V) Calculus of Variations

A) (Ordinary) Multi-variate Calculus: Consider $F(x,y)$

1) If dx and dy are independent then F is stationary ($dF=0$) at (x_0,y_0) if and only if $\left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0)} = 0 = \left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0)}$. That is

$$0 = dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \quad \text{at } (x_0,y_0)$$

iff $\left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0)} = 0 = \left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0)}$ when dx, dy are indep.

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2) Suppose there is a constraint among (x,y):

① $\boxed{G(x,y)=0}$ with G a given function.

$$\left.\begin{aligned} &\Rightarrow \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy = 0 \\ &0 = dF \Rightarrow \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dx = 0 \end{aligned}\right\} \text{ at } (x_0,y_0)$$

Interpret geometrically as $\vec{\nabla}G \cdot d\vec{r} = 0$ at (x_0,y_0)

$\vec{\nabla}F \cdot d\vec{r} = 0$

$\Rightarrow \vec{\nabla}F \parallel \vec{\nabla}G$ or $\vec{\nabla}F = -\lambda\vec{\nabla}G$ at (x_0,y_0)

or ② $\boxed{\frac{\partial F}{\partial x} = -\lambda\frac{\partial G}{\partial x}}$ at $(x_0,y_0) \Rightarrow$ 3 eq's and 3 unknowns (x_0,y_0,λ)

③ $\boxed{\frac{\partial F}{\partial y} = -\lambda\frac{\partial G}{\partial x}}$

V A2) Thus equivalent problems are

F is stationary at (x_0, y_0) with (x,y) subject to the constraint $G(x,y)=0$ if and only if

$(F+\lambda G)$ is stationary at (x_0,y_0) for a unique λ and for $(dx, dy, d\lambda)$ independent.

i.e. $0 = d(F+\lambda G) = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \lambda \frac{\partial G}{\partial x} + \lambda \frac{\partial G}{\partial y}$
$+ G\, d\lambda = 0$

$(dx, dy, d\lambda)$ indep.

$\Rightarrow$ ① $G = 0$

② $\frac{\partial F}{\partial x} + \lambda \frac{\partial G}{\partial x} = 0$

③ $\frac{\partial F}{\partial y} + \lambda \frac{\partial G}{\partial y} = 0$

B) Calculus of Variations

1) Functionals $J[y]$ map the set of (suitably defined functions $y(x)$) into the real numbers $\mathbb{R}$

$J: \Omega \to \mathbb{R}$; Ω = space or set of functions y

Consider

$$J[y] \equiv \int_{x_1}^{x_2} f(y(x), y'(x); x)\, dx$$

IV B 1) with x_1, x_2 fixed endpoints and where f is a given function of y, $y' = \frac{dy}{dx}$ and x.

A necessary condition for J to be stationary ($\delta J = 0$) (i.e. y an extremum) is that $y(x)$ obeys the Euler-Lagrange equation

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0}$$

2) Let $J[y_1, y_2, \ldots, y_n]$ be a functional of independent functions $y_1(x), \ldots, y_n(x)$:

$$J[y_1, \ldots, y_n] \equiv \int_{x_1}^{x_2} f(y_1, \ldots, y_n, y_1', \ldots, y_n'; x)\, dx.$$

A necessary condition for J to be stationary ($\delta J = 0$) is that

$$\boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0 \quad \text{for each } i = 1, \ldots, n}$$

V C.) Constrained Systems

1) Isoperimetric Constraints (integral constraints)

Find $y(x)$ so that
$J[y] = \int_a^b f(y, y'; x)\,dx$ is stationary

and $L = \int_a^b g(y, y'; x)\,dx$ is constant

with g also a given function of y, y' & x.
L = constant is the constraint.

This is equivalent to the problem of introducing a constant Lagrange multiplier λ and finding the stationary condition for

$$J[y] + \lambda L[y] \quad \text{for arbitrary } \lambda$$

i.e. $\delta(J[y] + \lambda L[y]) = 0 \Rightarrow y = y(x;\lambda)$
as if no constraint. Then determine value of λ so that $L[y(x;\lambda)] = L$ = constant the constraint is satisfied. Hence $\lambda L[y(x;\lambda)] = L$ for that λ and so

$\delta(J + \lambda L) = 0 \Rightarrow$ $\boxed{\delta J = 0 \text{ since } L = \text{constant}}$
which was the original problem.

V.C.2.) Holonomic Constraints (algebraic relations among coordinates)

Find the extremal path $\{y_1(x), \dots, y_n(x)\}$,

i.e. path which makes J stationary $\delta J = 0$,

for

$$J[y_1, \dots, y_n] = \int_{x_1}^{x_2} f(y_1, \dots, y_n; y_1', \dots, y_n'; x)\, dx$$

subject to the constraints

$$g^r(y_1, \dots, y_n; x) = 0 \qquad r = 1, \dots, m < n$$

a) Method 1: Use $g^r = 0$ directly to eliminate m of the n coordinates y in terms of (n-m) independent ones.

Express J in terms of these (n-m) independent coordinates only. Then they obey Euler's equation. For example let $\{y_1, \dots, y_{(n-m)}\}$ be independent. That is use g^r, $r = 1, \dots, m$ to write

m-dependent coordinates

$$\left\{ \begin{aligned} y_{(n-m+1)} &= y_{(n-m+1)}(y_1, \dots, y_{(n-m)}; x) \\ &\vdots \\ y_n &= y_n(y_1, \dots, y_{(n-m)}; x) \end{aligned} \right.$$

V(2a) Substitute these relations into $f(y_i; y_i'; x)$;

$$\begin{aligned}&\hat{f}(y_1,\dots,y_{(n-m)},y_1',\dots,y_{(n-m)}';x)\\ &\equiv f(y_1,\dots,y_{(n-m)},y_{(n-m+1)}(y_1,\dots,y_{(n-m)};x),\dots;x)\end{aligned}$$

write m-dependent y's in terms of $(n-m)$ independent y's.

Then $\delta J = 0 \Rightarrow$

$$\frac{\partial \hat{f}}{\partial y_i} - \frac{d}{dx}\frac{\partial \hat{f}}{\partial y_i'} = 0 \quad \text{for } i = 1,\dots,(n-m)$$

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b) Method 2: Lagrange Multipliers $\lambda^r(x)$

The problem is equivalent to introducing new coordinates $\lambda^r(x)$ for each constraint $r = 1,\dots,m$, called Lagrange multipliers. Then introduce the new functional of $\{y_i, \lambda^r\}$

$$I[y_1,\dots,y_n,\lambda,\dots,\lambda_m] \equiv \int_{x_1}^{x_2} dx \left[f(y_i; y_i'; x) + \sum_{r=1}^{m} \lambda_r(x) g^r(y_i; x) \right]$$

The solution to the original problem is the extremal $\{y_i, \lambda_r\}$ that make I stationary, $\delta I = 0$, where we are treating $\{\delta y_i, \delta\lambda^r\}$ as independent variations in coordinates

V C(2b) This yields $(n+m)$ Euler-Lagrange equations for the $(n+m)$ coordinates (y_i, λ_r):

$$1)\ \frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} + \sum_{r=1}^{m} \lambda_r \frac{\partial g^r}{\partial y^i} = 0 \quad , i=1,\dots,n$$

$$2)\ \frac{\partial}{\partial \lambda_r}\left(f + \sum_{s=1}^{m} \lambda_s g^s\right) = g^r(y_i;x) = 0 \ ; r=1,\dots,m$$

The details of this result were that $g^r = 0$

$$\Rightarrow \quad \sum_{i=1}^{n} \frac{\partial g^r}{\partial y_i}\delta y_i = 0 \ , r=1,\dots,m .$$

This is used to eliminate <u>m-dependent</u> variations δy's in terms of $(n-m)$ <u>independent</u> variations. For example let $(\delta y_1, \dots, \delta y_{(n-m)})$ be independent; then

$$\underbrace{\begin{pmatrix} \frac{\partial g^1}{\partial y_{(n-m+1)}} & \frac{\partial g^1}{\partial y_{(n-m+2)}} & \cdots & \frac{\partial g^1}{\partial y_n} \\ \frac{\partial g^2}{\partial y_{(n-m+1)}} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial g^m}{\partial y_{(n-m+1)}} & \cdots & \cdots & \frac{\partial g^m}{\partial y_n} \end{pmatrix}}_{(m\times m)\text{ matrix}} \underbrace{\begin{pmatrix} \delta y_{(n-m+1)} \\ \delta y_{(n-m+2)} \\ \vdots \\ \delta y_n \end{pmatrix}}_{\text{dependent variations}} =$$

IVc 2b)

$$= -\begin{pmatrix} \sum_{a=1}^{(n-m)} \frac{\partial g^1}{\partial y^a}\delta y_a \\ \vdots \\ \sum_{a=1}^{(n-m)} \frac{\partial g^m}{\partial y^a}\delta y_a \end{pmatrix}$$

independent variations

Cryptically we can write this as ...

$$\left(\frac{\partial g^r}{\partial y_\alpha}\right)\left(\delta y_\alpha\right) = -\left(\frac{\partial g^r}{\partial y^a}\delta y^a\right) \qquad \begin{aligned}\alpha &= (n-m+1),\dots,n \\ a &= 1,\dots,(n-m) \\ r &= 1,\dots,m\end{aligned}$$

$$\Rightarrow \quad \delta y_\alpha = -\left(\frac{\partial g^r}{\partial y_\alpha}\right)^{-1}\left(\frac{\partial g^r}{\partial y^a}\bar{\delta} y^a\right)$$

Hence when considering the variation of J

$$0 = \bar{\delta} J = \int_{x_1}^{x_2} dx \left[\left(\frac{\partial f}{\partial y_a} - \frac{d}{dx}\frac{\partial f}{\partial y_a'}\right)\delta y_a + \left(\frac{\partial f}{\partial y_\alpha} - \frac{d}{dx}\frac{\partial f}{\partial y_\alpha'}\right)\delta y_\alpha\right]$$

$$= \int_{x_1}^{x_2} dx \left[\left(\frac{\partial f}{\partial y_a} - \frac{d}{dx}\frac{\partial f}{\partial y_a'}\right) - \left(\frac{\partial f}{\partial y_\alpha} - \frac{d}{dx}\frac{\partial f}{\partial y_\alpha'}\right)\left(\frac{\partial g^r}{\partial y_\alpha}\right)^{-1}\left(\frac{\partial g^r}{\partial y^a}\right)\right] \times \bar{\delta} y_a$$

Since the $\bar{\delta} y_a$; $a = 1,\dots,(n-m)$ are independent J is stationary $\bar{\delta} J = 0$ requires the integrand to vanish for each $\delta y_a \Rightarrow$

IIc(2b)
$$\boxed{\left(\frac{\partial f}{\partial y_a} - \frac{d}{dx}\frac{\partial f}{\partial y'_a}\right) = \left(\frac{\partial f}{\partial y_\alpha} - \frac{d}{dx}\frac{\partial f}{\partial y'_\alpha}\right)\left(\frac{\partial g^r}{\partial y_\alpha}\right)^{-1}\left(\frac{\partial g^r}{\partial y^a}\right)}$$

$a = 1, \ldots, (n-m)$

but
$$\boxed{\left(\frac{\partial f}{\partial y_\alpha} - \frac{d}{dx}\frac{\partial f}{\partial y'_\alpha}\right)\left(\frac{\partial g^r}{\partial y_\alpha}\right)^{-1} \equiv -\lambda^r(x)}$$
is a function of x call it $\lambda^r(x)$

$r = 1, \ldots, m$

Hence we have

1) $$\left(\frac{\partial f}{\partial y_a} - \frac{d}{dx}\frac{\partial f}{\partial y'_a}\right) + \sum_{r=1}^{m} \lambda^r(x)\frac{\partial g^r}{\partial y^a} = 0$$
$a = 1, \ldots, (n-m)$

and

2) $$\left(\frac{\partial f}{\partial y_\alpha} - \frac{d}{dx}\frac{\partial f}{\partial y'_\alpha}\right) + \sum_{r=1}^{m} \lambda^r(x)\frac{\partial g^r}{\partial y_\alpha} = 0$$

for $\alpha = (n-m+1), \ldots, n$

(this follows from multiplying the definition of λ^r above by the $(m \times m)$ matrix $\frac{\partial g^r}{\partial y}$ & summing on $r = 1, \ldots, m$.

These 2 equations are just

$$\boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y'_i} + \sum_{r=1}^{m} \lambda^r(x)\frac{\partial g^r}{\partial y_i} = 0 \qquad i = 1, \ldots, n}$$

IIc.2b) along with $\boxed{g^r(y_1,\dots,y_n;x) = 0 \qquad r=1,\dots,m}$

are $(n+m)$ equations for the $(n+m)$ unknowns (y_i, λ_r).

These Euler-Lagrange equations are equivalent to the variational problem of making the functional $I[y,\lambda]$ stationary with (y,λ) treated as independent coordinates where

$$I[y_1,\dots,y_n,\lambda_1,\dots,\lambda_m]$$
$$= \int_{x_1}^{x_2} dx \left[f(y_i, y_i'; x) + \sum_{r=1}^{m} \lambda_r(x)\, g^r(y_i;x) \right].$$

So $\delta I = 0 \implies$

1) $$\frac{\partial(f+\lambda_r g^r)}{\partial y_i} - \frac{d}{dx}\frac{\partial(f+\lambda_r g^r)}{\partial y_i'} = 0$$

$$\implies \boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} + \sum_{r=1}^{m} \lambda_r g^r = 0 \quad , i=1,\dots,n}$$

2) $$\frac{\partial(f+\lambda_r g^r)}{\partial \lambda_r} - \frac{d}{dx}\cancel{\frac{\partial(f+\lambda_r g^r)}{\partial \lambda_r'}} = 0$$

$$\implies \boxed{g^r(y_i;x) = 0 \qquad r=1,\dots,m}$$

V.C.3) <u>Non-Holonomic Constraints</u>: Constraints are not expressed as algebraic relations among the coordinates. Suppose the system requires n-generalized coordinates q^a to describe the configuration of our system at any time t

$$x_{\alpha i} = x_{\alpha i}(q^1, \ldots, q^n; t). \text{ In}$$

addition the system's motions are constrained further so that there exists relations among the generalized velocities

$$\sum_{a=1}^{n} n^r_a \, dq^a + m^r dt = 0 \qquad r = 1, \ldots, m.$$

(i.e. $n^r_a \dot{q}^a + m^r = 0$) Note: if $m^r = \frac{\partial q^r}{\partial t}$

and $n^r_a = \frac{\partial q^r}{\partial q^a} \Rightarrow \frac{dq^r}{dt} = 0 \Rightarrow q^r = \text{constant}$ and the constraints are holonomic.

For virtual displacements, the coordinate variations are instantaneous $\delta q^a \neq 0$ while $dt = 0$ in the above. Hence the constraints can be handled similarly to the holonomic case; $\sum_{a=1}^{n} n^r_a \delta q^a = 0$.

Since the virtual work performed by the forces of constraint during these allowed displacements is zero

I.c.3) $\delta W = \sum_{a=1}^{n} Q^c_a \, \delta q^a = 0$ we have that

Q^c_a can be expanded in terms of the constrained directions n^r_a :

$$Q^c_a = \sum_{r=1}^{m} \lambda_r(t) \, n^r_a$$

Lagrange's equations are then

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} = -\hat{Q}_a - \sum_{r=1}^{m} \lambda_r n^r_a \; ; a=1,\dots,n$$

along with the differential equations of constraint

$$\sum_{a=1}^{n} n^r_a \dot{q}^a + m^r = 0 \; ; r=1,\dots,m$$

These are $(n+m)$ equations for the $(n+m)$ unknowns (q^a, λ_r).

If $\hat{Q}_a = 0$, the Lagrange's equations become

$$\boxed{\begin{aligned} &\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} + \sum_{r=1}^{m} \lambda_r n^r_a = 0 \quad ; a=1,\dots,n \\ &\sum_{a=1}^{n} n^r_a \dot{q}^a + m^r = 0 \quad ; r=1,\dots,m \end{aligned}}$$

with the constraint equation