VI.) Hamilton's Principle: Lagrangian & Hamiltonian Dynamics

A.) Lagrange's Equations: Unconstrained System with N-particles of mass m_α, $\alpha = 1, 2, \ldots, N$ and position vectors $\vec{r}_\alpha$ wrt some inertial frame. Using Cartesian rectangular coordinates

$$\vec{r}_\alpha = x_\alpha \hat{i} + y_\alpha \hat{j} + z_\alpha \hat{k}$$
$$= x_{\alpha 1} \hat{e}_1 + x_{\alpha 2} \hat{e}_2 + x_{\alpha 3} \hat{e}_3$$

N-2:

$$m_\alpha \ddot{\vec{r}}_\alpha = \vec{F}_\alpha \quad ; \ \alpha = 1, 2, \ldots, N.$$

$$\text{i.e.} \quad m_\alpha \ddot{x}_{\alpha i} = F_{\alpha i} .$$

This can be re-cast as

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_{\alpha i}} T\right) = m_\alpha \ddot{x}_{\alpha i}$$
$$= F_{\alpha i}$$

with the total KE of the system

$$T = \sum_{\beta=1}^{N} \frac{1}{2} m_\beta \vec{v}_\beta^{\,2}$$
$$= \sum_{\beta=1}^{N} \sum_{j=1}^{3} \frac{1}{2} m_\beta (\dot{x}_{\beta j})^2 .$$

VI. A) Since $T = T(\dot{x}_{\alpha i})$ and $\vec{F}_\alpha$ can be written as

$$F_{\alpha i} = -\frac{\partial}{\partial x_{\alpha i}} U(x_{\beta j}, t) + \hat{F}_{\alpha i}$$

we find N-2 becomes Lagrange's Equations
(Euler-Lagrange Equations of Motion)

$$\frac{\partial L}{\partial x_{\alpha i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha i}} = -\hat{F}_{\alpha i} \quad ; \alpha = 1 \ldots, N \quad i = 1, 2, 3.$$

with Lagrangian $L \equiv T - U$
$= L(x_{\alpha i}, \dot{x}_{\alpha i}; t)$.

(In the case of Lorentz Force Law $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$
$F_{\alpha i}$ can be expressed as

$$F_{\alpha i} = -\frac{\partial}{\partial x_{\alpha i}} U - \left(\frac{\partial}{\partial x_{\alpha i}} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_{\alpha i}}\right) M + \hat{F}_{\alpha i}$$

then $L = T - U - M$ & N:2 becomes

$$\frac{\partial L}{\partial x_{\alpha i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha i}} = -\hat{F}_{\alpha i} \, .)$$

VI.B) Hamilton's Principle:

If all the forces are derivable from a potential so that $\tilde{F}_{\alpha i} = 0$, then

$$\frac{\partial L}{\partial x_{\alpha i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha i}} = 0,$$

The Euler-Lagrange Equations. Hence, Hamilton's Principle States

Of all possible paths along which a dynamical system may move from one point to another within a specified time interval, the actual path followed is that which minimizes the time integral of the difference between the KE and P.E. That is

$$\delta \int_{t_1}^{t_2} L(x, \dot{x}; t)\, dt = 0$$

where $L = T - U$, the Lagrangian and

$$\int_{t_1}^{t_2} L\, dt \equiv I(t_1, t_2) \text{ is the action.}$$

(Calculus of Variations: $\delta I = 0 \Rightarrow$ Euler-Lagrange Eq. $\frac{\partial L}{\partial x_{\alpha i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha i}} = 0$)

VI.C) Transform to Generalized Coordinates:

q^A, $A = 1, 2, \ldots, 3N$ and generalized velocities $\dot{q}^A$.

$$x_{\alpha i} = x_{\alpha i}(q^1, q^2, \ldots, q^{3N}; t)$$

&

$$q^A = q^A(x_{11}, x_{12}, \ldots, x_{N3}; t)$$

$\left(\left| \frac{\partial x_{\alpha i}}{\partial q^A} \right| \neq 0 \right.$ except at isolated points$\left. \right)$

using $\frac{\partial x_{\alpha i}}{\partial \dot{q}^A} = 0$

&

$$\frac{d}{dt} x_{\alpha i} = \frac{\partial x_{\alpha i}}{\partial q^A} \frac{dq^A}{dt} + \frac{\partial x_{\alpha i}}{\partial t}$$

$$\Rightarrow \quad \frac{\partial \dot{x}_{\alpha i}}{\partial \dot{q}^A} = \frac{\partial x_{\alpha i}}{\partial q^A}$$

We find that

$$\frac{\partial L}{\partial q^A} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^A} = \sum_{\alpha i} \frac{\partial x_{\alpha i}}{\partial q^A} \left(\frac{\partial L}{\partial x_{\alpha i}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_{\alpha i}} \right)$$

$$= -\sum_{\alpha i} \frac{\partial x_{\alpha i}}{\partial q^A} \hat{F}_{\alpha i}$$

Define generalized forces

$$Q_A \equiv \sum_{\alpha i} \frac{\partial x_{\alpha i}}{\partial q^A} F_{\alpha i}$$

$$\hat{Q}_A \equiv \sum_{\alpha i} \frac{\partial x_{\alpha i}}{\partial q^A} \hat{F}_{\alpha i}$$

VI.C.) If $F_{\alpha i}$ is derivable from a potential, then

$$Q_A = -\sum_{\alpha i} \frac{\partial x_{\alpha i}}{\partial q^A} \frac{\partial U(x,t)}{\partial x_{\alpha i}} = -\frac{\partial U(q,t)}{\partial q^A}.$$

Interpret Q_A by use of Principle of Virtual Work
Consider virtual (instantaneous) displacements $\delta x_{\alpha i}$

$$\delta W = \sum_{\alpha} \vec{F}_\alpha \cdot \delta \vec{r}_\alpha = \sum_{\alpha i} F_{\alpha i} \delta x_{\alpha i}$$

$$= \sum_{\alpha i} F_{\alpha i} \sum_A \frac{\partial x_{\alpha i}}{\partial q^A} \delta q^A$$

$$= \sum_A \left(\sum_{\alpha i} \frac{\partial x_{\alpha i}}{\partial q^A} F_{\alpha i} \right) \delta q^A$$

$$= \sum_A Q_A \delta q^A \quad \left(= -\delta U \text{ if } Q_A = -\frac{\partial U}{\partial q^A} \right).$$

So Lagrange's Eq. becomes

$$\frac{\partial L}{\partial q^A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A} = -\hat{Q}_A$$

and if $\hat{Q}_A = 0 \Rightarrow$ Euler-Lagrange Eq. $\frac{\partial L}{\partial q^A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A} = 0$
Which leads to Hamilton's Principle in any coordinate system

$$\delta \int_{t_1}^{t_2} L(q, \dot{q}; t)\, dt = 0 \quad \text{with}$$

VI.c) $L = T - U$; $T = T(q, \dot{q})$; $U = U(q, t)$.

D.) Constrained Motion: forces of constraint acting on system so that its configuration at any time t can be specified by $n < 3N$ independent variables, denoted q^a, $a = 1, 2, \dots, n < 3N$. Thus there exists relations amongst the $3N$ coordinates $x_{\alpha i}$, or q^A

$$x_{\alpha i} = x_{\alpha i}(q^a; t)$$

n degrees of freedom

$$q^A = q^A(q^a, t).$$

Although the δq^a are independent, the δq^A are dependent; there are $(3N - n)$ relations amongst them

$$\sum_{A=1}^{3N} n^r_A \, \delta q^A = 0 \quad , \quad r = 1, 2, \dots, (3N - n)$$

Integrating these we have the holonomic constraints amongst the coordinates

$$g^r(q^A; t) = 0 \quad \text{that is } n^r_A = \frac{\partial g^r}{\partial q^A}.$$

Suppose the forces of constraint are known: $F^c_{\alpha i}$. Then Lagrange's Eq's are

$$\frac{\partial L}{\partial x_{\alpha i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha i}} = -\left(\hat{F}_{\alpha i} + F^c_{\alpha i}\right)$$

VI.D.) or in generalized coordinates again

$$\frac{\partial L}{\partial q^A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A} = -\left(\hat{Q}_A + Q^c_A\right).$$

So proceeding as in the unconstrained case for $x_{\alpha i} = x_{\alpha i}(q^a; t)$ now so $\delta x_{\alpha i} = \frac{\partial x_{\alpha i}}{\partial q^a}\delta q^a$

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} = \sum_{\alpha i} \frac{\partial x_{\alpha i}}{\partial q^a}\left(\frac{\partial L}{\partial x_{\alpha i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha i}}\right)$$

or multiplying by the independent δq^a

$$\sum_{a=1}^{n} \delta q^a \left(\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a}\right) = \sum_{\alpha i} \delta x_{\alpha i}\left(\frac{\partial L}{\partial x_{\alpha i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha i}}\right)$$

$$= -\sum_{\alpha i} \delta x_{\alpha i}\left(\hat{F}_{\alpha i} + F^c_{\alpha i}\right)$$

Two Approaches

1) Eliminate dependent constrained coordinates, that is work with the q^a only.

Forces of constraint do no work since allowed displacements are orthogonal to the forces of constraint : $\delta\vec{r}_\alpha \cdot \vec{F}^c_\alpha = 0$

$$\Rightarrow \quad 0 = \sum_{\alpha i} \delta x_{\alpha i} F^c_{\alpha i} = 0$$

So Principle of Virtual Work yields

$$\delta W = 0 = \sum_{\alpha i} \delta x_{\alpha i}\left(F_{\alpha i} + \cancel{F}^c_{\alpha i}\right) = \sum_{\alpha i} \delta x_{\alpha i} F_{\alpha i}$$

VI (D.1.) $\Rightarrow$ $$\sum_{a=1}^{n_1} \delta q^a \left(\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} \right) = -\sum_{\alpha i} \delta x_{\alpha i} \hat{F}_{\alpha i}$$
$$= -\sum_{a=1}^{n_1} \delta q^a \frac{\partial x_{\alpha i}}{\partial q^a} \hat{F}_{\alpha i}$$
$$= -\sum_{a=1}^{n_1} \delta q^a \hat{Q}_a$$

but the δq^a are independent $\Rightarrow$
$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} = -\hat{Q}_a \;, \quad a = 1, 2, \ldots, n$$

& $$\delta W = \sum_{\alpha i} \delta x_{\alpha i} F_{\alpha i} = \sum_a \delta q^a Q_a \,.$$

If $\hat{Q}_a = 0 \Rightarrow$ Euler-Lagrange eq. $\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} = 0$

which is derivable from Hamilton's Principle
$$\delta \int_{t_1}^{t_2} L(q^a, \dot{q}^a; t)\, dt = 0 \,,$$ the independent coordinates q^a only.

The configuration of the system is then determined from the $q^a(t)$ and $q^r = 0$, to yield
$$x_{\alpha i} = x_{\alpha i}(q^a; t) \,.$$

VI.D.2) <u>Lagrange Multipliers</u>: Lagrange's Eq's:

$$\left(\frac{\partial L}{\partial x_{\alpha i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha i}}\right) = -\left(\hat{F}_{\alpha i} + F^{c}_{\alpha i}\right) \text{ with}$$

Constraint relations $g^r(x_{\alpha i};t) = 0 \Rightarrow$

$$\frac{\partial g^r}{\partial x_{\alpha i}}\delta x_{\alpha i} = 0 = \vec{n}^r \cdot \delta\vec{x} = \sum_{\alpha i} n^r_{\alpha i}\delta x_{\alpha i}$$

with the 3N-dimensional vectors $\delta\vec{x}$ and $\vec{n}^r$, $r = 1, \ldots, (3N-n) \equiv m$; $n^r_{\alpha i} \equiv \frac{\partial g^r}{\partial x_{\alpha i}}$.

virtual work $\Rightarrow$ $\sum_{\alpha i} F^c_{\alpha i}\delta x_{\alpha i} = 0 \equiv \vec{F}^c \cdot \delta\vec{x}$

Thus $\vec{F}^c \perp \delta\vec{x}$. Since $\vec{n}^r$ span the m-dimensional subspace of vectors $\perp$ to $\delta\vec{x}$ we can expand $\vec{F}^c$ in terms of the basis vectors $\vec{n}^r$

$$\vec{F}^c \equiv \sum_{r=1}^{m} \lambda_r(t)\,\vec{n}^r$$

Lagrange multipliers (indep. of $\delta x_{\alpha i}$)

$$\Rightarrow \left(\frac{\partial L}{\partial x_{\alpha i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha i}}\right) = -\hat{F}_{\alpha i} - \sum_{r=1}^{m}\lambda_r n^r_{\alpha i}$$

where $n^r_{\alpha i} = \frac{\partial g^r}{\partial x_{\alpha i}}$ and $g^r(x_{\alpha i};t) = 0$.

VI.D.2) Hence we have $(3N+m)$ equations for the $3N$ $(x_{\alpha i})$ + m (λ_r) unknowns.

Multiply by $\frac{\partial x_{\alpha i}}{\partial q^A}$ and sum over αi $\Rightarrow$

$$\sum_{\alpha i}' \frac{\partial x_{\alpha i}}{\partial q^A}\left(\frac{\partial L}{\partial x_{\alpha i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\alpha i}}\right) = -\sum_{\alpha i}' \frac{\partial x_{\alpha i}}{\partial q^A}\hat{F}_{\alpha i} - \sum_{r=1}^{m}{}' \lambda_r \sum_{\alpha i}' n^r_{\alpha i}\frac{\partial x_{\alpha i}}{\partial q^A}$$

That is

$$\boxed{\frac{\partial L}{\partial q^A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A} + \sum_{r=1}^{m}{}' \lambda_r N^r_A = -\hat{Q}_A \quad \text{where} \quad N^r_A = \frac{\partial g^r}{\partial q^A} \text{ and } g^r(q^A;t) = 0.}$$

generalized Forces of constraint are given by $Q^c_A = \sum_{r=1}^{m}{}' \lambda_r N^r_A$.

If $\hat{Q}_A = 0 \Rightarrow \begin{cases} \frac{\partial L}{\partial q^A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A} + \sum_{r=1}^{m}{}' \lambda_r N^r_A = 0 \\ g^r(q^A;t) = 0 \end{cases}$

which is derivable from Hamilton's principle with the modified action:

$$I(t_1,t_2) \equiv \int_{t_1}^{t_2}\left[L(q^A,\dot{q}^A;t) + \sum_{r=1}^{m}{}' \lambda_r g^r(q^A;t)\right]dt$$

and $\delta I = 0$ with $\{q^A, \lambda_r\}$ as independent

VI. D.2) coordinates i.e. δq^A & $\delta\lambda_r$ are indep.

E.) Non-holonomic Constraints:

$$x_{\alpha i} = x_{\alpha i}(q^a; t)$$

but constraints are amongst velocities

$$\sum_{a=1}^{n} n^r_a \, dq^a + m^r \, dt = 0 \quad ; \; r = 1, 2, \ldots, m.$$

i.e. $\sum_a n^r_a \dot{q}^a + m^r = 0$

(if $m^r = \frac{\partial f_r}{\partial t}$; $n^r_a = \frac{\partial f_r}{\partial q^a} \Rightarrow \frac{df_r}{dt} = 0 \Rightarrow f_r = \text{constant}$ and this becomes the holonomic case)

Proceed as holonomic case since virtual displacements are instantaneous for: δq^a, $dt = 0$, hence $n^r_a \delta q^a = 0$. The Q^c_a do no work $\delta W = \sum_a Q^c_a \delta q^a = 0$ so

$$Q^c_a = \sum_{r=1}^{m} \lambda_r(t) n^r_a \quad \text{and}$$

Lagrange's Equations are

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} + \sum_{r=1}^{m} \lambda_r n^r_a = -\hat{Q}_a \quad \text{along}$$

with the constraint

$$\sum_{a=1}^{n} n^r_a \, dq^a + m^r \, dt = 0 .$$

VI. E.) So there are $(n+m)$ unknowns $\{q^a, \lambda_r\}$ and $(n+m)$ equations above. If $\hat{Q}_a = 0$ the equations of motion are derivable from Hamilton's principle

$$\delta \int_{t_1}^{t_2} L\,dt = 0$$ in

which we must eliminate the constrained variables

$$\delta \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} \right) \delta q^a\,dt$$

but δq^a are not indep : $n^r_a\,\delta q^a = 0$ hence

$$\int_{t_1}^{t_2} \lambda_r n^r_a\,\delta q^a\,dt = 0 .$$ So

add this to $\delta \int_{t_1}^{t_2} L\,dt$ above

$$0 = \int_{t_1}^{t_2} \sum_{a=1}^{n} \left[\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} + \sum_{r=1}^{m} \lambda_r n^r_a \right] \delta q^a\,dt .$$

So $(n-m)$ of the δq^a are indep. and set the integrand for these to zero, the remaining m are dependent, use the λ_r to set the integrand to zero in these cases. Hence we obtain

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} + \sum_{r=1}^{m} \lambda_r n^r_a = 0$$

$$n^r_a\,dq^a + m^r\,dt = 0$$

VI. F.) Symmetry & Conservation Laws:
Hamilton's Principle yields the dynamics for the system

$$\delta \int_{t_1}^{t_2} L(x_{\alpha i}, \dot{x}_{\alpha i}; t)\, dt = 0$$

Invariances (symmetries) of L under coordinate transformations imply conservation laws.
Closed system has covariance wrt space-time translations & space Rotations

$$\vec{x}'_\alpha(t') = \vec{x}_\alpha(t) + \vec{a} + \delta\vec{\theta} \times \vec{x}_\alpha(t) + \epsilon\, \dot{\vec{x}}_\alpha(t)$$

where infinitesimal space translations are given by $\vec{a}$, $x'_{\alpha i} = x_{\alpha i} + a_i$, space rotations are given by $\delta\vec{\theta}$, $x'_{\alpha i} = x_{\alpha i} + \epsilon_{ijk}\, \delta\theta_j\, x_{\alpha k}$ and time translations $t' = t + \epsilon$.
So we can define the total variation

$$\Delta t \equiv t' - t$$

$$\Delta \vec{x}_\alpha \equiv \vec{x}'_\alpha(t') - \vec{x}_\alpha(t)$$

Under these symmetry transformations the total change in the Lagrangian is given by Noether's theorem q.II.F.)

$$\Delta L \equiv L(x_{\alpha i}+\Delta x_{\alpha i}, \dot{x}_{\alpha i}+\Delta \dot{x}_{\alpha i}; t+\Delta t) - L(x_{\alpha i}, \dot{x}_{\alpha i}; t)$$

$$= \frac{\partial L}{\partial t}\Delta t + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_{\alpha i}}\Delta x_{\alpha i}\right)$$

(having used the Euler-Lagrange eq.'s of motion)

Conservation Laws:

1) Homogeneity of time $\Leftrightarrow$ Energy Conservation

Let $\Delta t = \epsilon$ only $\Rightarrow \Delta x_{\alpha i} = \dot{x}_{\alpha i}\Delta t$

The dynamics is independent of the origin of time hence L should not depend on time <u>explicitly</u>

$$L(x, \dot{x}; t+\epsilon) = L(x, \dot{x}; t)$$

that is $\frac{\partial L}{\partial t} = 0$.

The total variation $\Delta L = L(t+\epsilon) - L(t)$

$$= L(x(t+\epsilon), \dot{x}(t+\epsilon); t+\epsilon) - L(x(t), \dot{x}(t); t)$$

$$= \frac{dL}{dt}\Delta t$$

VI. F.1.) But Noether's thm.

$$\Delta L = \cancel{\frac{\partial L}{\partial t}}^{\to 0} \Delta t + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_{\alpha i}} \Delta x_{\alpha i}\right)$$

$$= \frac{dL}{dt} \Delta t$$

$$\Rightarrow \quad \Delta t \left[\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_{\alpha i}} \dot{x}_{\alpha i}\right) \right]$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_{\alpha i}} \dot{x}_{\alpha i} - L\right) = 0$$

Hence
$$\boxed{\sum_{\alpha i} \frac{\partial L}{\partial \dot{x}_{\alpha i}} \dot{x}_{\alpha i} - L \equiv H = \text{constant}}$$

H is called the <u>Hamiltonian</u>.

If U is $\dot{x}$ independent and conservative $\Rightarrow$

$$H = T + U = E = \text{total Energy}$$

Energy is conserved $E = \text{constant}$

Likewise if $x_{\alpha i} = x_{\alpha i}(q^A)$ only and $\frac{\partial U}{\partial \dot{q}^A} = 0$

$$\boxed{\begin{gathered}\text{Then time homogeneity } \frac{\partial L}{\partial t} = 0 \Leftrightarrow \\ H = \sum_A \frac{\partial L}{\partial \dot{q}^A} \dot{q}^A - L = E = \text{constant}\end{gathered}}$$

VI. F. 2.) Space Homogeneity $\Leftrightarrow$ Conservation of linear Momentum

let $\Delta x_{\alpha i} = a_i = \text{constant}$ only $\Rightarrow \Delta t = 0,\ \Delta \dot{x}_{\alpha i} = 0$

The origin of space does not change the dynamics of the system, that is

$$L(x_{\alpha i} + \Delta x_{\alpha i}, \dot{x}_{\alpha i}; t) = L(x_{\alpha i}, \dot{x}_{\alpha i}; t)$$

$$\Rightarrow \quad \Delta L = \sum_{\alpha i}' \frac{\partial L}{\partial x_{\alpha i}} a_i = 0$$

On the other hand Noether's Thm. $\Rightarrow$

$$\Delta L = \frac{d}{dt}\left(\sum_{\alpha i}' \frac{\partial L}{\partial \dot{x}_{\alpha i}} a_i \right)$$

Hence $\quad \frac{d}{dt}\left(\sum_{\alpha i}' \frac{\partial L}{\partial \dot{x}_{\alpha i}} a_i \right) = 0$ but a_i

is arbitrary $\Rightarrow \quad \boxed{\sum_{\alpha}' \frac{\partial L}{\partial \dot{x}_{\alpha i}} = \text{constant}_i} \quad i = 1, 2, 3.$

But $\quad \frac{\partial L}{\partial \dot{x}_{\alpha i}} = \frac{\partial T}{\partial \dot{x}_{\alpha i}} = m_\alpha \dot{x}_{\alpha i} = p_{\alpha i}$

$$\boxed{\begin{array}{c}\text{So homogeneity of space} \Leftrightarrow \text{momentum conservation} \\ \sum_{\alpha}' p_{\alpha i} = P_i = \text{constant}\end{array}}$$

VI. F.3.) Isotropy of space $\Leftrightarrow$ angular momentum conservation

let $\Delta x_{\alpha i} = \epsilon_{ijk}\,\delta\theta_j\, x_{\alpha k} \Rightarrow \Delta t = 0$ but

$$\Delta \dot{x}_{\alpha i} = \epsilon_{ijk}\,\delta\theta_j\,\dot{x}_{\alpha k}.$$

The laws of physics (dynamics) should remain unchanged under rotations of the coordinate system (L is rotationally invariant) that is

$$L(\vec{x}_\alpha + \vec{\delta\theta}\times\vec{x}_\alpha,\ \dot{\vec{x}}_\alpha + \vec{\delta\theta}\times\dot{\vec{x}}_\alpha; t) = L(\vec{x}_\alpha, \dot{\vec{x}}_\alpha; t)$$

Again this means $\Delta L = 0$ but Noether's Thm $\Rightarrow$

$$\Delta L = \frac{d}{dt}\left(\sum_{\alpha i}\frac{\partial L}{\partial \dot{x}_{\alpha i}}\Delta x_{\alpha i}\right) = 0$$

Thus

$$\sum_{\alpha i}\frac{\partial L}{\partial \dot{x}_{\alpha i}}\Delta x_{\alpha i} = \text{constant}$$

$$\| $$

$$\sum_{\alpha i} p_{\alpha i}\,\epsilon_{ijk}\,\delta\theta_j\, x_{\alpha k}$$

$$= \delta\theta_j\left[\sum_\alpha \vec{x}_\alpha \times \vec{p}_\alpha\right]_j \equiv \delta\theta_j L_j = \text{constant}$$

$$\boxed{\begin{array}{c}\text{Hence isotropy of space} \Leftrightarrow \text{Angular Momentum Conservation}\\ L_j = \sum_\alpha (\vec{r}_\alpha \times \vec{p}_\alpha)_j = \text{constant.}\end{array}}$$

VI. F. 4.) Clausius Virial Theorem: a statistical Conservation Law

$$S \equiv \sum_{\alpha}{}' \vec{p}_\alpha \cdot \vec{r}_\alpha$$

$$\Rightarrow \quad \frac{dS}{dt} = \sum_{\alpha}{}' \left(\dot{\vec{p}}_\alpha \cdot \vec{r}_\alpha + \vec{p}_\alpha \cdot \dot{\vec{r}}_\alpha \right)$$

Time average $\frac{dS}{dt}$ over time interval $[0, T]$

$$\left\langle \frac{dS}{dt} \right\rangle = \frac{1}{T} \int_0^T \frac{dS}{dt}\, dt = \frac{S(T) - S(0)}{T}$$

a) If the motion is periodic and T is an integer multiple of the period $S(T) = S(0)$ and $\left\langle \frac{dS}{dt} \right\rangle = 0$

b) If the motion is not periodic but is bounded then S is bounded and for large $T \to \infty$ $\left\langle \frac{dS}{dt} \right\rangle = 0$ also.

$$\Rightarrow \quad \left\langle \sum_{\alpha}{}' \vec{p}_\alpha \cdot \dot{\vec{r}}_\alpha \right\rangle = -\left\langle \sum_{\alpha}{}' \dot{\vec{p}}_\alpha \cdot \vec{r}_\alpha \right\rangle$$

But $\vec{p}_\alpha \cdot \dot{\vec{r}}_\alpha = 2T_\alpha$ & $\dot{\vec{p}}_\alpha = \vec{F}_\alpha$ $\Rightarrow$

$$\boxed{\langle T \rangle = -\frac{1}{2} \left\langle \sum_{\alpha}{}' \vec{F}_\alpha \cdot \vec{r}_\alpha \right\rangle \equiv \text{Virial of the system}}$$

Clausius's Virial Thm.

If $\vec{F}_\alpha = -\vec{\nabla}_\alpha U \Rightarrow$

$$\langle T \rangle = \frac{1}{2} \left\langle \sum_{\alpha}{}' \vec{r}_\alpha \cdot \vec{\nabla}_\alpha U \right\rangle$$

VI.F.4.) ex. Central Forces between 2 bodies $U = kr^{n+1}$

$$\sum_\alpha' \vec{r}_\alpha \cdot \vec{\nabla}_\alpha U = \vec{r} \cdot \vec{\nabla} U(|\vec{r}|) = (n+1) U$$

$$\Rightarrow \quad \langle T \rangle = \frac{n+1}{2} \langle U \rangle$$

for $\frac{1}{r^2}$-force $n=-2 \Rightarrow \langle T \rangle = -\frac{1}{2} \langle U \rangle$

G.) <u>Hamiltonian Dynamics</u>: treat δq^a and δp_a as independent with $p_a \equiv \frac{\partial L}{\partial \dot{q}^a}$

The momentum canonically conjugate to q^a.

Thus $\dot{q}^a = \dot{q}^a(q^b, p^b; t)$ is implicitly given by $p_a = \frac{\partial L}{\partial \dot{q}^a}$.

View Hamiltonian as a function of q^a, p_a and t

$$H = H(q^a, p_a; t) \equiv \sum_a' p_a \dot{q}^a - L(q^a, \dot{q}^a; t)$$

with $\dot{q}^a = \dot{q}^a(q^b, p_b; t)$ on the RHS.

This is a Legendre transformation changing variables from $(q, \dot{q}, t)$ to (q, p, t).

VI. G.) If (q^a, p_a, t) are independent, then

$$dH = \sum_a \left(\frac{\partial H}{\partial q^a} dq^a + \frac{\partial H}{\partial p_a} dp_a \right) + \frac{\partial H}{\partial t} dt$$

But on the other hand from the definition of H

$$dH = \sum_a \left(dp_a \dot{q}^a + p_a d\dot{q}^a - \frac{\partial L}{\partial q^a} dq^a - \frac{\partial L}{\partial \dot{q}^a} d\dot{q}^a \right) - \frac{\partial L}{\partial t} dt$$

Using $p_a \equiv \frac{\partial L}{\partial \dot{q}^a}$ and the Euler-Lagrange eq's $\dot{p}_a = \frac{\partial L}{\partial q^a}$

$\Rightarrow$

$$dH = \sum_a \left(\dot{q}^a dp_a - \dot{p}_a dq^a \right) - \frac{\partial L}{\partial t} dt$$

Equating these two expressions for $dH \Rightarrow$

Hamilton's Equations of Motion

$$\left\{ \begin{array}{l} \dot{q}^a = \dfrac{\partial H}{\partial p_a} \\ -\dot{p}_a = \dfrac{\partial H}{\partial q^a} \end{array} \right\} \quad \text{and} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Substituting this into the first dH expression

$\Rightarrow$

$$dH = \frac{\partial H}{\partial t} dt \quad \Rightarrow \quad \boxed{\frac{dH}{dt} = \frac{\partial H}{\partial t}}$$

VI.G.) So if $\frac{\partial H}{\partial t} = 0 \Rightarrow H = \text{constant}$ and if U is velocity independent & $x = x(q)$ only $\Rightarrow$ $H = E$.

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1) If q^a does not appear in H it is called <u>cyclic</u>

$$\Rightarrow \quad -\dot{p}_a = \frac{\partial H}{\partial q^a} = 0 \Rightarrow p_a = \text{constant} \equiv \Pi_a$$

i.e. $$\dot{q}_a = \frac{\partial H}{\partial \Pi_a} \equiv \omega_a \Rightarrow q_a(t) = \int^t \omega_a \, dt$$

So actually H is a function of $(2n-2)$ variables only in this case

$$H = H(q^1, \ldots, \cancel{q^a}, \ldots, q^n, p_1, \ldots, p_a = \Pi_a, \ldots, p_n; t)$$

So the number of degrees of freedom ~~is~~ reduced to $(2n-2)$. q^a is said to be <u>ignorable</u>.

If q^a is cyclic in H it is also cyclic in L that is

$$\frac{\partial L}{\partial q^a} = 0 = -\frac{\partial H}{\partial q^a} \Rightarrow p_a = \frac{\partial L}{\partial \dot{q}^a} = \text{constant}$$

for cyclic variables.

VI. G. 2.) Hamilton's Principle in terms of q^a, p_a Variables

$$\delta \int_{t_1}^{t_2} \left(\sum_a p_a \dot{q}^a - H(q,p;t) \right) dt = 0$$

where δq^a and δp_a are independent.

$\Rightarrow$ Hamilton's Equations of Motion:

$$\dot{q}^a = \frac{\partial H}{\partial p_a}$$

$$-\dot{p}_a = \frac{\partial H}{\partial q^a} .$$