Physics 410: Review Outline

I) Newton's Laws: the framework of classical Mechanics: experimentally we find that

N-1: There exists a set of frames of reference with respect to which all bodies not subjected to forces are unaccelerated — these are inertial frames of reference and the

The principle of relativity states that all inertial systems are equivalent for the description of Nature.

Since all inertial frames are related by Galilean transformations

(space Rotations + space translations + Galilean boosts)

$$x'_i = \lambda_{ij} x_j + a_i - v_i t$$

$$t' = t$$

The laws of Newtonian mechanics must be covariant wrt these Gal. transformations

I) N-2: A body acted upon by a force moves in such a manner that the time rate of change of momentum equals the force

Newton defined momentum as

$$\vec{p} = m\vec{v} \qquad \vec{v} = \dot{\vec{x}}$$

where $\vec{x}$ = coordinate of particle in an inertial frame

Then N-2 $\Leftrightarrow$ $\vec{F} = \frac{d\vec{p}}{dt}$ (defines force)

N-3: If two bodies exert forces on each other then the accelerations of these bodies are in opposite direction

(use this to define mass)

Thus by studying the trajectory of a particle we can determine the forces it is subjected to & visa-versa.

Examples:

1) $\vec{F} = +m\vec{g} - b\vec{v}$ Projectile with air resistance

1) $m\ddot{x} = -b\dot{x}$

2) $m\ddot{y} = -b\dot{y}$

3) $m\ddot{z} = -mg - b\dot{z}$

} 3 - 2^{nd} order Diff eq. we need

6 initial or B.C. Conditions

3 positions + 3 vel. for example

let $x(0) = y(0) = z(0) = 0$

$\dot{x}(0) = V\cos\alpha \qquad \dot{y}(0) = 0.$

$\dot{z}(0) = V\sin\alpha$

$b = km$

$$\Rightarrow \boxed{\begin{aligned} x(t) &= \frac{mV\cos\alpha}{b}\left(1 - e^{-\frac{b}{m}t}\right) \\ y(t) &= 0 \\ z(t) &= -\frac{mgt}{b} + \frac{\frac{b}{m}V\sin\alpha + g}{b^2/m^2}\left(1 - e^{-\frac{b}{m}t}\right) \end{aligned}}$$

Spatial traj. $z = z(x)$

$$z = \left[\tan\alpha + \frac{mg}{bV\cos\alpha}\right]x + \frac{m^2 g}{b^2}\ln\left(1 - \frac{bx}{mV\cos\alpha}\right)$$

$$\simeq \underbrace{x\tan\alpha - \frac{1}{2}\frac{gx^2}{V^2\cos^2\alpha}}_{\text{parabola: friction free}} - \underbrace{\frac{1}{3}\frac{gbx^3}{mV^3\cos^3\alpha}}_{\text{shortens Range}} + \cdots$$

$$R_0 = \frac{V^2}{g}\sin 2\alpha$$ Range in absence of air resistance

$$R = R_0 - \Delta R$$ Range with air resistance

$$\Delta R \simeq \frac{4}{3}\frac{bV^3}{mg^2}\sin\alpha\sin 2\alpha .$$

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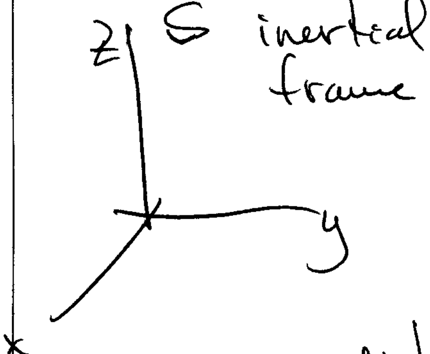
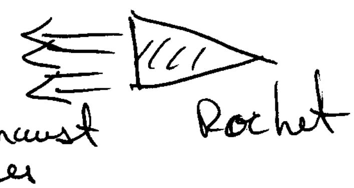
2) Rocket Motion





Rocket vel. wrt S $= \vec{v}(t)$

exhaust vel. wrt Rocket $= \vec{V} =$ const.

" " " S $= \vec{V} + \vec{v}(t)$

mass of Rocket + fuel $= m(t)$

$\vec{p}$ = momentum of system = Rocket + fuel!

$$p(t) = m(t)\vec{v}(t)$$

2nd order small

$$p(t+\Delta t) = (m-\Delta m)(\vec{v}+\Delta\vec{v}) + \Delta m\left(\vec{V}+\vec{v}+\cancel{\Delta\vec{v}}\right)$$

$$\Rightarrow \quad \frac{d\vec{p}}{dt} = m\dot{\vec{v}} - \dot{m}\vec{V}$$

where $\dot{m} \equiv -\frac{\Delta m}{\Delta t}$

So $\vec{F} = \frac{d\vec{p}}{dt} = m\dot{\vec{v}} - \dot{m}\vec{V}$

$$= \frac{d}{dt}(m\vec{v}) - \underbrace{\dot{m}(\vec{V}+\vec{v})}_{\text{We must include these terms!!}}$$

The rate of change of momentum is not simply $\frac{d(m\vec{v})}{dt}$!! (only in non-inertial frame $\vec{V}+\vec{v}=0$!)

II) Conservations Laws: If Newton's laws are true then

1) The total linear momentum of the system is conserved if there is no total external force $\vec{F}=0$

$$\vec{P} = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = M\dot{\vec{R}} \quad ; \quad \vec{F} = \sum_\alpha \vec{F}_\alpha^{(e)}$$

$\vec{R} = \frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha$ = CM position

$$\dot{\vec{P}} = \vec{F} \quad \text{if } \vec{F}=0 \Rightarrow \vec{P} = \text{const.}$$

II) 2) If the total external torque vanishes then the total $\measuredangle$ momentum of the system is conserved.

$$\vec{L} = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha = \vec{R} \times \vec{P} + \sum_\alpha \bar{\vec{r}}_\alpha \times \bar{\vec{p}}_\alpha$$

where

$$\vec{r}_\alpha = \vec{R} + \bar{\vec{r}}_\alpha$$

$$\vec{p}_\alpha = \frac{m_\alpha}{M} \vec{P} + \bar{\vec{p}}_\alpha$$

$$\dot{\vec{L}} = \underbrace{\vec{N}^{(e)}}_{\text{total ext. torque}} \quad \text{with } \vec{N}^{(e)} = \sum_\alpha \vec{r}_\alpha \times \vec{F}^{(e)}_\alpha$$

and force on particle α $\vec{F}_\alpha = \vec{F}^{(e)}_\alpha + \sum_{\beta \neq \alpha} \vec{f}_{\alpha\beta}$

with N3 for $\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$; $\vec{f}_{\alpha\beta} \times (\vec{r}_\alpha - \vec{r}_\beta) = 0$.

Thus if $\vec{N}^{(e)} = 0$ then $\vec{L} = \text{const}$.

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3) If all forces, internal and external, are conservative (irrotational + time indep.) then the total energy is conserved

II.3) total energy $E = T + U$

$$T = \sum_{\alpha} \tfrac{1}{2} m_\alpha v_\alpha^2 = \tfrac{1}{2} M V^2 + \sum_{\alpha} \tfrac{1}{2} m_\alpha \bar{v}_\alpha^2$$

$= \text{total KE}$

$$U = \sum_{\alpha} U_\alpha + \sum_{\beta > \alpha} \bar{U}_{\alpha\beta}$$

U = total PE

U_α = total PE due to ext. forces; $\vec{F}_\alpha^{(e)} = -\vec{\nabla}_\alpha U_\alpha$

$\bar{U}_{\alpha\beta}$ = total PE due to internal forces

$\vec{f}_{\alpha\beta} = -\vec{\nabla}_\alpha \bar{U}_{\alpha\beta}$

$\bar{U}_{\alpha\beta} = \bar{U}_{\beta\alpha}$

$$\left(dW = \sum_{\alpha} \vec{F}_\alpha \cdot \vec{v}_\alpha \, dt = dT \right)$$

then $$\frac{dE}{dt} = \frac{\partial U}{\partial t}.$$

thus if $\vec{F}_\alpha^{(e)}$ & $\vec{f}_{\alpha\beta}$ are conservative

$\Rightarrow E = \text{const.}$

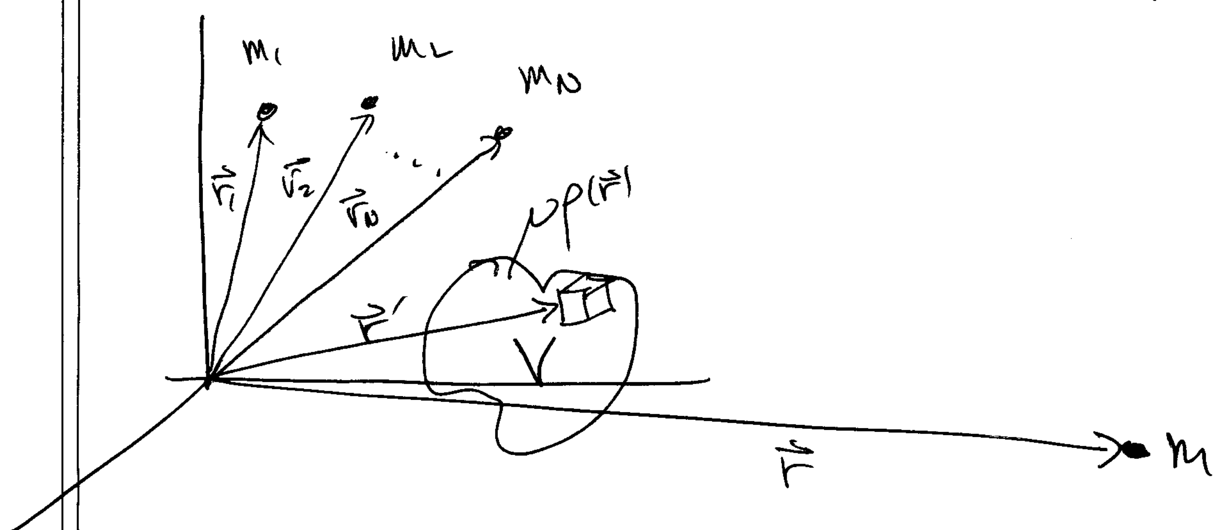
III) Newton's Universal Law of Gravitation

1) $$\vec{F}_{12} = \overset{\text{attractive}}{-}\, G \frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_1 - \vec{r}_2)$$

grav. force on particle 1 due to particle 2.

2) $\vec{F}_{12} = -\vec{F}_{21}$; 1) $G = 6.67 \times 10^{-8} \frac{\text{dyne-cm}^2}{\text{gm}^2}$

3) The gravitational force obey law of superposition



$$\vec{F}_m = -Gm \left\{ \sum_{i=1}^{N} \frac{m_i}{|\vec{r} - \vec{r}_i|^3} (\vec{r} - \vec{r}_i) + \int_V \frac{\rho(\vec{r}')(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d^3 r' \right\}$$

III 2) Gravitational field : $\vec{g}(\vec{r}) = \frac{\vec{F}}{m}$ ← test mass

0) units of acceleration

1) $$\vec{g}_m(\vec{r}) = -G\left\{\sum_{i=1}^{N_f} \frac{m_i}{|\vec{r}-\vec{r}_i|^3}(\vec{r}-\vec{r}_i) + \int_V \frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, d^3r'\right\}$$

Newton

2) As $\vec{F}$, $\vec{g}$ is conservative:

$$\vec{\nabla}\times\vec{g}(\vec{r}) = 0 \quad ; \quad \frac{\partial \vec{g}}{\partial t} = 0.$$

$$\Rightarrow \quad \vec{g}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$$

↑ gravitational potential

$$\Rightarrow \quad \phi(\vec{r}) = -G\left\{\sum_{i=1}^{N_f} \frac{m_i}{|\vec{r}-\vec{r}_i|} + \int_V \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3r'\right\}$$

III 3) Treat point masses as localized mass densities given by Dirac delta function $\delta(\vec{r})$

$$\delta(\vec{r}) = 0 \quad \text{for } \vec{r} \neq 0$$

$$\int_V d^3r\, \delta(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r}=0 \in V \\ 0 & \text{if } \vec{r}=0 \notin V \end{cases}$$

$$\Rightarrow \int_V d^3r'\, \delta(\vec{r}-\vec{r}')\, f(\vec{r}') = f(\vec{r})$$

Thus the mass density for a point mass m_i at position $\vec{r}_i$ is

$$\rho(\vec{r}) = m_i\, \delta(\vec{r}-\vec{r}_i)$$

For masses $m_1, \dots, m_N$ at $\vec{r}_1, \dots, \vec{r}_N$ plus continuous mass distribution $\hat{\rho}$

$$\rho(\vec{r}) = \sum_{i=1}^{N} m_i\, \delta(\vec{r}-\vec{r}_i) + \hat{\rho}(\vec{r})$$

then

$$\phi(\vec{r}) = -G \int_V d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

III.4) Since

$$\nabla^2 \frac{1}{|\vec{r}-\vec{r}\,'|} = -4\pi\,\delta(\vec{r}-\vec{r}\,')$$

$\Rightarrow$

$$\boxed{\begin{array}{l} 1)\ \vec{\nabla}\cdot\vec{g}(\vec{r}) = 4\pi G\rho(\vec{r}) \\ 2)\ \vec{\nabla}\times\vec{g}(\vec{r}) = 0 \end{array}}$$

&

gravitational field equations (Newtonian limit of Einstein GR field eq.)

integrate 1) + Gauss's theorem $\Rightarrow$ $\oint_S \vec{g}(\vec{r})\cdot d\vec{S} = 4\pi G M$

total mass in V

$M = \int_V \rho(\vec{r})\,d^3r$

5) Gravitational PE $\quad U = m\phi$

IV) Simple Harmonic Oscillator: single particle in one dimensional motion

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

$b\dot{x}$: frictional force
kx: harmonic osc. linear restoring force
$F(t)$: driving force

Need 2 B.C. since 2nd order Linear DE.

A) Solution to homogeneous eq. first

$$m\ddot{x} + b\dot{x} + kx = 0$$

or $$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$ with

$$\beta = \frac{b}{2m} \; ; \; \omega_0^2 = k/m$$

i) Underdamped: $\omega_0^2 > \beta^2$

$$x(t) = A e^{-\beta t} \cos(\omega_1 t + \theta)$$

$$\omega_1^2 \equiv \omega_0^2 - \beta^2 > 0$$

A, θ determined by 2 initial or B. Conditions.

IV A. 2) Critically Damped: $\omega_0^2 = \beta^2$

$$x(t) = (C_1 + C_2 t)e^{-\beta t}$$

C_1, C_2 determined by 2 B.C.

3) Overdamped Osc. $\beta^2 > \omega_0^2$

$$x(t) = e^{-\beta t}\left[C_1 e^{\omega_2 t} + C_2 e^{-\omega_2 t}\right]$$

$$\omega_2^2 = \beta^2 - \omega_0^2 > 0 .$$

C_1, C_2 determined by 2 B.C.

B) Driven Osc.: $m\ddot{x} + b\dot{x} + kx = F(t)$

$$x(t) = x_h + x_p$$

where $m\ddot{x}_h + b\dot{x}_h + kx_h = 0$ and is generically, for small damping, given by $x_h(t) = Ae^{-\beta t}\cos(\omega_1 t + \delta)$

A, δ determined by 2 B.C.

This is the homogeneous solution or transient sol.

IV B) $x_p(t)$ is the particular solution which yields $m\ddot{x}_p + b\dot{x}_p + kx_p = F$. It is a steady state sol. and is indep. of B.C.

Different methods for finding $x_p(t)$

1) First solve $m\ddot{x} + b\dot{x} + kx = F_0 \begin{pmatrix} \cos(\omega t + \theta_0) \\ \sin(\omega t + \theta_0) \end{pmatrix}$

$$x_p(t) = \frac{F_0/m}{\sqrt{[\omega_0^2 - \omega^2]^2 + 4\beta^2\omega^2}} \begin{pmatrix} \cos(\omega t + \theta_0 - \delta) \\ \sin(\omega t + \theta_0 - \delta) \end{pmatrix}$$

$$\tan\delta = \left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

a) $x_p(t)$ has maximum at amplitude resonance freq.

$$\omega = \omega_R = \sqrt{\omega_0^2 - 2\beta^2}.$$

b) $\left\langle \frac{dW}{dt} \right\rangle = \frac{1}{2} F_0 \dot{x}_{max} \sin\delta$ max at $\omega = \omega_0$

power absorption.

IV B 2) Solution by Principle of superposition for particular solution

if $L x_n(t) = F_n(t)$

$$L = m\frac{d^2}{dt^2} + b\frac{d}{dt} + k$$

then the solution of $L x(t) = F(t)$

with $F(t) = \sum_n F_n(t)$

is simply $x(t) = \sum_n x_n(t)$.

3) For periodic driving forces

$F(t+\tau) = F(t)$ $\tau = \frac{2\pi}{\omega}$ = period

then Fourier's Thm. $\Rightarrow$

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos n\omega t + b_n \sin n\omega t\right]$$

with $a_n = \frac{2}{\tau}\int_0^\tau dt\, F(t) \cos n\omega t$

$b_n = \frac{2}{\tau}\int_0^\tau dt\, F(t) \sin n\omega t$

IV.B3) Then the general solution to

$$m\ddot{x} + b\dot{x} + kx = F(t) \text{ is given by}$$

$$x(t) = A e^{-\beta t} \cos(\omega_1 t + \theta) + x_p(t)$$

$$x_p(t) = \frac{a_0}{m}\frac{1}{2\omega_0} + \sum_{n=1}^{\infty} \frac{\frac{a_n}{m}\cos(\omega_n t - \delta_n) + \frac{b_n}{m}\sin(\omega_n t - \delta_n)}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\beta^2\omega_n^2}}$$

where $\omega_n = n\omega = \frac{2\pi n}{\tau}$

$$\tan\delta_n = \left(\frac{2\omega_n\beta}{\omega_0^2 - \omega_n^2}\right)$$

with A, θ determined by 2 B.C.

4) General expansion in terms of complete sets of othogonal functions

If $\{e_i(t)\}$ is a complete set of functions over interval concerned $t_0 \leq t \leq t_1$

Then $F(t) = \sum_i f_i \, e_i(t)$

where $f_i = \langle e_i | F \rangle = \int_{t_0}^{t_1} dt \, e_i(t) F(t)$

VII. B. 4)

ex. Legendre Poly. $P_n(t)$
Hermite Poly $H_n(t)$
Eigenfunction of Diff. L

Note: 1) $\langle e_i | e_j \rangle = \int_{t_0}^{t_1} dt\, e_i(t)\, e_j(t) = \delta_{ij}$
orthonormality

2) Completeness: $\sum_i |e_i(t)\rangle\langle e_i(t')| = \sum_i e_i(t)\, e_i(t')$
$= \delta(t'-t)$

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5) In particular consider particular solutions to $L x(t) = F(t)$ by means of expansions in terms of eigenfunctions of L

$$L u_n(t) = \lambda_n u_n(t)$$

eigenfunctions

eigenvalues assume $\lambda_n \neq 0$

+ B.C.

IV B5) $u_n(t)$ are normalized

$$\int_{t_0}^{t_1} dt\, u_m(t) u_n(t) = \delta_{mn}$$

and complete

$$\sum_n{}' u_n(t) u_n(t') = \delta(t-t')$$

Then the general solution to $Lx(t) = F(t)$ + B.C. is given by

$$x(t) = \sum_n{}' \frac{f_n}{\lambda_n} u_n(t)$$

where

$$F(t) = \sum_n{}' f_n u_n(t)$$

$$f_n = \int_{t_0}^{t_1} dt\, F(t) u_n(t)$$

$x(t)$ has B.C. built in since $u_n(t)$ does.

IV.B6.) general Solution to $L_t x(t) = F(t) + B.C.$

by Green function methods:

The Green function $G(t,t')$ for L_t is defined as a solution of

$$L_t G(t,t') = \delta(t-t')$$

plus causal initial conditions

$$G(t,t') = 0 \quad \text{for } t \leq t'$$

$$\dot{G}(t,t') = 1 \quad \text{for } t = t'+\epsilon$$

Then the most general solution to $L_t x(t) = F(t)$

$\Rightarrow$

$$x(t) = \int_{t_0}^{t_1} dt'\, G(t,t')\, F(t')$$

$$+ A\cos(\omega_1 t + \theta)$$

if particle is initially at rest at $x=0$, $A=0$

other conditions need A, θ.

III.B) 1) in general just solve $L_t G = \delta$ directly

or

2) Eigenfunction expansion for G

$$G(t,t') = \sum_n \frac{1}{\lambda_n} x_n(t)\, x_n(t')$$

$$L_t x_n(t) = \lambda_n x_n(t).$$

C) Green function for Damped HO:

$$\left(\frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega_0^2\right] G(t,t') = \delta(t-t')$$

$$G(t,t') = \begin{cases} \frac{1}{\omega_1} e^{-\beta(t-t')} \sin\omega_1(t-t') & t \geq t' \\ 0 & t \leq t' \end{cases}$$

$$= \frac{1}{\omega_1} \Theta(t-t')\, e^{-\beta(t-t')} \sin\omega_1(t-t_1)$$

So

$$x(t) = A e^{-\beta t} \cos(\omega_1 t + \theta)$$
$$+ \frac{1}{m\omega_1} \int_{-\infty}^{t} dt'\, e^{-\beta(t-t')} \sin\omega_1(t-t') F(t')$$

Jc) where

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{1}{m} F(t)$$

& if $X(0) = \dot{x}(0) = 0$ $\quad A = \theta = 0$

otherwise we need A, θ to be determined by B.C.