. In general we will have integrals which depend on more than one function of a real variable to extremize; that is we will consider functionals of several variables.

\[ J[y_1, y_2, \ldots, y_n] = \int_1^x f(y_1, \ldots, y_n, y_1', \ldots, y_n'; x) \, dx \]

where \( y_i = y_i(x) \). We desire the form of the function \( y_i = y_i(x) \) which extremize \( J \).

Towards this end consider paths in this multi-dimensional space

\[ y_i(x, \alpha) = y_i(x, \alpha) + \alpha \eta_i(x) \quad \text{with} \]

\( \eta_i \) independent \( C^1 \) functions \( \exists \eta_i'(x) = \eta_i(x) = 0 \).

Then as before

\[ \delta J = \alpha \sum_{i=1}^n \int_1^x \left[ \frac{d}{dy_i} \left( \frac{d}{dx} \delta y_i \right) - \frac{d}{dx} \frac{df}{dy_i} \right] \eta_i(x) \, dx \]

\[ \text{in out} \]

If \( J \) is extremized by \( \{ y_i \} \), then \( \delta J = 0 \)

\[ \implies \frac{df}{dy_i} - \frac{d}{dx} \frac{df}{dy_i} = 0 \quad \text{for each} \quad i = 1, \ldots, n \]
Since the \( y_i \) are independent variations of the functions.

Even this "n-dimensional" functional problem is simple. In many cases we desire the shortest path between two points when the path is additionally constrained. For example a particle moving on the surface of a sphere, then \( x^2 + y^2 + z^2 = R^2 \), \( x, y, z \) are not independent variables.

Let's consider the case of ordinary functions of 2-variables \( F = F(x, y) \).

If \( F(x, y) \) is stationary at \((x_0, y_0)\) then
\[
\langle dF = 0 \rangle = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy \text{ at } (x_0, y_0)
\]

Since \( dx \) & \( dy \) are indep.

\( F \) is stationary \((dF=0)\) iff \( \frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y} \) at \((x_0, y_0)\)
Now suppose that \( x \) and \( y \) are not independent, but there is a constraint among \((x, y)\)

\[ G(x, y) = 0. \]

\[ \implies \frac{dG}{dx} = 0 = \frac{\partial G}{\partial x} \frac{dx}{dx} + \frac{\partial G}{\partial y} \frac{dy}{dy} \]

Suppose \( \frac{\partial G}{\partial y} \neq 0 \) then

\[ dy = -\frac{\partial G}{\partial x} \frac{dx}{\partial y} \]

So

\[ dF = \left[ \frac{\partial F}{\partial x} \frac{\partial G}{\partial x} - \frac{\partial G}{\partial x} \frac{\partial F}{\partial y} \right] \frac{dx}{\partial y} \]

For \( F \) to be stationary a nec. condition is since \( dx \) is independent variable

\[ dF = 0 \implies \]

\[ \frac{\partial F}{\partial x} \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \frac{\partial G}{\partial y} = 0 \]

This is true if

\[ \frac{\partial F}{\partial x} = -\lambda \frac{\partial G}{\partial x} \]

and

\[ \frac{\partial F}{\partial y} = -\lambda \frac{\partial G}{\partial y} \]

\( \lambda \) is a number called it \( \lambda \).
Alternative said:

1) \[ \frac{dF}{db} = 0 \Rightarrow \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} \left( \frac{dx}{dy} \right) = 0 \]

\[ s_h \begin{vmatrix} dx & 0 \\ dy & 0 \end{vmatrix} = 0 \Rightarrow \det \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} = 0 \]

\[ = G \Rightarrow F_x = -\lambda G_x, \quad F_y = -\lambda G_y \quad \text{1st row is a multiple of 2nd row.} \]

2) Geometrically, \[ \frac{dF}{db} = 0 \Rightarrow \frac{\nabla F \cdot d\vec{r}}{d\vec{r}} = 0 \]

\[ \Rightarrow \nabla F \parallel d\vec{r} \quad \Rightarrow \quad \nabla F || \nabla G \]

\[ = G \Rightarrow F_x = -\lambda G_x, \quad F_y = -\lambda G_y \]

So we have an equation for \( \lambda \).

\[ \lambda = \frac{F_x}{G_x} \]

So we have our necessary condition

\[ F_x = -\lambda G_x \]

\[ F_y = -\lambda G_y \quad \text{where} \quad \lambda = -\frac{F_y}{G_y} \quad \text{(Unique)} \]

This is the combination of conditions.
If \( \frac{\partial F}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \) and \( \frac{\partial F}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \) at \( (x_0, y_0) \), then with a constraint at \( (x_0, y_0) \), \( G(x_0, y_0) = 0 \) (1).

So, \( dx \) and \( dy \) are consistent with the constraint.

So then

\[
\begin{align*}
dx \left[ \frac{\partial F}{\partial x} + \lambda \frac{\partial g}{\partial x} \right] &= 0 \\
+ dy \left[ \frac{\partial F}{\partial y} + \lambda \frac{\partial g}{\partial y} \right] &= 0
\end{align*}
\]

\( \Rightarrow \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0 \Rightarrow dF = 0 \) at \( (x_0, y_0) \).

This is sufficient condition for the stationarity of \( F \) at \( (x_0, y_0) \) subject to constraint \( G = 0 \).

So we have recast the problem to an equivalent problem.

\( F \) is stationary at \( (x_0, y_0) \) with \( (dx, dy) \) subject to the constraint that \( G(x_0, y_0) = 0 \).

If \( F + \lambda G \) is stationary at \( (x_0, y_0) \) for some (unique) \( \lambda \), for arbitrary \( (dx, dy) \) and \( dx \).
That is,
\[ d(F + \lambda G) = 0 \left( \frac{\partial H}{\partial x} \frac{dx}{d\lambda} + \frac{\partial H}{\partial y} \frac{dy}{d\lambda} + \frac{\partial H}{\partial \lambda} \frac{d\lambda}{d\lambda} \right) \]

\[ = \frac{\partial F}{\partial x} + \lambda \frac{\partial G}{\partial x} = 0 \]
\[ \frac{\partial F}{\partial y} + \lambda \frac{\partial G}{\partial y} = 0 \]

AND
\[ \frac{d}{dx} (F + \lambda G) = 0 \implies G = 0 \]
So it is like 3 coord. \((x, y, \lambda)\) with no constraint but a new function
\[ H = F + \lambda G \] to extremize
in terms of \((x, y, \lambda)\).

Now consider the example

1) \(F(x, y) = xy\) and \(G(x, y) = x^2 + y^2 - 1 = 0\)

a) \(y = \pm \sqrt{1 - x^2} \implies \frac{dF}{dx} = \pm (1 - 2x^2) \quad \frac{dy}{dx} = \pm 2x \frac{dF}{dx} = \mp (1 - 2x^2) \quad \frac{dy}{dx} = \pm 2x \sqrt{1 - x^2} \)

So \(dF = 0\) at \(x = \pm \frac{\sqrt{2}}{2}, y = \pm \frac{\sqrt{2}}{2}\) \(4\) points \(= 0 \implies x = \pm \frac{1}{\sqrt{2}}\)
\((\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \implies F = \frac{\sqrt{2}}{2} \quad (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) \implies F = -\frac{\sqrt{2}}{2} \quad \Rightarrow y = \pm \frac{1}{\sqrt{2}}\)
1. (b) Equivalent method: Lagrange Multiplier

\[ H = F + \lambda G \]

1) \[ \frac{\partial H}{\partial x} = y + 2\lambda x = 0 \]
2) \[ \frac{\partial H}{\partial y} = x + 2\lambda y = 0 \Rightarrow \lambda = -\frac{x}{2y} \quad (y \neq 0) \]
3) \[ \frac{\partial H}{\partial \lambda} = G = x^2 + y^2 - 1 = 0 \]

2) \[ y - 2 \frac{x}{2y} x = 0 \Rightarrow y^2 - x^2 = 0 \Rightarrow x^2 = y^2 \]
3) \[ x^2 = 1 - x^2 \Rightarrow 2x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{2}} \]

\[ y = \pm \frac{1}{\sqrt{2}} \quad \text{and} \quad \lambda = \pm \frac{1}{\sqrt{2}} \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( \lambda )</th>
<th>( E )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( -\frac{1}{\sqrt{2}} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( -\frac{1}{\sqrt{2}} )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( -\frac{1}{2} )</td>
</tr>
<tr>
<td>( -\frac{1}{\sqrt{2}} )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>( -\frac{1}{\sqrt{2}} )</td>
<td>( -\frac{1}{\sqrt{2}} )</td>
<td>( -\frac{1}{\sqrt{2}} )</td>
<td>( -\frac{1}{2} )</td>
</tr>
</tbody>
</table>

The Lagrange multiplier technique works in more general case.
2) Suppose $F(x,y,z) = xyz$ and $G = x^2 + y^2 + z^2 - 2 = 0$

Then $H = F + \lambda G$

1) $\frac{\partial H}{\partial x} = yz + 2\lambda x = 0 \Rightarrow xyz + 2\lambda x = 0$

2) $\frac{\partial H}{\partial y} = xz + 2\lambda y = 0 \Rightarrow xyz + 2\lambda y = 0$

3) $\frac{\partial H}{\partial z} = xy + 2\lambda z = 0 \Rightarrow xyz + 2\lambda z = 0$

4) $\frac{\partial H}{\partial \lambda} = G = x^2 + y^2 + z^2 - 2 = 0$

Add 1, 2, 3 \Rightarrow $3xyz + 2\lambda (x^2 + y^2 + z^2) = 0$

4) $1 = \lambda = -\frac{1}{2}(3xyz)$

1) $xyz(x^2 - 3x^2) = 0$

2) $xyz(1 - 3yz) = 0$

4) $x^2 + y^2 + z^2 = 1$

(3) $xyz(x^2 - 3z^2) = 0 = xyz[-2 + 3x^2 + 3y^2]$

This is just sum of 1 & 2 nothing new
Cases:
\[ x = \pm 1, \quad y = 0, \quad z = 0 \]
\[ x = 0, \quad y = \pm 1, \quad z = 0 \] part of circle below
\[ x = 0, \quad y = 0, \quad z = \pm 1 \]
\[ x = \pm \frac{1}{\sqrt{3}}, \quad y = \pm \frac{1}{\sqrt{3}}, \quad z = \pm \frac{1}{\sqrt{3}} \]

Circles
\[
\begin{align*}
2 = 0 & \Rightarrow x^2 + y^2 = 1 \quad \text{circle} \\
2 = 0 & \Rightarrow x^2 + z^2 = 1 \\
y = 0 & \Rightarrow x + z = 1 \end{align*}
\]
\[ x = \pm \frac{1}{\sqrt{3}}, \quad y = 0, \quad z = \pm \frac{2}{\sqrt{3}} \]
\[ \text{etc.} \]
A simpler warm up constraint problem involves an overall constraint.

**Ex.** A heavy chain of fixed length $L$ is hung between the points $(0,0)$ and $(a,b)$ as shown.

It has uniform mass density $\rho$ and total mass $M$ and is subject to a constant gravitational force in the $(-y)$-direction.

**What will be its equilibrium shape?**
(The curve is called a catenary)

1) First we have the constraint that the length of the chain is $L$. If $y = y(x)$, then the arc length along the curve is

$$ds = \sqrt{1 + (\frac{dy}{dx})^2} \, dx$$

So,

$$L = \int_0^a \sqrt{1 + (\frac{dy}{dx})^2} \, dx \quad (= \int_0^a g(y,y';x) \, dx)$$