Physics 271 Electricity & Magnetism

Review Notes

I) Electrostatics

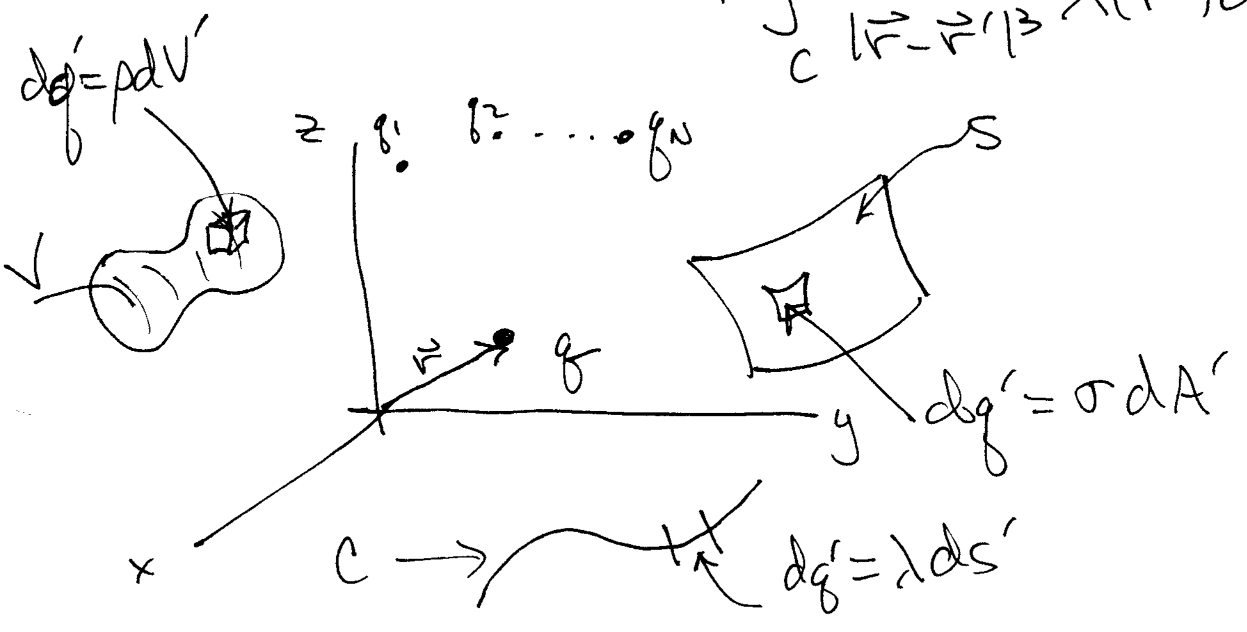
A) Coulomb's Law: Experimentally observed force between charges

$$\vec{F}_q = \frac{q}{4\pi\epsilon_0}\left\{ \sum_{i=1}^{N} \frac{q_i}{|\vec{r}-\vec{r}_i|^2}\frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|} \right.$$

$$+ \int_V \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\rho(\vec{r}\,')dV'$$

$$+ \int_S \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\sigma(\vec{r}\,')dA'$$

$$\left. + \int_C \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\lambda(\vec{r}\,')ds' \right\}$$



$$\left(\epsilon_0 = 8.85\times10^{-12}\frac{C^2}{N\cdot m^2} = 8.85\ pF/m\right)$$

IA1) SI units $\frac{1}{4\pi\epsilon_0} = 8.99\times10^{9}\ \frac{N\cdot m^2}{C^2}$

2) Principle of superposition: Total force is vector sum of pairwise individual forces on q.

B) Electric Field: $\vec{E} = \lim_{q_0\to 0}\frac{\vec{F}_{q_0}}{q_0}$

with q_0 a positive test charge

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left\{\sum_{i=1}^{N} q_i\frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3} + \int_V \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\rho(\vec{r}\,')\,dV' + \int_S \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\sigma(\vec{r}\,')\,dA' + \int_C \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\lambda(\vec{r}\,')\,ds'\right\}$$

1) As did the force, $\vec{E}$ obeys the principle of superposition, the total Electric field is the vector sum of the individual charges) electric fields

IB2) Lines of Force (= field lines): a) end & start on charges, b) tangent to line is direction of $\vec{E}$ (as a positive test charge would move), c) # of lines / unit cross-sectional area is proportional to magnitude of $\vec{E}$ in that region, d) lines are continuous and do not cross in charge free regions of space

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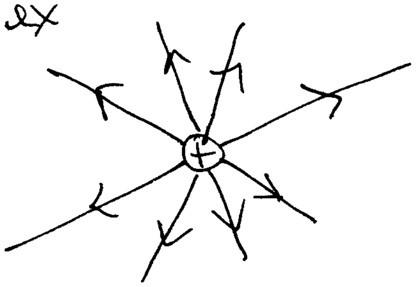

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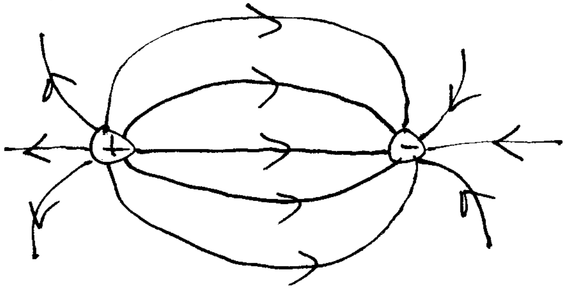
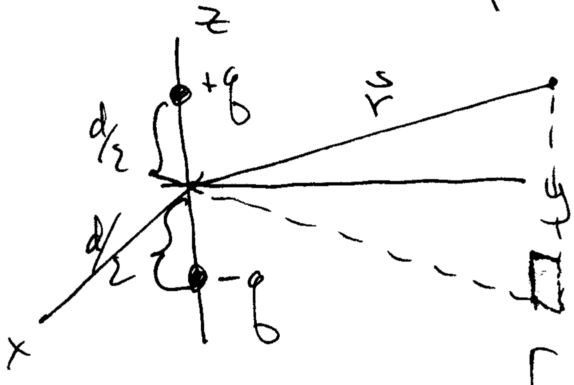


3) Electric Dipole:



$\vec{p} \equiv q d \hat{k}$, $\vec{E}$ given by $\vec{p}$

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0} \times \left[\frac{\vec{r} - \frac{d}{2}\hat{k}}{(x^2+y^2+(z-\frac{d}{2})^2)^{3/2}} - \frac{\vec{r} + \frac{d}{2}\hat{k}}{(x^2+y^2+(z+\frac{d}{2})^2)^{3/2}} \right]$$

IB3) $\Rightarrow$ a) $\vec{E}(x,y=0,z=0) = \dfrac{-\vec{p}}{4\pi\epsilon_0\left(x^2+\left(\frac{d}{2}\right)^2\right)^{3/2}}$

b) $x >> d$: Taylor Expand: $\epsilon << 1$

$$f(\epsilon) = f(0) + \epsilon f'(0) + \frac{\epsilon^2}{2} f''(0) + \ldots$$

$$\vec{E}(x,y=0,z=0) \overset{x>>d}{\approx} \frac{-\vec{p}}{4\pi\epsilon_0 x^3}$$

c) General Result for $r >> d$

$$\vec{E}(\vec{r}) \approx \frac{1}{4\pi\epsilon_0}\left[3\frac{\vec{p}\cdot\vec{r}}{r^5}\vec{r} - \frac{\vec{p}}{r^3}\right]$$

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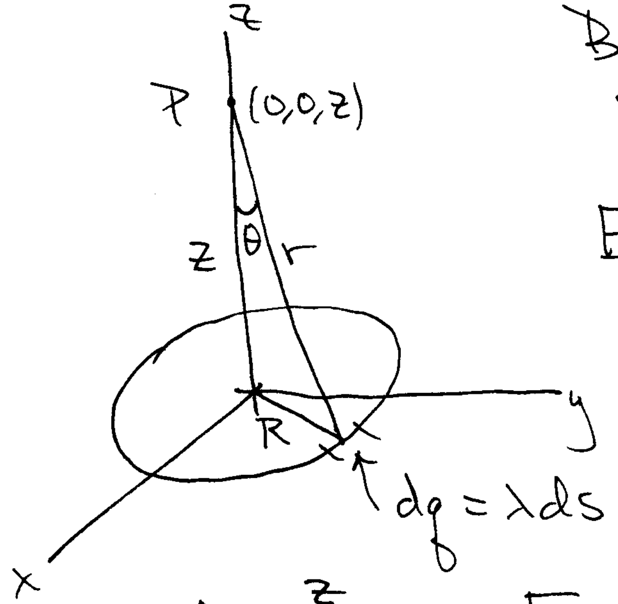
4) Ring of Charge:



By symmetry $E_x = E_y = 0$ at P!

$$E_z = \int_{\text{ring}} dE_z$$

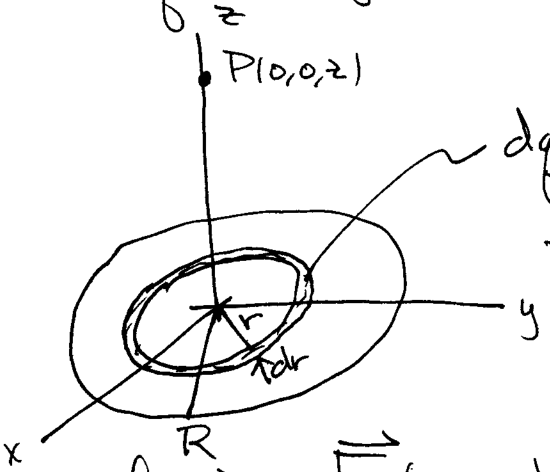
$$= \frac{\lambda}{4\pi\epsilon_0}\int_{\varphi=0}^{2\pi}\frac{R\,d\varphi}{(z^2+R^2)}\cos\theta$$

where $\cos\theta = \frac{z}{r}$

$$E_z \doteq \frac{q}{4\pi\epsilon_0}\frac{z}{(z^2+R^2)^{3/2}}$$

with $q = 2\pi R\lambda$

IB5) Disk of Charge:



$$dq = \sigma\, dA$$
$$= \sigma (2\pi r)\, dr$$

Ring of Charge

$$d\vec{E}(0,0,z) = \frac{dq\, z\, \hat{k}}{4\pi\epsilon_0 (z^2+r^2)^{3/2}}$$

integrate ⇒ $$\vec{E}(0,0,z) = \frac{2\pi\sigma z\,\hat{k}}{4\pi\epsilon_0}\int_{r=0}^{R}\frac{r\,dr}{(z^2+r^2)^{3/2}}$$

$$\vec{E}(0,0,z) = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right]\hat{k} \quad (z>0)$$

a) Close to disk $R >> z$: $$\frac{z}{\sqrt{z^2+R^2}} = \frac{z/R}{\sqrt{1+(z/R)^2}}$$
$$\approx 0$$

$$\Rightarrow \quad \vec{E} = \frac{\sigma}{2\epsilon_0}\hat{k}$$

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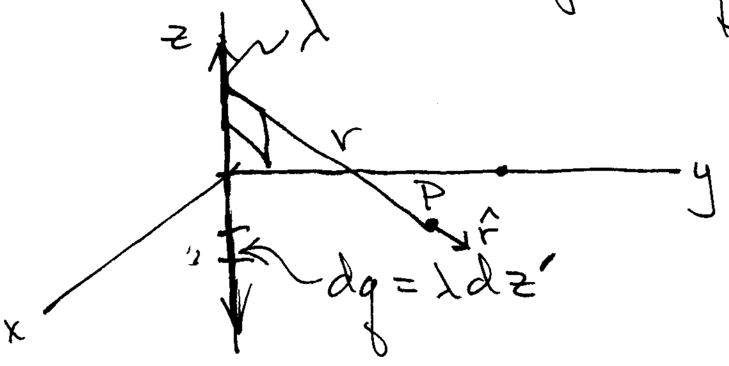
6) Line of Charge: By symmetry $\vec{E}$ is ⊥ to z-axis



$$\vec{E} = E\,\hat{r}$$

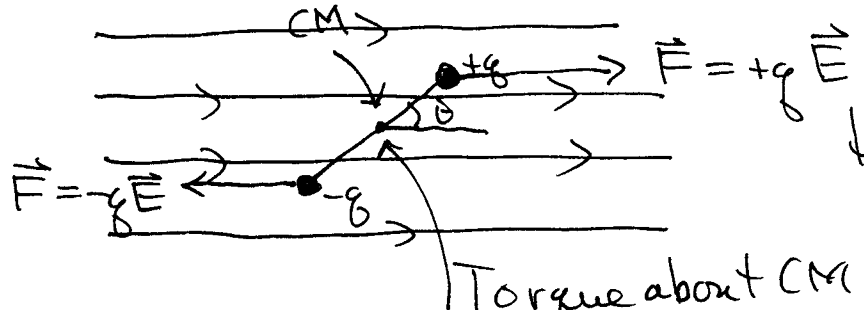
$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$

I C) Given the electric field $\vec{E}(\vec{r})$ at a point in space $\vec{r}$, the electric force of a point charge q located at $\vec{r}$ is simply

$$\vec{F}_q = q\vec{E}.$$

If classical, non-relativistic particle, it will then move according to Newton's 2nd Law $\vec{F}_q = m\vec{a}$.

1) Dipole in a uniform $\vec{E}$-field



Torque tends to align $\vec{p}$ with $\vec{E}$

Torque about CM $\underset{\otimes}{\vec{\tau}}$ points into page.

$$\vec{\tau} = \sum_i \vec{r}_i \times \vec{F}_i$$

$$|\vec{\tau}| = \tau = F\frac{d}{2}\sin\theta + F\frac{d}{2}\sin\theta$$
$$= Fd\sin\theta$$

$$F = qE \implies \tau = qdE\sin\theta$$
$$= pE\sin\theta$$

Which vectorially becomes

$$\boxed{\vec{\tau} = \vec{p}\times\vec{E}}$$

final ∡ $\quad \rightarrow -$

IC 1) Work done by electric field $= W = \displaystyle\int_{\theta_0}^{\theta} \vec{\tau}\cdot d\vec{\theta} = \int_{\theta_0}^{\theta} -\tau\, d\theta$

initial ∡ $\qquad\qquad$ $\vec{\tau}$ & $d\vec{\theta}$ opposite directions since ∡ θ decreases

$$W = -pE\int_{\theta_0}^{\theta} \sin\theta\, d\theta$$

$$W = pE\left[\cos\theta - \cos\theta_0\right],$$

Change in potential energy of system

$$\Delta U = U(\theta) - U(\theta_0) = -W$$

$$= -pE\left[\cos\theta - \cos\theta_0\right]$$

Choose arbitrary reference point $\theta = \frac{\pi}{2}$ for zero of PE : $U(\frac{\pi}{2}) \equiv 0$.

$$\Rightarrow \quad \boxed{U(\theta) = -pE\cos\theta = -\vec{p}\cdot\vec{E}}$$

PE is a minimum when $\vec{p}$ & $\vec{E}$ align.

I.D.) Gauss' Law:

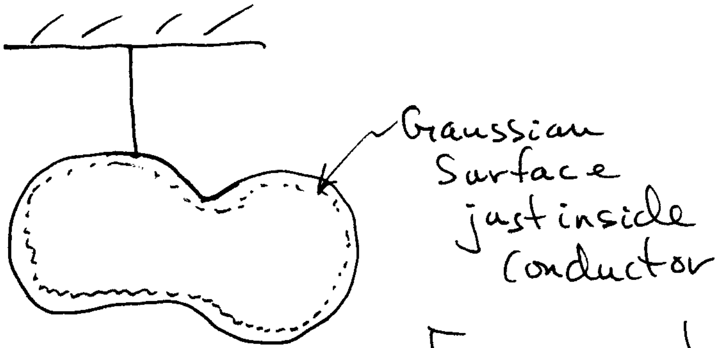
$$\oint_S \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0} Q_{enclosed}$$

S = closed Gaussian surface bounding volume V
$Q_{enclosed}$ = all charge inside Gaussian surface
i.e. in volume V.

$$\Phi_E = \oint_S \vec{E}\cdot d\vec{A} = \text{flux of } \vec{E} \text{ through } S$$

1) Isolated conductor with excess charge q

a)

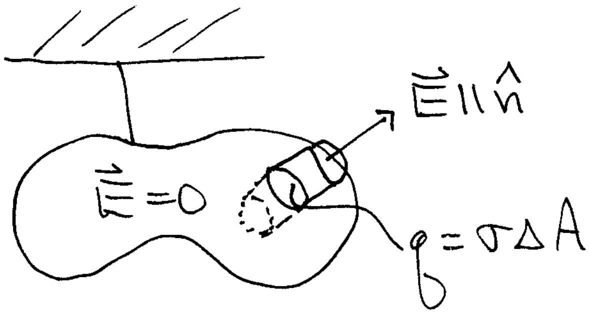


Experimental fact:
no currents in conductor ⇒ $\vec{E}=0$ inside conductor

Hence $\Phi_E = 0 = \frac{1}{\epsilon_0} Q_{enclosed}$

⇒ **All q on outer surface of conductor**

ID1b)



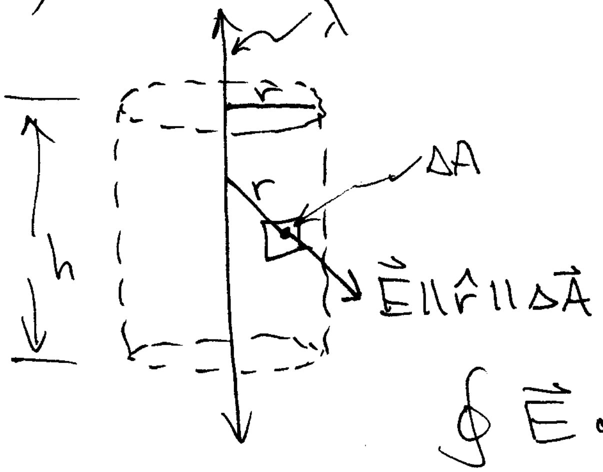
$\vec{E}\perp$ to surface (no currents)

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{enclosed}}{\epsilon_0}$$

$$E\Delta A = \frac{\sigma\Delta A}{\epsilon_0}$$

$$\Rightarrow \boxed{\vec{E}=\frac{\sigma}{\epsilon_0}\hat{n}}$$

2) Infinite line of charge:



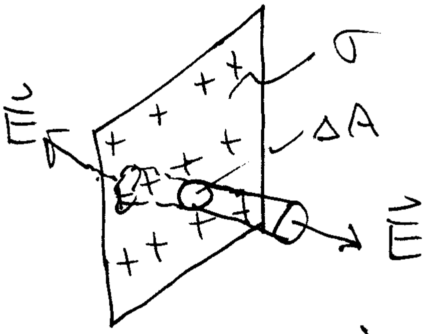
Cylindrical symmetry
$\Rightarrow \vec{E}=E(r)\,\hat{r}$

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0}Q_{enclosed}$$

$$\underbrace{0+0}_{\vec{E}\perp d\vec{A}\text{ on endcaps}}+E(r)2\pi r h = \frac{1}{\epsilon_0}\lambda h$$

$$\Rightarrow \boxed{E(r)=\frac{\lambda}{2\pi\epsilon_0 r}}$$

ID 3) Infinite Sheet of Charge



Planar symmetry $\vec{E} \perp$ plane

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0} Q_{enclosed}$$

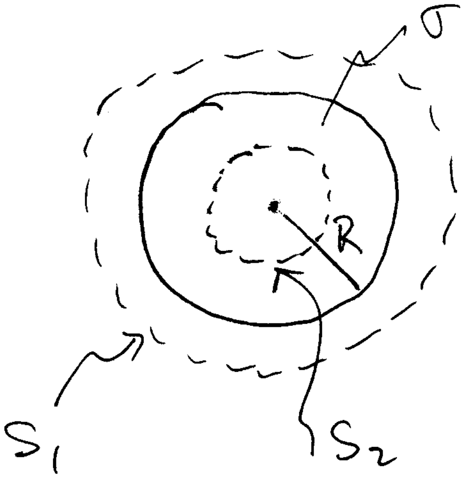
$$0 + E\Delta A + E\Delta A = \frac{1}{\epsilon_0}\sigma\Delta A$$

cylindrical side

2 endcaps

$$\Rightarrow \quad E = \frac{\sigma}{2\epsilon_0} \quad \text{\& } \perp \text{ to sheet}$$

4) Spherical Shell of Charge



Spherical symmetry

$$\Rightarrow \vec{E} = E(r)\hat{r}$$

1) $S_1 : r \geq R$

$$\oint_{S_1} \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0} Q_{enclosed}$$

$$E(r)\,4\pi r^2 = \frac{1}{\epsilon_0}\underbrace{4\pi R^2\sigma}_{\equiv q}$$

$$\Rightarrow \boxed{\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}}$$

I D 4 2) S_2: $r<R$ No charge is enclosed in S_2

$$\oint_{S_2} \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0} Q_{enclosed} = 0$$

$$\Vert$$

$$E(r)\oint_{S_2} dA = E(r)\,4\pi r^2 = 0$$

$$\Rightarrow \boxed{\vec{E} = 0 \text{ for } r<R \text{ inside shell.}}$$

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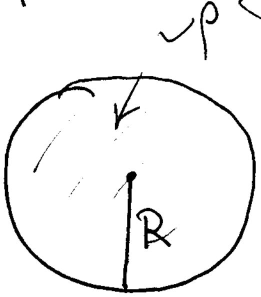
5) Spherically Symmetric Charge Distribution:



Spherical symmetry ⇒

$$\vec{E} = E(r)\,\hat{r}$$

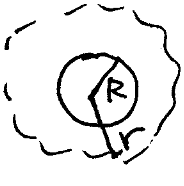
1) $r \geq R$ a) Add up shells $dq = 4\pi r'^2 dr' \rho$

$$E = \int_0^R \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2} = \frac{\rho}{4\pi\epsilon_0 r^2}\underbrace{\int_0^R 4\pi r'^2 dr'}_{=\frac{4}{3}\pi R^3}$$

$$\boxed{E = \frac{q}{4\pi\epsilon_0 r^2}\ ;\ r>R \quad q = \frac{4}{3}\pi R^3 \rho}$$

Acts like pt. q at center for $r \geq R$

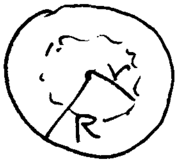
b) Gauss' Law directly



$$\oint_S \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0} Q_{enclosed}$$

$$= E(r)\,4\pi r^2 = \frac{1}{\epsilon_0} q \quad \Rightarrow \boxed{E = \frac{q}{4\pi\epsilon_0 r^2},\ r>R}$$

ID5 2) $r \leq R$ Gauss's law:



$$\oint_S \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} Q_{\text{enclosed}}$$

$$\| \qquad\qquad \|$$

$$E(r) \oint_S dA = \frac{1}{\epsilon_0}\left(\frac{4}{3}\pi r^3 \rho\right)$$

$$\|$$

$$E(r)\, 4\pi r^2 = \frac{1}{\epsilon_0}\left(\frac{4}{3}\pi R^3 \rho\right)\left(\frac{r^3}{R^3}\right)$$

$$\Rightarrow \boxed{E(r) = \frac{q}{4\pi\epsilon_0 R^2}\left(\frac{r}{R}\right), \quad r \leq R}$$

Note: $r = R$ both expressions must be =, and they are $E(R) = \frac{q}{4\pi\epsilon_0 R^2}$

E.) Potential Energy: Coulomb force is conservative

$\Rightarrow$

Work to go a→b $= W_{ab} = \int_a^b \vec{F} \cdot d\vec{s}$ is path independent

Potential Energy difference $\Delta U = U_b - U_a$
$\equiv - W_{ab}$

$$\Delta U = U_b - U_a \equiv -\int_a^b \vec{F} \cdot d\vec{s}$$

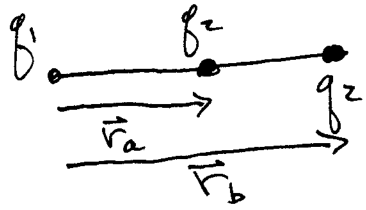
I.E) Potential Energy at a point $= U_b \equiv -W + U_a$

where U_a is an arbitrarily defined value at the reference point a. Usually choose $U_a \equiv 0$ at point a. For finite charges we choose $a = \infty$ i.e. all charges infinitely separated and $U_\infty \equiv 0$.

For $\vec{F} = q\vec{E}$ we have

$$\boxed{\Delta U = U_b - U_a = -q\int_a^b \vec{E}\cdot d\vec{s}}$$

1) Two point charges: q_1 fixed, q_2 moves from $\vec{r}_a$ to $\vec{r}_b$

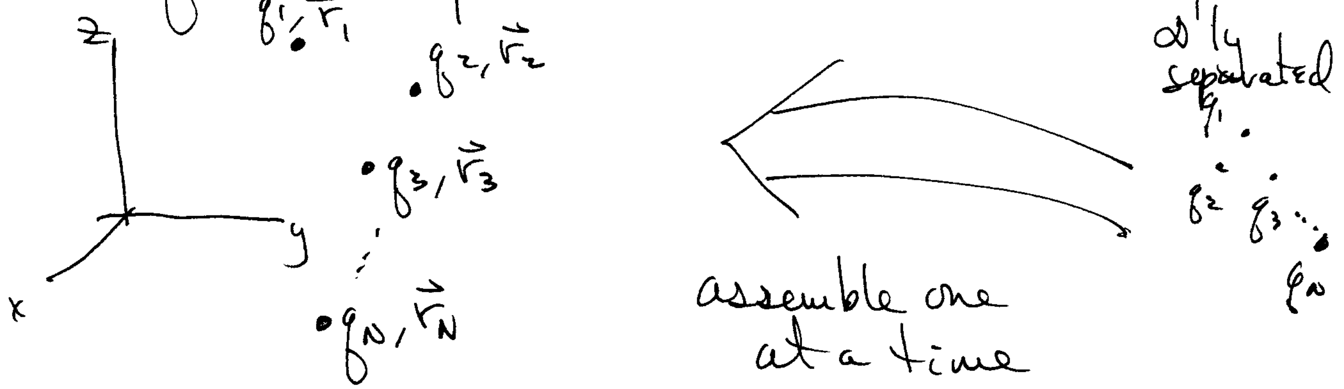


$$U_b - U_a = \frac{q_1 q_2}{4\pi\epsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$

Let $r_a \to \infty$ and define $U_a = 0$ there

$$\Rightarrow \quad \boxed{U(r) = \frac{q_1 q_2}{4\pi\epsilon_0 r}} \quad (r = \text{separation of } q_1 \text{ \& } q_2)$$

I.E.2) System of point charges: Potential energy equals the work done by an external force to assemble the charge distribution from ∞ separation to final locations



$$U = U_1 + U_2 + \cdots + U_N$$

$$= \sum_{i=1}^{N} U_i$$

work done to bring q_i in when $q_1, \dots, q_{i-1}$ are already in place.

$$U = \sum_{i=1}^{N} \sum_{j=1}^{i-1} \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}$$

$$= \frac{1}{2} \sum_{i=1}^{N} q_i \underbrace{\left(\sum_{j=1}^{N}{}' \frac{q_j}{4\pi\epsilon_0 |\vec{r}_i - \vec{r}_j|} \right)}_{\equiv V(\vec{r}_i)}$$

skip $i=j$

= electric potential at $\vec{r}_i$ due to other charges $q_1 \cdots q_N$ excluding q_i

I.F.) Electric Potential: $V(\vec{r}) \equiv \lim_{q_0 \to 0} \frac{U(\vec{r})}{q_0}$

V = PE/unit charge

$$\boxed{\Delta V = V_b - V_a = -\int_a^b \vec{E} \cdot d\vec{s}}$$

1) SI units: $1\,\text{volt} \equiv \frac{1\,\text{Joule}}{1\,\text{Coulomb}} \Rightarrow E\,\text{units} = \frac{1\,\text{Volt}}{1\,\text{m}} = \frac{1\text{N}}{1\text{C}}$

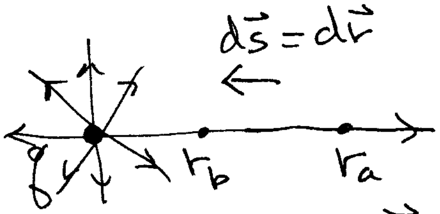
2) Move charge q between 2 points at fixed potential difference ΔV then the PE difference is

$$\Delta U = q \Delta V.$$

3) Point Charge q:

$$V_b - V_a = -\int_a^b \vec{E} \cdot d\vec{s}$$

$$= -\int_{r_a}^{r_b} \frac{q}{4\pi\epsilon_0} \frac{dr}{r^2}$$



$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3}$$

$$\boxed{V_b - V_a = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)}$$

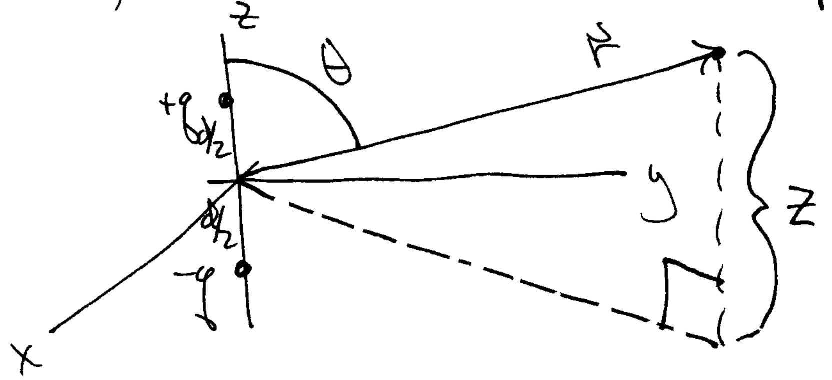
Convention: $r_a = \infty \quad V_a \equiv 0$

$$\boxed{V(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{r}}$$

IF4) Principle of superposition: $V = \sum_{i=1}^{N} V_i = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i}$

<u>Scalar</u> addition of individual charge potentials.

5) Potential due to a dipole: use superposition of 2 pt. charges



$\hat{r}\cdot\hat{k} = \cos\theta = \frac{z}{r}$

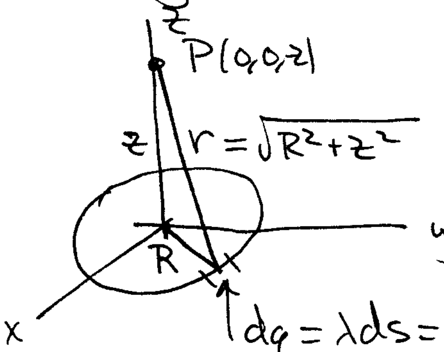
$$V = V_+ + V_- = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{r}-\frac{d}{2}\hat{k}|} - \frac{1}{|\vec{r}+\frac{d}{2}\hat{k}|}\right]$$

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0 r}\left[\frac{1}{\sqrt{1-\frac{d}{r}\cos\theta+\frac{d^2}{4r^2}}} - \frac{1}{\sqrt{1+\frac{d}{r}\cos\theta+\frac{d^2}{4r^2}}}\right]$$

a) $r >> d$: $\frac{1}{\sqrt{1+\epsilon}} = 1 - \frac{1}{2}\epsilon + \cdots$

$$\boxed{V = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3}}$$

I.F 6) Ring of charge



$dq = \lambda ds = \lambda R d\varphi$

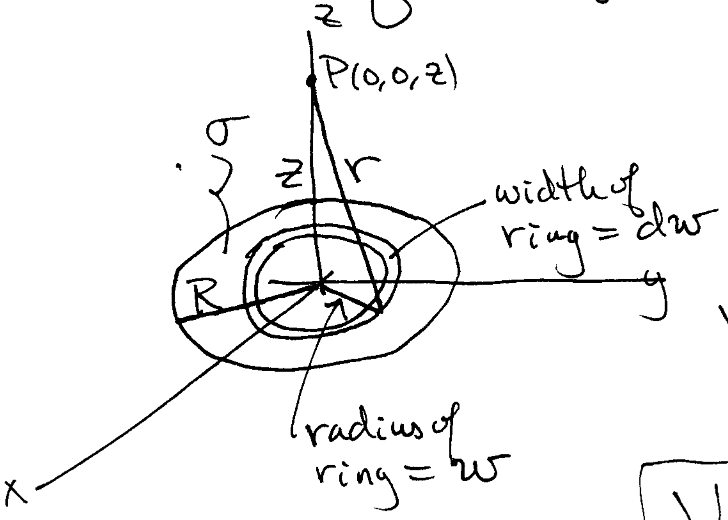
$$V(0,0,z) = \frac{1}{4\pi\epsilon_0}\int_{ring} \frac{dq}{r}$$

r is same around ring

$$= \frac{1}{4\pi\epsilon_0 r}\int_{ring} dq$$

$$\frac{\overset{=2\pi R\lambda}{q}}{4\pi\epsilon_0\sqrt{R^2+z^2}} = \boxed{V(0,0,z) = \frac{2\pi R\lambda}{4\pi\epsilon_0\sqrt{R^2+z^2}}}$$

7) Disk of Charge



Ring of charge

$$dV(P) = \frac{dq}{4\pi\epsilon_0 r}$$

$r = \sqrt{w^2+z^2}$

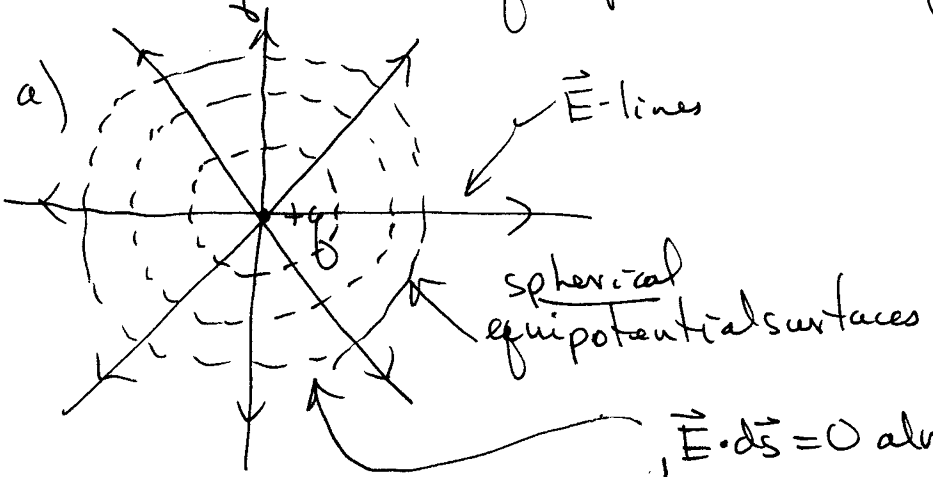
$$V(P) = \int_{w=0}^{R} dV(P)$$

$$\boxed{V(P) = \frac{\sigma}{2\epsilon_0}\left[\sqrt{R^2+z^2} - z\right]}$$

I.F. 8) Equipotential Surfaces: family of surfaces with each surface having the same electric potential.

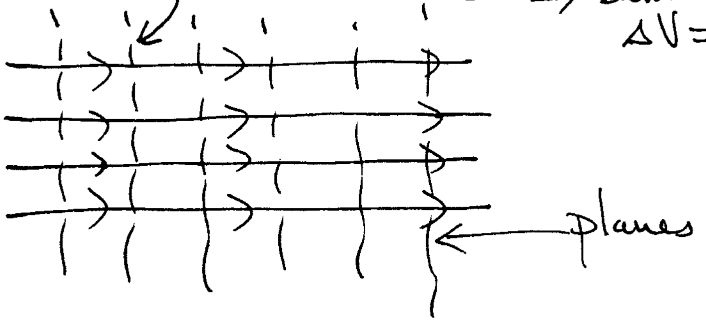
lines of $\vec{E} \perp$ equipotential surfaces

a)



$\vec{E}\cdot d\vec{s} = 0$ along surface $\Rightarrow$ same potential $\Delta V = 0$.

b) Uniform $\vec{E}$

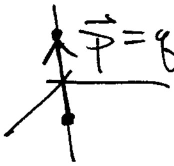


9) Gradient of electric potential.

$$\underset{\vec{\nabla} V\cdot d\vec{s}}{\underset{\parallel}{dV}} = -E\cdot d\vec{s} \quad \Rightarrow \quad \boxed{\begin{aligned} \vec{E} &= -\vec{\nabla} V \\ E_x &= -\frac{\partial V}{\partial x} \\ E_y &= -\frac{\partial V}{\partial y} \\ E_z &= -\frac{\partial V}{\partial z} \end{aligned}}$$

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

I.F.9a) Electric Dipole: $\boxed{V = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3}}$



$\vec{p} = qd\hat{k} = p\hat{k}$

$$= \frac{pz}{4\pi\epsilon_0 (x^2+y^2+z^2)^{3/2}}$$

$$\vec{E} = -\vec{\nabla} V$$

$$E_x = -\frac{\partial V}{\partial x} = \frac{3pzx}{4\pi\epsilon_0 r^5}$$

$$E_y = -\frac{\partial V}{\partial y} = \frac{3pzy}{4\pi\epsilon_0 r^5}$$

$$E_z = -\frac{\partial V}{\partial z} = -\frac{p}{4\pi\epsilon_0 r^3} + \frac{3pz^2}{4\pi\epsilon_0 r^5}$$

$$\Rightarrow \quad \boxed{\vec{E} = \frac{3(\vec{p}\cdot\vec{r})\,\vec{r}}{4\pi\epsilon_0 r^5} - \frac{\vec{p}}{4\pi\epsilon_0 r^3}}$$

---

10) All isolated conductor is an equipotential.

I G) Capacitors: two conducting plates carrying equal but opposite charge; plates are equipotentials.

$$\boxed{q = CV}$$

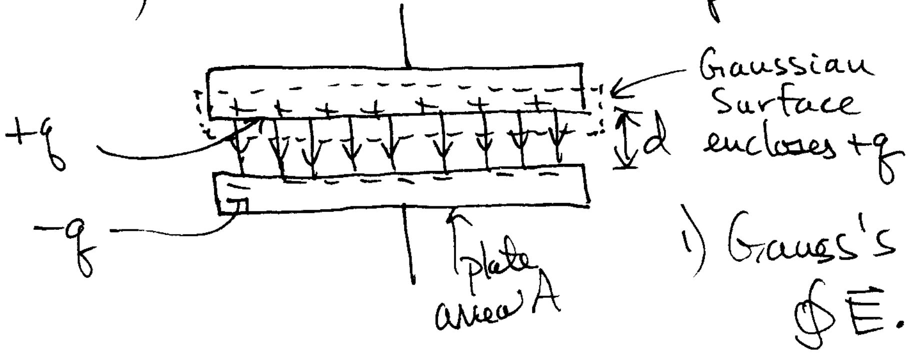
q = magnitude of charge

V = magnitude of Potential difference

C = capacitance of capacitor

1) SI units of capacitance 1 Farad $= \frac{1\text{ Coulomb}}{1\text{ Volt}}$.

2) Parallel Plate Capacitor: Neglect fringing



1) Gauss's Law

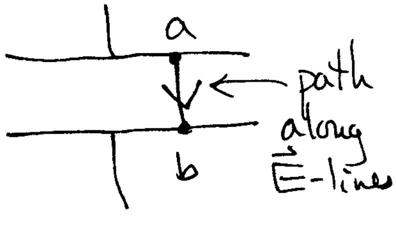
$$\oint_S \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0} q$$

$$\|$$

$$EA = \frac{q}{\epsilon_0}$$

$$E = \frac{q/A}{\epsilon_0} \left(= \frac{\sigma}{\epsilon_0} \right)$$

2) Electric Potential Difference



$$-V = V_b - V_a = -\int_a^b \vec{E}\cdot d\vec{s}$$

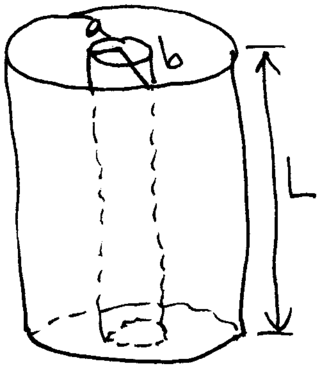
$$\Rightarrow +V = +\int_a^b \vec{E}\cdot d\vec{s} = E\int_0^d ds = Ed$$

IG 2.2) $V = Ed = \left(\frac{q}{\epsilon_0 A}\right) d = \left(\frac{d}{\epsilon_0 A}\right) q = \frac{1}{C} q$

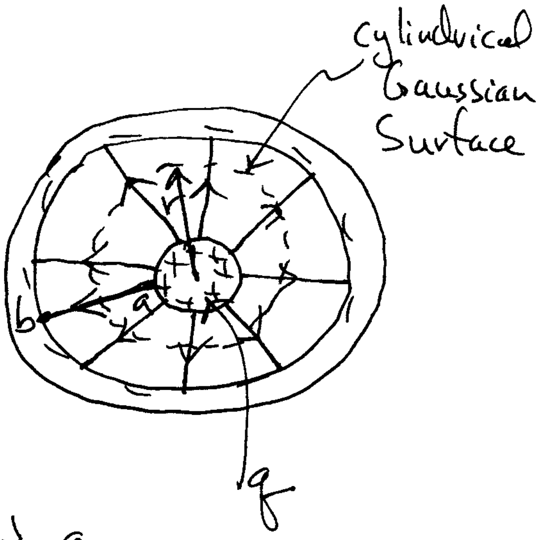
$$\Rightarrow \boxed{C = \frac{\epsilon_0 A}{d} \quad \text{||-plate capacitor}}$$

---

3) Cylindrical Capacitor:



L>>b to ignore fringing



1) Gauss' Law

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0} q$$

$$\|$$

$$\underbrace{0+0}_{\substack{\text{endcaps}\\ \vec{E}\perp d\vec{A}}} + E(r)\underbrace{2\pi r L}_{\substack{\text{cylindrical}\\ \text{area at}\\ \text{radius } r}}$$

$$\Rightarrow \quad E(r) = \frac{q}{2\pi\epsilon_0 r L}$$

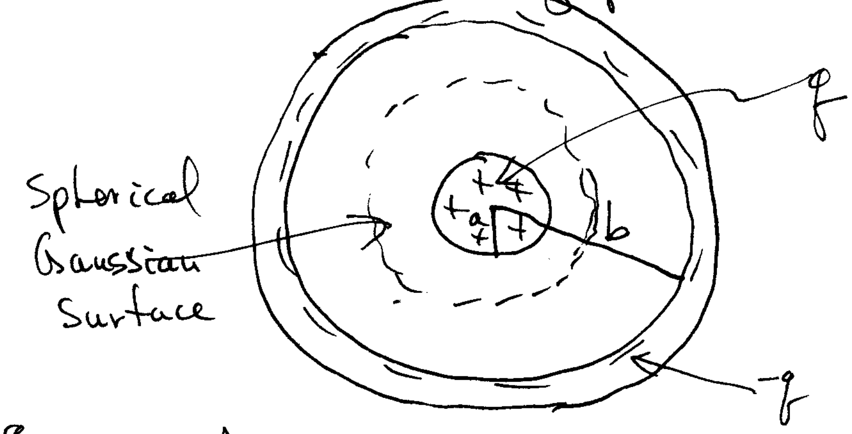
2) Potential Difference

$$V = \int_a^b \vec{E}\cdot d\vec{s} = \frac{q}{2\pi\epsilon_0 L}\int_a^b \frac{dr}{r}$$

$$= \frac{q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)$$

IG3) $$\Rightarrow \boxed{C = 2\pi\epsilon_0 \frac{L}{\ln(b/a)} \quad \text{cylindrical capacitor}}$$

4) Spherical Capacitor: Cross-Section of sphere



1) Gauss' Law

$\oint \vec{E} \cdot d\vec{A} = q/\epsilon_0$

S ||

$E(r)4\pi r^2$

$$\Rightarrow \vec{E}(r) = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$$

2) Potential difference:

$$V = \int_a^b \vec{E} \cdot d\vec{s} = \int_a^b E(r)dr = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

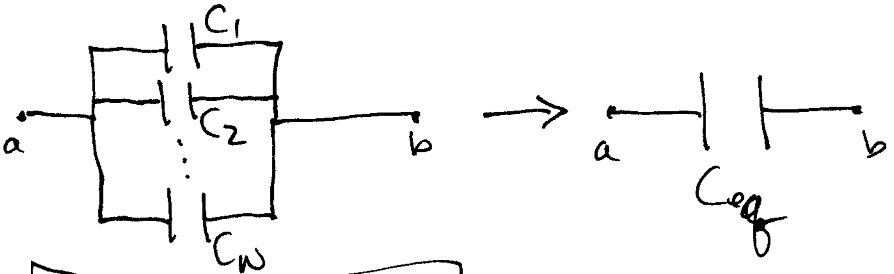
3) $$\boxed{C = q/V = 4\pi\epsilon_0\left(\frac{ab}{b-a}\right) \quad \text{spherical Capacitor}}$$

5) Isolated Sphere: let $b \to \infty$ above and call $a = R$

$$\boxed{C = 4\pi\epsilon_0 R \quad \text{isolated sphere}}$$

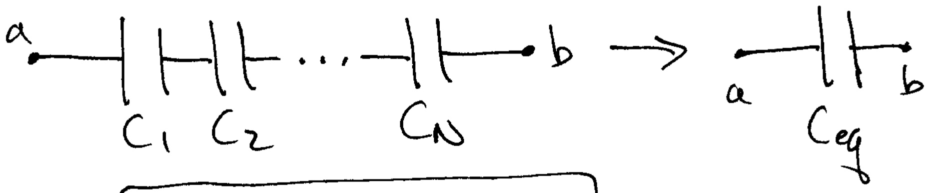
I.G.6) Equivalent Capacitance:

a) Capacitors in Parallel



$$\boxed{C_{eq} = \sum_{i=1}^{N} C_i}$$

b) Capacitors in Series



$$\boxed{\frac{1}{C_{eq}} = \sum_{i=1}^{N} \frac{1}{C_i}}$$

H.) Energy Storage in an Electric Field

Charge capacitor, at time t', q' charge is on plates with potential difference V' across them, so add dq' more charge the increase in PE is

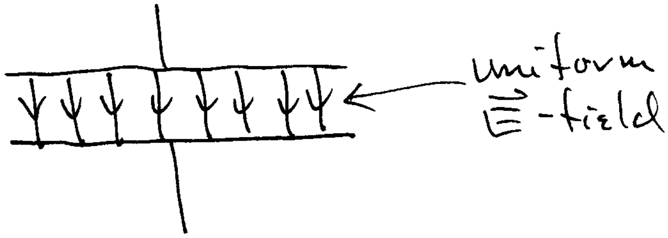
$$dU = dq' V' \quad \text{but } V' = q'/C$$

I.H) $dU = \frac{q'\,dq'}{C}$; Continue until totally charged ⇒

$$U = \int_0^U dU = \int_0^q \frac{1}{C} q'\,dq' = \frac{1}{2} q^2/C$$

$$\boxed{U = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}CV^2}$$ since $q = CV$

1) ||-plate:



Energy density $\equiv u$ = const. between plates since $\vec{E}$ is uniform

$$u = \frac{U}{Ad} = \frac{\frac{1}{2}CV^2}{Ad} = \boxed{\frac{1}{2}\epsilon_0 E^2 = u}$$

since $C = \frac{\epsilon_0 A}{d}$ and $E = \frac{V}{d}$

|| If $\vec{E}$ exists at a point in space, then the energy density stored at that point is

$$\boxed{u = \frac{1}{2}\epsilon_0 |\vec{E}|^2}$$

general result

IH2) Isolated Conducting sphere with charge q

1) $U = \frac{q^2}{2C} = \frac{1}{2}\frac{q^2}{4\pi\epsilon_0 R}$

Total energy stored in $\vec{E}$-field

2) Energy density at point r from center

$$u = \frac{1}{2}\epsilon_0 E^2_{(r)} = \frac{q^2}{32\pi^2\epsilon_0 r^4}$$

3) One Half energy stored within radius R_0

$$dU = u\,dV = u(r)\,4\pi r^2 dr$$

$$= \frac{q}{8\pi\epsilon_0}\frac{dr}{r^2}$$

spherical shell
Volume $= dU$
$= (4\pi r^2) dr$

Require

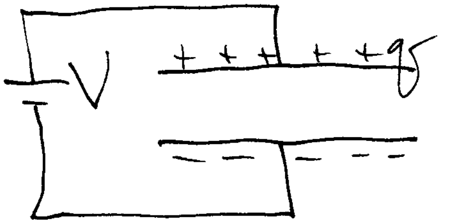
$$\int_R^{R_0} dU = \frac{1}{2}\int_R^{\infty} dU$$

$$\Rightarrow \boxed{R_0 = 2R}$$

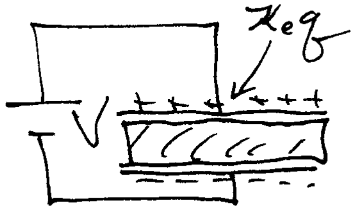
I.I.) Dielectrics: Intermediate materials between conductors and insulators: the charged particles in dielectrics displace their positions slightly in response to an $\vec{E}$-field.

1) Faraday: Experimental results

a) V fixed



insert dielectric slab with battery attached V maintained



For same V, stored charge q increases

$\Rightarrow$ Capacitance $C = q/V$ increases

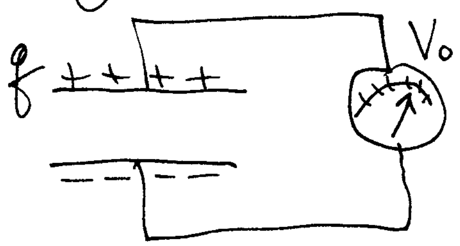
Define dielectric constant

$$\kappa_e \equiv \frac{C \leftarrow \text{with dielectric}}{C_0 \leftarrow \text{vacuum between plates}}$$

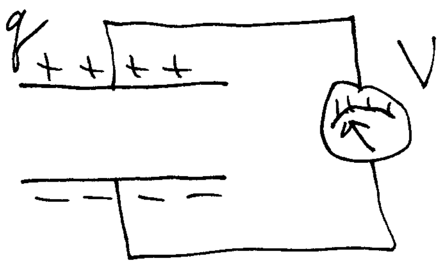
($\epsilon \equiv \kappa_e \epsilon_0$ = permittivity of material)

So $q = CV = \kappa_e C_0 V = \kappa_e q_0$.

- II 1b) q - fixed



insert
Dielectric
slab



$$q = C_o V_o$$

$$q = CV$$

but $C = \kappa_e C_o$

$$q = \kappa_e C_o V$$

$\Rightarrow$

$$q = C_o V_o = \kappa_e C_o V \Rightarrow \boxed{V = \frac{1}{\kappa_e} V_o}$$

for fixed charge q, the voltage decreases as the slab is inserted.

---

2) Fill Capacitors with dielectric $\kappa_e \Rightarrow$ $C = \kappa_e C_o$

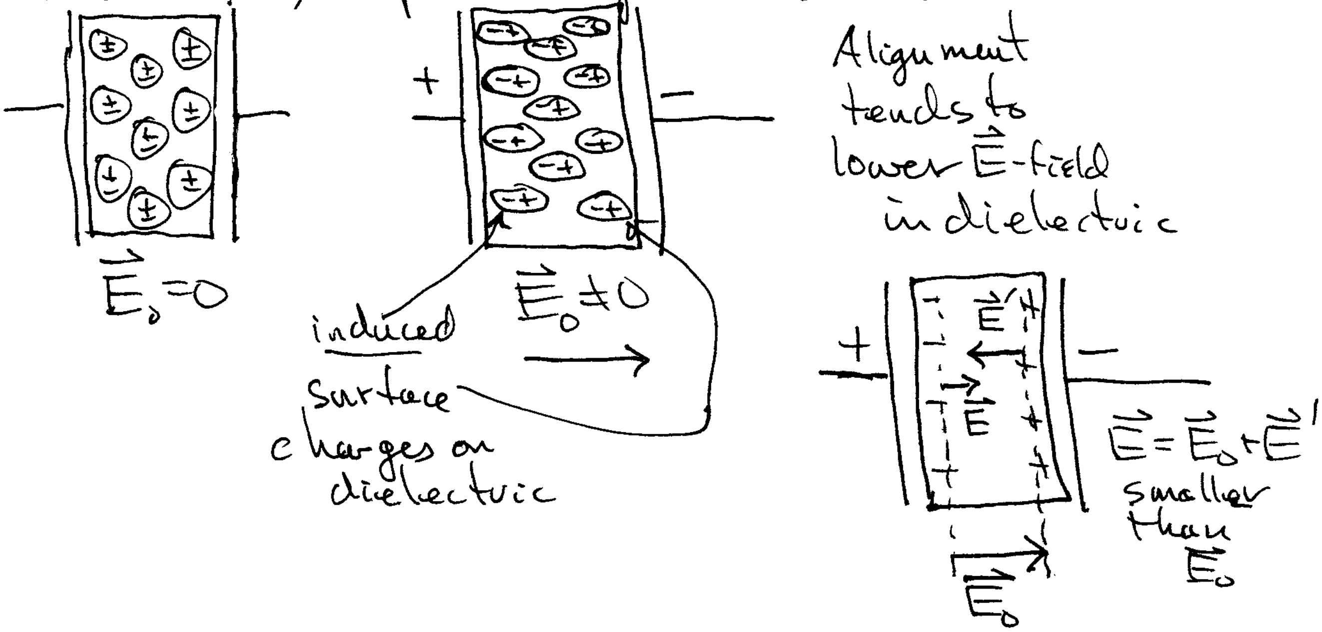
a) ‖-plate : $C = \frac{\kappa_e \epsilon_o A}{d}$

b) Cylindrical : $C = 2\pi \kappa_e \epsilon_o \frac{L}{\ln(b/a)}$

c) Spherical : $C = 4\pi \kappa_e \epsilon_o \frac{ab}{b-a}$

d) Isolated Sphere : $C = 4\pi \kappa_e \epsilon_o R$.

-I.I.3) Microscopic View of Dielectrics: Atomic (molecular) dipoles align in $\vec{E}$-field.

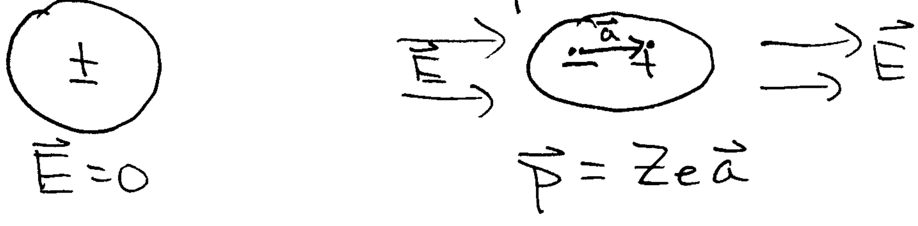


Alignment tends to lower $\vec{E}$-field in dielectric

$\vec{E} = \vec{E}_0 + \vec{E}'$ smaller than $\vec{E}_0$

a) Molecular dipole moment $\vec{p}$ & Polarization Charge

ex. Atomic # = Z



$$\vec{p} = Ze\vec{a}$$

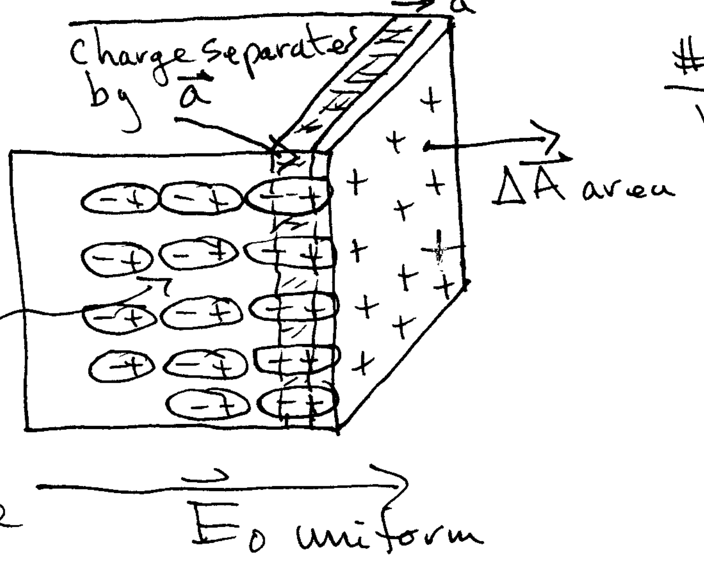
Polarization Vector $= \vec{P} \equiv N\vec{p} = \dfrac{\text{dipole moment}}{\text{Volume}}$

N: # of atoms / Volume

$\left(= \dfrac{1}{\Delta V} \sum_{\substack{\text{molecules} \\ \text{in } \Delta V}} \vec{p} \right.$

sum molecular dipole moments in macro small but micro large volume ΔV

This depends on size ΔV, so divide by ΔV $\left. \right)$



II.3a) Negative charges shift away from surface, no new negative charge enters surface since it is the external surface of the dielectric. Hence the box at the surface gains positive charge equal to the amount of negative charge that left the surface box

$$= \underbrace{N}_{\frac{\text{atoms}}{\text{Vol.}}}\underbrace{Ze}_{\frac{\text{Charge}}{\text{atom}}}\underbrace{(a\Delta A)}_{\text{Vol. of box}}$$

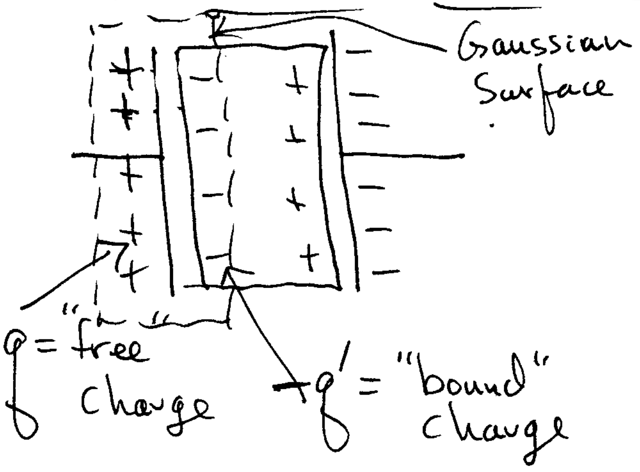
Hence the surface acquires a Polarization surface charge density

$$\sigma_P = \frac{NZe\,a\Delta A}{\Delta A} = NZea = N|\vec{p}| = |\underline{P}|$$

or in terms of vectors and in general

$$\boxed{\sigma_P \Delta A = \underline{\vec{P}} \cdot \Delta\vec{A}}$$

I.I.4) Gauss' Law Re-Visited



$$\underset{\overset{\|}{EA}}{\oint_S \vec{E}\cdot d\vec{A}} = \frac{1}{\epsilon_0}\left[q - q'\right]$$

$$\Rightarrow \boxed{E = \frac{q}{\epsilon_0 A} - \frac{q'}{\epsilon_0 A}}$$

(Vacuum: $E_0 = \frac{q}{\epsilon_0 A}$)

a) Charges

$$E = \frac{1}{\kappa_e} E_0$$ (Faraday's Exp. linear dielectric matrials)

$$\| \qquad\qquad \|$$

$$\frac{q}{\epsilon_0 A} - \frac{q'}{\epsilon_0 A} = \frac{1}{\kappa_e}\frac{q}{\epsilon_0 A}$$

$$\Rightarrow \boxed{q' = q\left(1 - \frac{1}{\kappa_e}\right) < q}$$

induced Polarization surface charge < free charge

likewise $\sigma_p = \sigma\left(1 - \frac{1}{\kappa_e}\right)$

$\left(\begin{array}{l} \sigma_p = q'/A \\ \sigma = q/A \end{array}\right)$

b) Gauss' Law $\epsilon_0 \oint_S \vec{E}\cdot d\vec{A} = q - q' = q/\kappa_e$

$$\Rightarrow \boxed{\epsilon_0 \oint_S \kappa_e \vec{E}\cdot d\vec{A} = q}$$

← free charge only.

II.4. c) Electric Displacement Vector $\vec{D}$

1) $$E = \frac{q}{\epsilon_0 A} - \frac{q'}{\epsilon_0 A} = \frac{\sigma}{\epsilon_0} - \frac{\sigma_p}{\epsilon_0}$$

$$= \frac{\sigma}{\epsilon_0} - \frac{P}{\epsilon_0} \Rightarrow \underset{q/A}{\underset{\parallel}{\sigma}} = \epsilon_0 E + P$$

free charge density

Define: $\boxed{\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}}$

So $q/A = \sigma = D$; D is given by free charge only.

Now $\sigma = \kappa_e \epsilon_0 E$ hence

$$\boxed{\vec{D} = \kappa_e \epsilon_0 \vec{E}}$$

2) Gauss' Law for $\vec{D}$

$$\boxed{\oint_S \vec{D} \cdot d\vec{A} = q}$$

d.) Constitutive Equation: Linear, isotropic dielectrics (Faraday's exp.)

$$\vec{P} = \epsilon_0 \chi_E \vec{E}$$

↑ electric susceptibility

I.I.4.d) Now $\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}$

$$= \underbrace{\epsilon_0(1+\chi_E)}_{\equiv \epsilon = \epsilon_0 \chi_e = \text{permittivity}} \vec{E}$$

Thus $\boxed{\chi_e = \epsilon/\epsilon_0 = 1 + \chi_E}$ = dielectric constant

hence $\vec{P} = \epsilon_0(\chi_e - 1)\vec{E}$.

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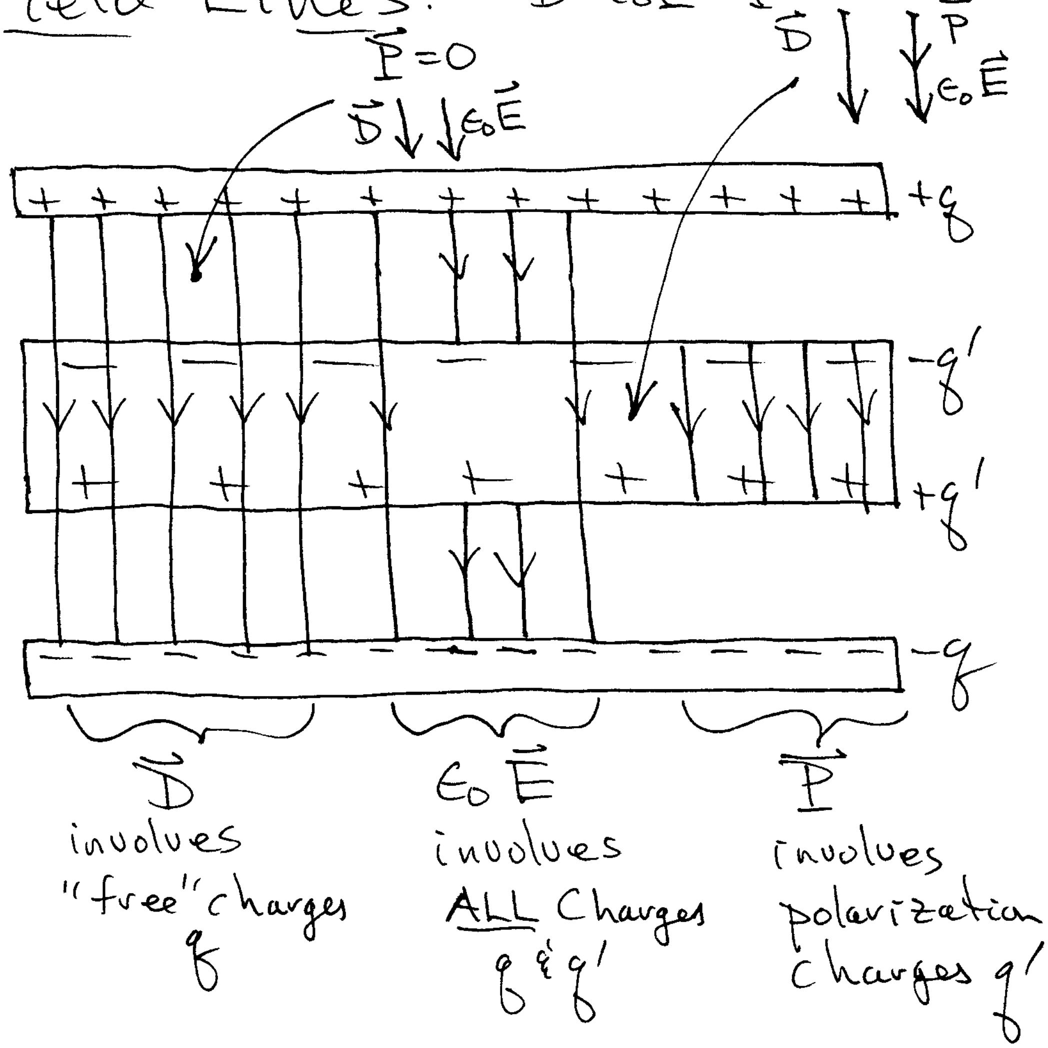
e) Field Lines: $\vec{D} = \epsilon_0\vec{E} + \vec{P}$



I.I.5) Energy stored in field - Re-Visited

Dielectric filled capacitor $q = CV$, $C = \kappa_e C_0$

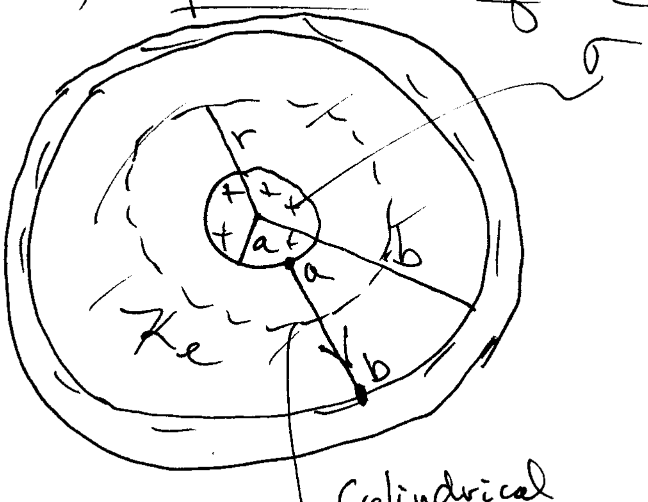
$$\Rightarrow \quad U = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}CV^2 \quad \text{as previously}$$

energy density stored in fields $\Rightarrow$ $\boxed{u = \frac{1}{2}\vec{D}\cdot\vec{E}}$

Hence energy stored in fields

$$\boxed{U = \frac{1}{2}\int_V \vec{D}\cdot\vec{E}\, dV}$$

6) Capacitance of Coaxial Cable



1) Gauss' Law

$$\oint_S \vec{D}\cdot d\vec{A} = Q_{free}$$

$$D(r)2\pi r l = 2\pi a l \sigma$$

$$\Rightarrow D(r) = \frac{a\sigma}{r}$$

2) $E(r) = \frac{1}{\epsilon_0 \chi_e} D(r)$

$$= \frac{a\sigma}{\kappa_e \epsilon_0 r}$$

II.6) 3) Potential Difference:

$$V = \int_a^b \vec{E}\cdot d\vec{s} = \int_a^b E(r)\,dr$$
$$= \frac{a\sigma}{\kappa_e \epsilon_0} \ln(b/a)$$

4) Capacitance $C = q/V$; $q = 2\pi a l \sigma$

$$\boxed{C = \frac{2\pi \kappa_e \epsilon_0}{\ln(b/a)}\, l} = \kappa_e C_0$$

$$\boxed{C/l = \frac{2\pi \kappa_e \epsilon_0}{\ln(b/a)}} = \text{Capacitance}/\text{unit length}.$$

---

7) Forces on Charges

a) Isolated System: Q = fixed

$$\underset{\substack{\text{Work performed}\\ \text{by } \vec{E}\text{-field}}}{dW} = \vec{F}\cdot d\vec{s} = -\underset{\substack{\text{loss}\\ \text{in P.E.}}}{dU} = -\vec{\nabla}U\cdot d\vec{s}$$

$$\Rightarrow \boxed{\vec{F} = -\vec{\nabla}U \Big|_{Q=\text{fixed}}}$$

I.I.7 b) Energy source attached to system:
fix: potential difference : V-fixed

$$\vec{F}\cdot d\vec{s} = \underset{\substack{\nearrow \\ \text{work performed} \\ \text{by battery \& } \vec{E}\text{-field}}}{dW} = \underset{\substack{\uparrow \\ \text{work} \\ \text{performed} \\ \text{by battery}}}{dW_{\text{battery}}} - \underset{\substack{\nwarrow \\ \text{PE} \\ \text{gain}}}{dU}$$

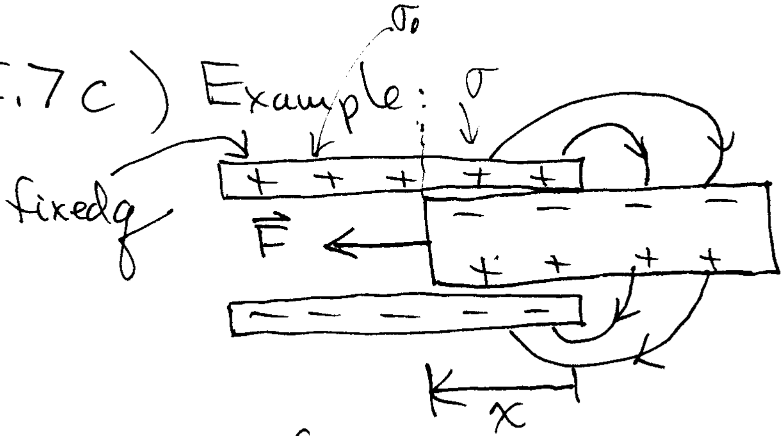
$$\left.\begin{aligned} dW_{\text{battery}} &= dq\,V \\ dU &= \tfrac{1}{2}\,dq\,V \end{aligned}\right\} \Rightarrow dW = \tfrac{1}{2}\,dq\,V = dU$$

So

$$\vec{F}\cdot d\vec{s} = +dU = +\vec{\nabla}U\cdot d\vec{s}$$

$$\Rightarrow \boxed{\vec{F} = +\vec{\nabla}U\,\Big|_{V=\text{fixed}}}$$

I.I.7c) Example:



q - fixed

dielectric slab pulled in by $\vec{E}$-field

$$U = \frac{1}{2}\int_{\substack{\text{capacitor}\\\text{Volume}}} \vec{D}\cdot\vec{E}\,dV = \frac{1}{2}\epsilon_0\chi_e E^2(xbd) + \frac{1}{2}\epsilon_0 E^2\left[(l-x)bd\right]$$

$$= \frac{1}{2}\epsilon_0 E^2 bd\left[\kappa_e x + (l-x)\right]$$

q is fixed, but V changes with x

$$\begin{cases} q = \sigma x b + \sigma_0(l-x)b \\ E = \dfrac{\sigma}{\kappa_e\epsilon_0} = \dfrac{\sigma_0}{\epsilon_0} = V/d \Rightarrow \sigma = \kappa_e\sigma_0 \end{cases}$$

$$\Rightarrow \quad \sigma_0 = \frac{q}{\kappa_e x b + (l-x)b}$$

Hence

$$U = \frac{1}{2}\frac{q^2}{C} \quad \text{with}$$

$$C = \frac{\epsilon_0 b}{d}\left[(l-x) + \chi_e x\right]$$

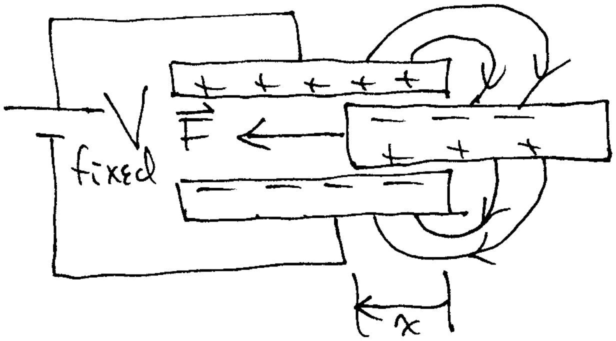
So

$$F = -\frac{dU}{dx} \quad \text{since } q \text{ is fixed}$$

$$\boxed{F = +\frac{q^2}{2C^2}\frac{dC}{dx} > 0}$$

(x-dependent force in q fixed case)

II 7d) Example: V-fixed



dielectric slab is pulled in.

$$E = V/d = \text{constant}$$

$$U = \frac{1}{2}\int_{\substack{\text{Capacitor}\\ \text{Volume}}} \vec{D}\cdot\vec{E}\, dV = \frac{1}{2}\epsilon_0 \chi_e E^2 (xbd) + \frac{1}{2}\epsilon_0 E^2 [(l-x)bd]$$

$$\boxed{\begin{aligned} U &= \frac{1}{2}\epsilon_0 \left(\frac{V}{d}\right)^2 bd\,[\chi_e x + (l-x)] \\ &= \frac{1}{2}CV^2 \end{aligned}}$$

$$\boxed{\begin{aligned} F &= +\frac{dU}{dx} = \frac{1}{2}V^2 \frac{dC}{dx} \\ &= \frac{1}{2}V^2 \frac{\epsilon_0(\chi_e - 1)b}{d} > 0 \end{aligned}}$$

F is a constant indep. of x in fixed V case

Note:

$$F\Big|_{q\text{ fixed}} = \frac{q^2}{2C^2}\frac{dC}{dx} \quad \text{but } q = CV$$

$$= \frac{1}{2}V^2 \frac{dC}{dx} \quad \text{but } \underline{V = V(x)} \text{ in fixed } q \text{ case.}$$

$$F\Big|_{V\text{ fixed}} = \frac{1}{2}V^2\frac{dC}{dx} \quad \text{but } V = \text{constant in fixed } V \text{ case}$$

Similar form but different x-dependence.

II.) Currents

A) Electric current, i, is the rate at which charge is transported through a given surface

$$i = \frac{dq}{dt}$$

1) SI units: $[i] = 1\ \text{Ampere} = 1\ \frac{\text{Coulomb}}{\text{Second}}$

2) Convention: Current is in the direction of positive charge flow.

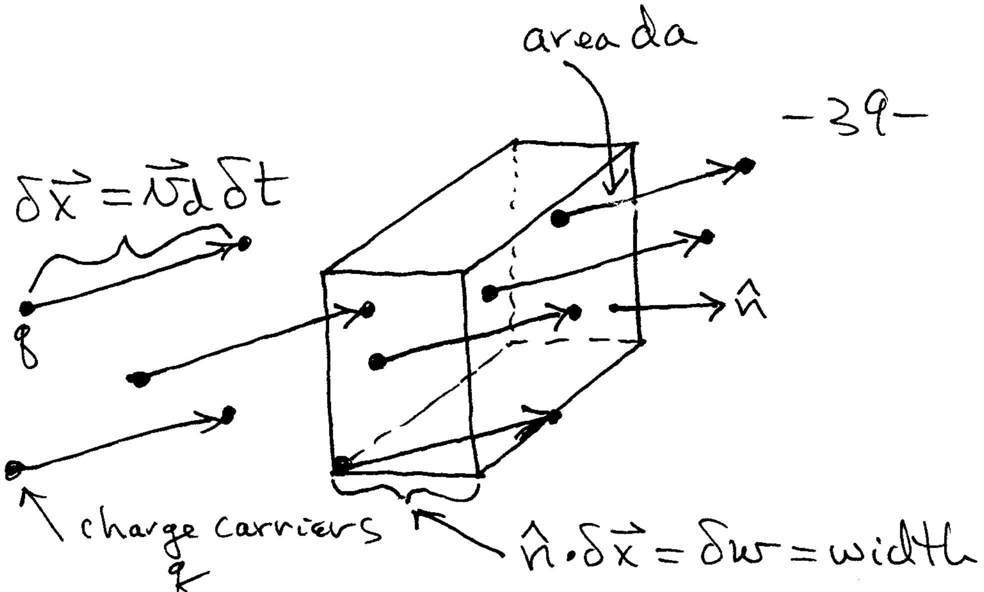
3) Current obeys scalar addition

$$\Rightarrow\ i_1 = i_2 + i_3$$

B) Conducting medium: free electron gas model electrons move at constant drift velocity $\vec{v}_d$ when an $\vec{E}$-field is established in conducting wire

1) Suppose conducting medium has one type of charge carrier q with $N = \frac{\#\text{ of charge carriers}}{\text{Volume}}$ each has drift velocity $\vec{v}_d$ in an $\vec{E}$-field

II B) 1)



In time δt, each charge carrier moves $\delta\vec{x} = \vec{v}_d \delta t$. Charge δQ passes through area da in time δt

$$\delta Q = (qN)(\delta\vec{x} \cdot \hat{n}\, da)$$

$$= qN\vec{v}_d \cdot d\vec{A}\, \delta t$$ where $d\vec{A} = da\,\hat{n}$

Hence, di = current transported through da

$$di = \frac{\delta Q}{\delta t} = Nq\vec{v}_d \cdot d\vec{A}$$

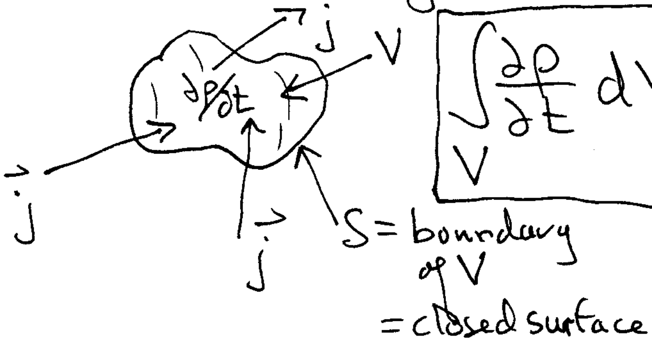
$$\boxed{di = \vec{j} \cdot d\vec{A}}$$

With the current density $\vec{j} \equiv Nq\vec{v}_d$

2) Current flowing through an arbitrarily shaped surface "S" of macroscopic size

$$i = \int_S \vec{j} \cdot d\vec{A}$$

II B3) Charge Conservation = Continuity Equation



$$\boxed{\int_V \frac{\partial\rho}{\partial t}\,dV + \oint_S \vec{j}\cdot d\vec{A} = 0}$$

---

c) Ohm's Law: Linear, isotropic conducting medium

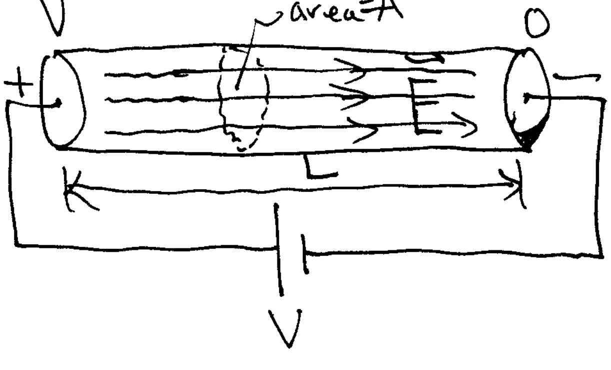
$$\vec{j} = \sigma\vec{E}$$

σ = conductivity of material

1) SI units: $1\ \text{ohm}\,(\Omega) \equiv 1\ \frac{\text{Volt}}{\text{Ampere}}$

$$[\sigma] = \Omega^{-1}m^{-1}$$

2) Wire of conductivity σ, length L, cross sectional area A



$$V = +\int_0^L \vec{E}\cdot d\vec{s}$$

$$= EL$$

$$\Rightarrow\ E = \frac{V}{L}$$

II.C.) 2) $i = \int_A \vec{j} \cdot d\vec{A} = jA = \sigma E A = \underbrace{\left(\sigma \frac{A}{L}\right)}_{\equiv R^{-1}} V$

$$\Rightarrow \boxed{V = iR \qquad \text{Macroscopic Ohm's Law}}$$

a) SI units of R : $[R] = \text{ohm}$

b) $\rho \equiv \frac{1}{\sigma}$ = resistivity of material

$[\rho] = \underline{\Omega} \cdot m$

$R = \frac{\rho L}{A}$

c) Circuit Symbol for <u>Resistor</u>

D) Work done by $\vec{E}$-field in moving dQ through potential difference V

$$dW = dQ\,V = i\,dt\,V$$

Rate of energy transfer = Power P

$$\boxed{P = \frac{dW}{dt} = iV}$$

(Units: 1 Amp·Volt = 1 watt)

1) For Ohmic Device $V = iR$

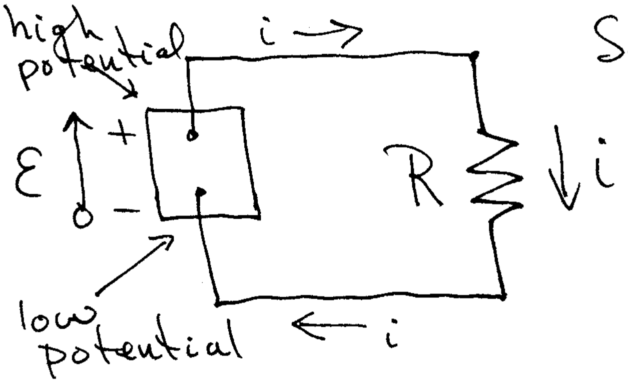
$$\Rightarrow P = iV = i^2 R = \frac{V^2}{R}$$

II.E.) Electromotive Force (emf) $\mathcal{E}$

$$\mathcal{E} = \frac{dW}{dq}$$

1) SI units: $[\mathcal{E}] = 1\,\frac{\text{Joule}}{\text{Coulomb}} = 1\,\text{volt}.$

2)



Seat of emf does work

$$dW = \mathcal{E}dq$$
$$= \mathcal{E} i\,dt$$

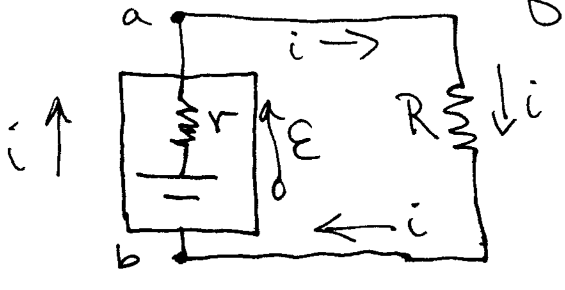
Conservation of energy: $\mathcal{E} i\,dt = i^2 R\,dt$

$$\Rightarrow \quad i = \mathcal{E}/R$$

3) Kirchhoff's Second Rule (Loop Rule):
(Conservation of energy)

The algebraic sum of the changes in potential encountered in a complete traversal of any closed circuit is zero.

4) Internal resistance of seat of emf



Loop Rule:

$$V_b + \mathcal{E} - ir - iR = V_b$$

$$\Rightarrow \boxed{i = \frac{\mathcal{E}}{R+r}}$$

II E.4.) Potential Difference V_{ab} : traverse part of circuit by any path that includes the two points a & b

a) Start at b and go CCW to a

$$V_b + iR = V_a$$

$$\Rightarrow V_a - V_b \equiv V_{ab} = iR$$

$$= \mathcal{E}\frac{R}{R+r}$$

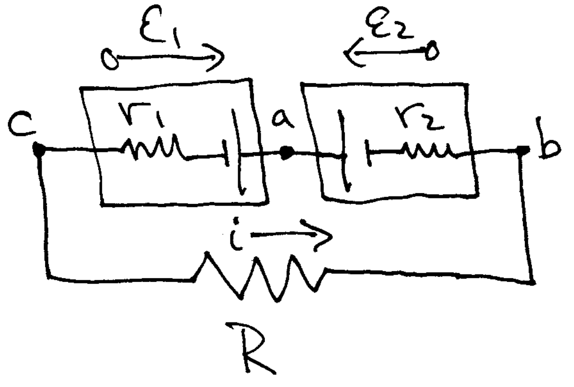
b) Start at a and go CCW to b

$$V_a + ir - \mathcal{E} = V_b$$

$$\Rightarrow V_{ab} = \mathcal{E} - ir = \mathcal{E}\frac{R}{R+r}$$

the same as above.

5.) Example



a) $i = ?$ Loop Rule: Start at "a" go CW back to "a"

$$-\mathcal{E}_2 + ir_2 + iR + ir_1 + \mathcal{E}_1 = 0$$

$$\Rightarrow \boxed{i = \frac{\mathcal{E}_2 - \mathcal{E}_1}{R + r_1 + r_2}}$$

II E5) a) The sign of i tells its direction of flow

b) $V_{ab} = ?$ Start at "b" go CCW to "a"

$$V_b - i r_2 + \mathcal{E}_2 = V_a$$

$$\Rightarrow \boxed{\begin{aligned} V_{ab} &= -i r_2 + \mathcal{E}_2 \\ &= \frac{\mathcal{E}_1 r_2 + \mathcal{E}_2 (R + r_1)}{R + r_1 + r_2} \end{aligned}}$$

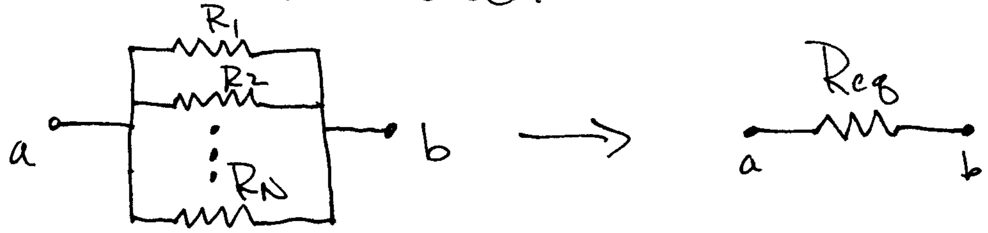
c) $V_{ac} = ?$ Start at "c" go CW to "a"

$$V_c + i r_1 + \mathcal{E}_1 = V_a$$

$$\Rightarrow \boxed{\begin{aligned} V_{ac} &= \mathcal{E}_1 + i r_1 \\ &= \frac{\mathcal{E}_1 (R + r_2) + \mathcal{E}_2 r_1}{R + r_1 + r_2} \end{aligned}}$$

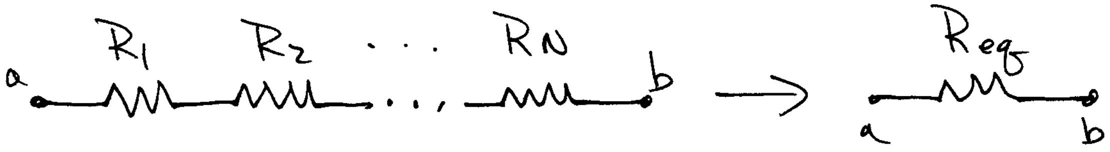
F) Equivalent Resistance

1) Resistors in Parallel



II F 1) $$\frac{1}{R_{eq}} = \sum_{i=1}^{N} \frac{1}{R_i} = \frac{1}{R_1} + \frac{1}{R_2} + \dots + \frac{1}{R_N}$$

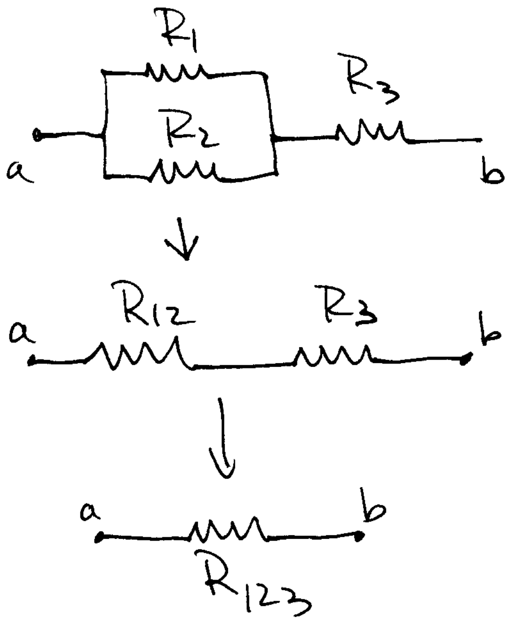
2) Resistors in Series



$$R_{eq} = \sum_{i=1}^{N} R_i = R_1 + R_2 + \dots + R_N$$

---

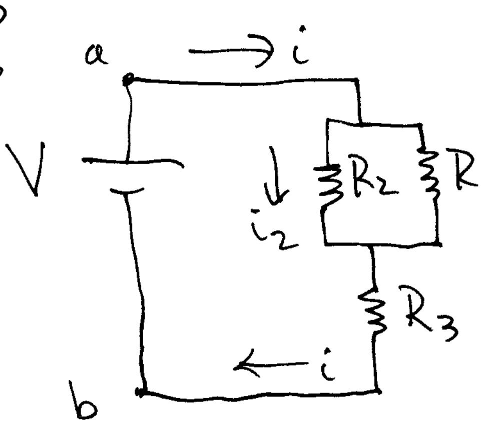
3) Example



$$R_{12} = \frac{R_1 R_2}{R_1 + R_2}$$

$$R_{123} = R_{12} + R_3$$
$$= \frac{R_1 R_2}{R_1 + R_2} + R_3$$

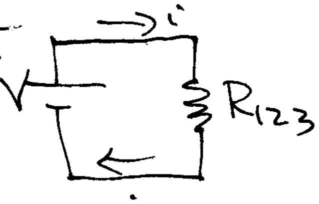
$i_1 = ?$



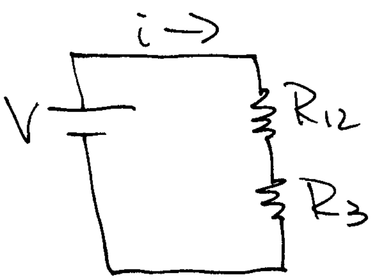
Steps:

a) $i = \frac{V}{R_{123}}$

Equivalent Circuit



- II F3) b) Equivalent Circuit



$$V_{12} = i R_{12}$$

c) Since R_1 & R_2 are in $\parallel$, V_{12} is the voltage that appears across both — so

$$(V_1 = V_2 = V_{12})$$

$$i_1 = \frac{V_{12}}{R_1}$$

$$= \frac{V R_{12}}{R_1 R_{123}}$$

G) Multiloop Circuits: Made up of junctions and branches

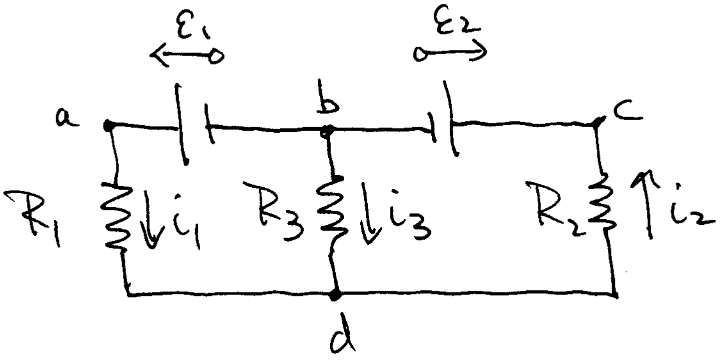
Junction = 3 or more wire segments meet

Branch = Circuit path that starts at one junction and ends at another.

1) Kirchhoff's First Rule (Junction Rule) (Conservation of Charge)

At any junction, the sum of the currents leaving the junction equal, the sum of the currents entering the junction.

- II G 2) Example:



a) 2 Junctions b & d

b) 3 Branches: Left Branch bad
Central Branch bd
Right Branch bcd

Find i_1, i_2, i_3:

1) Apply Junction Rule to d ⇒

$$\boxed{i_1 + i_3 = i_2}$$

2) Apply Loop Rule: CCW b-a-d-b

$$\boxed{\mathcal{E}_1 - i_1 R_1 + i_3 R_3 = 0}$$

3) Apply Loop Rule: CCW b-d-c-b

$$\boxed{-i_3 R_3 - i_2 R_2 - \mathcal{E}_2 = 0}$$

⇒ 3 equations for 3 unknowns i_1, i_2, i_3

(Use either junction to apply junction rule, get same equation, Use any 2 independent loops, get same information)

- II G2) Solve 3 simultaneous equations

$$i_3 = -\frac{R_2\mathcal{E}_1 + R_1\mathcal{E}_2}{R_1R_2+R_2R_3+R_1R_3}$$

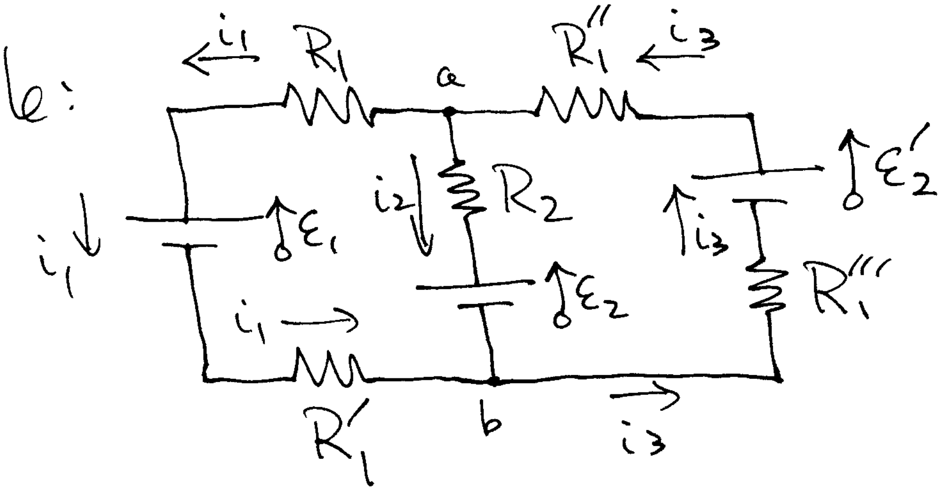
(negative result ⇒ i_3 flows from $d \rightarrow b$ opposite how we drew)

$$i_2 = \frac{\mathcal{E}_1R_3 - \mathcal{E}_2(R_1+R_3)}{R_1R_2+R_2R_3+R_1R_3}$$

$$i_1 = \frac{\mathcal{E}_1(R_2+R_3) - \mathcal{E}_2R_3}{R_1R_2+R_2R_3+R_1R_3}$$

---

3) Example:



1) Apply Junction Rule at b

$$i_1 + i_2 = i_3$$

2) Apply Loop Rule: CCW a-b-a (Left Loop)

$$-i_1R_1 - \mathcal{E}_1 - i_1R_1' + \mathcal{E}_2 + i_2R_2 = 0$$

(if $R_1' = R_1 \Rightarrow 2i_1R_1 - i_2R_2 = \mathcal{E}_2 - \mathcal{E}_1$)

II G3) 3) Apply Loop Rule: CW a-b-a (Right Loop)

$$i_3 R_1'' - \mathcal{E}_2' + i_3 R_1''' + \mathcal{E}_2 + i_2 R_2 = 0$$

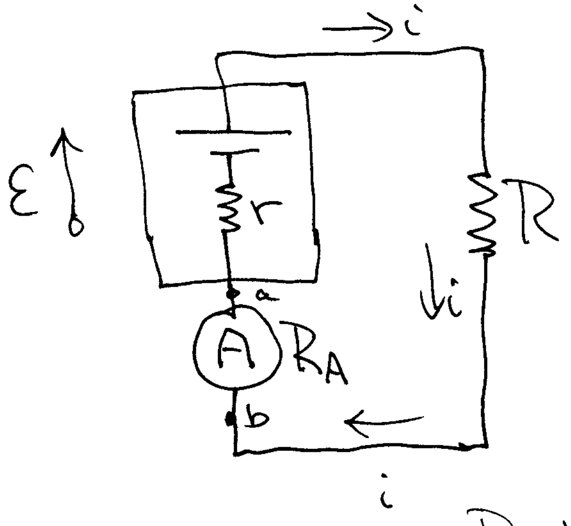
(if $R_1'' = R_1''' = R_1$ and $\mathcal{E}_2' = \mathcal{E}_2 \Rightarrow \quad i_2 R_2 + 2 i_3 R_1 = 0$)

Hence we have 3 equations for the 3 unknowns i_1, i_2, i_3 — solve simultaneously:

(See class notes for solution and $V_a - i_2 R_2 - \mathcal{E}_2 = V_b \Rightarrow V_{ab} = \mathcal{E}_2 + i_2 R_2$)

H) Measuring Instruments:

1) Ammeter measures current



Loop Rule

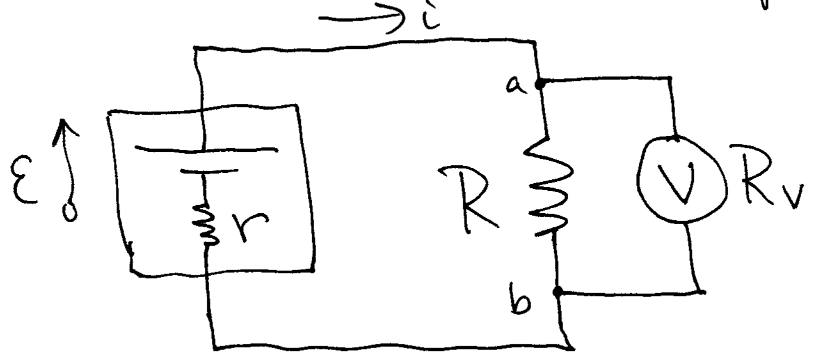
$$0 = \mathcal{E} - iR - iR_A - ir$$

$$\Rightarrow \quad i = \frac{\mathcal{E}}{r + R + R_A}$$

Ideal Ammeter $R_A = 0$

Realistic Ammeter $R_A \ll r + R$ to give small error

II H 2) Voltmeter measure potential difference



Loop Rule

$$\mathcal{E} - iR_{eq} - ir = 0$$

$$i = \frac{\mathcal{E}}{r + R_{eq}}$$

$$R_{eq} = \frac{R R_V}{R + R_V}$$

Potential Difference measured

$$V_a - iR_{eq} = V_b$$

$$\Rightarrow \quad V_{ab} = iR_{eq} = \mathcal{E}\frac{R_{eq}}{R_{eq} + r}$$

Without Voltmeter this would be $V^0_{ab} = \mathcal{E}\frac{R}{R+r}$

So ideal Voltmeter $R_V = \infty$ (Then $R_{eq} = R$)

Realistic Voltmeter $R_V >> R$ to give small error

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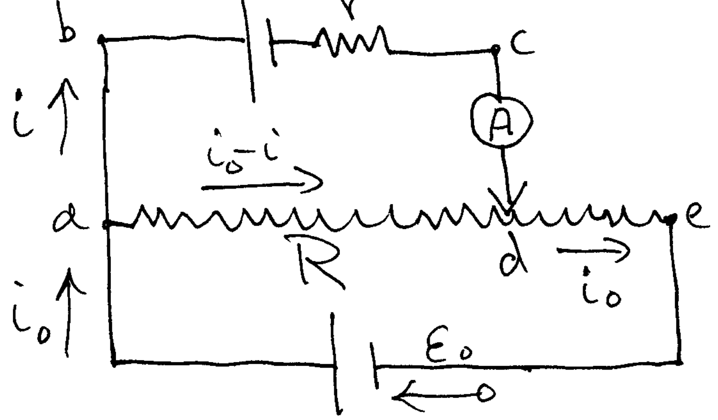
3) Potentiometer: Measures unknown emf $\mathcal{E}_x$ by comparing to standard emf $\mathcal{E}_S$



R = resistance between a-d

II H3) <u>Step</u> 1: Place $\mathcal{E}_S$ where $\mathcal{E}$ is and adjust R until $i = 0$ - potentiometer is "balanced"

call this value of $R = R_S$ at balanc

Loop Rule CW a-b-c-d-a

$$-\mathcal{E}_S - ir + (i_0 - i)R_S = 0$$

But $i = 0$ in balance $\Rightarrow$

$$\mathcal{E}_S = i_0 R_S .$$

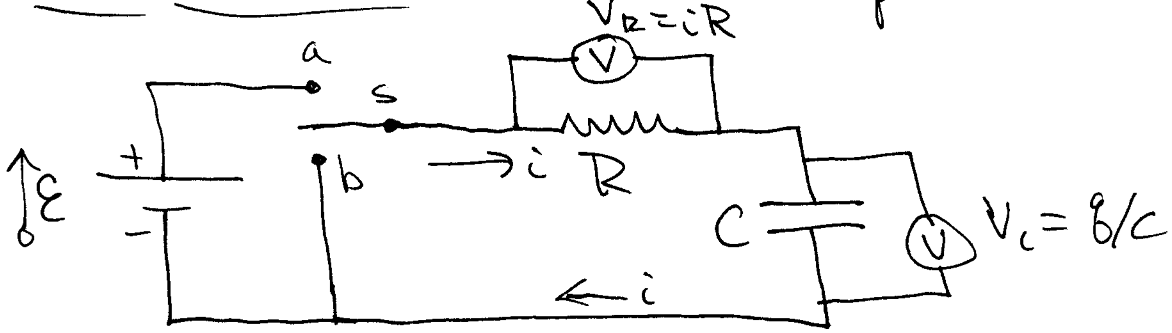
<u>Step</u> 2: Substitute $\mathcal{E}_x$ for $\mathcal{E}_S$ and adjust R to R_x for balance $i = 0$ again

Now loop Rule gives $\mathcal{E}_x = i_0 R_x$ (same i_0)

$$\Rightarrow \boxed{\mathcal{E}_x = \mathcal{E}_S \frac{R_x}{R_S}}$$

(independent of $\mathcal{E}_0$ - $i = 0$ "null instrument")

II.I) RC-circuits: Current will depend on time



a) Throw switch to a — charge capacitor
Loop Rule: CW start at a end at a
with $V_R = iR$ and $V_C = q/C$

$$\mathcal{E} = iR + q/C$$

$$= R\frac{dq}{dt} + q/C$$

This is DE for $q = q(t)$:

$$\int_0^{q(t)} \frac{dq}{\mathcal{E}C - q} = \frac{1}{RC}\int_0^t dt$$

$$-\ln\left[\mathcal{E}C - q\right]\Big|_0^{q(t)} = \frac{1}{RC}\; t$$

$$\Rightarrow \quad \ln\left[\frac{\mathcal{E}C - q(t)}{\mathcal{E}C}\right] = -\frac{t}{RC}$$

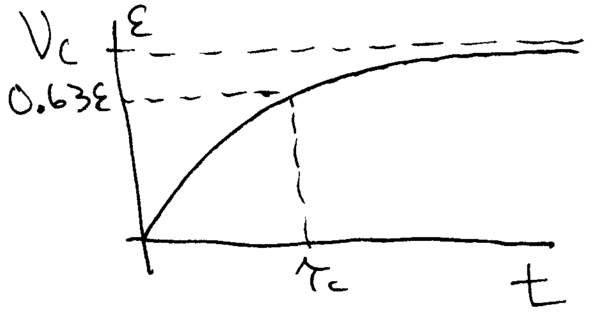
$$\Rightarrow \quad \frac{\mathcal{E}C - q(t)}{\mathcal{E}C} = e^{-t/RC}$$

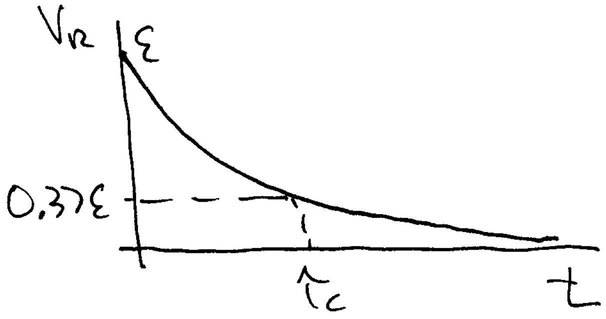
II a) $\Rightarrow$

$$q(t) = \mathcal{E}C\left[1 - e^{-t/\tau_c}\right]$$

with $\tau_c \equiv RC$ = Capacitive Time Constant

Hence $$i = i(t) = \frac{dq(t)}{dt} = \frac{\mathcal{E}}{R}e^{-t/\tau_c}$$





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b) Throw switch from a to b: Discharge Capacitor

Loop Rule:
$$iR + q/C = 0$$

$$\Rightarrow R\frac{dq}{dt} + \frac{1}{C}q = 0$$

$$\Rightarrow \int_{q_0}^{q(t)} \frac{dq}{q} = -\frac{1}{RC}\int_0^t dt$$

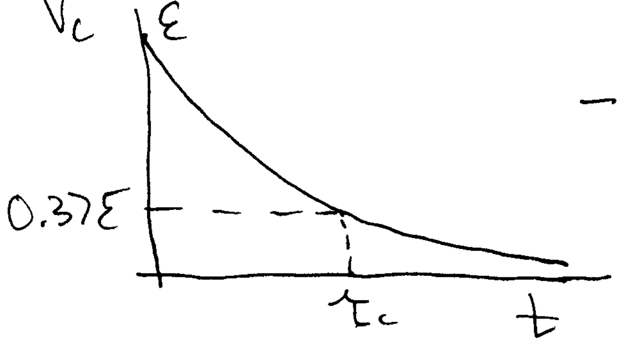
$$\Rightarrow \boxed{q(t) = q_0 e^{-t/\tau_c}}$$

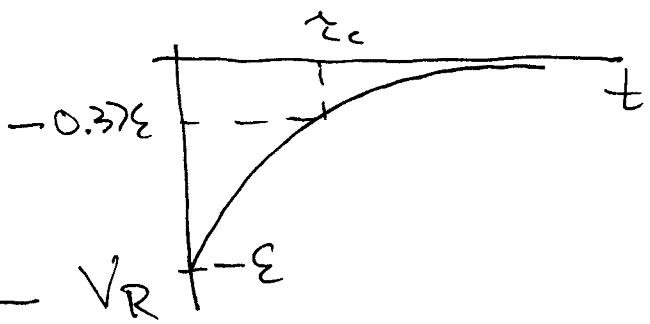
II.I.b) q_0 = initial charge on capacitor
for $t >> \tau_c$ in part a $q_0 \approx \varepsilon C$.

$$q(t) = \varepsilon C e^{-t/\tau_c}$$

and

$$i(t) = \frac{dq(t)}{dt} = -\frac{\varepsilon}{R} e^{-t/\tau_c}$$





III) Magnetic Field:

A) The magnetic field $\vec{B}$ is defi ed by the force law

$$\vec{F} = q\vec{v} \times \vec{B}$$

1) SI units: $[\vec{B}]$ = tesla (T)

$$= 1 \frac{\text{newton}}{\text{Ampere-meter}}$$

(non-SI units: gauss(G) $= 10^{-4}$ T)

2) $\vec{F} \perp \vec{v}$: magnetic force does no work only change direction of $\vec{v}$, not magnitude.

3) Lorentz Force Law: total force on charge q in $\vec{E}$- and $\vec{B}$-fields

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

(This defines $\vec{B}$ and $\vec{E}$)

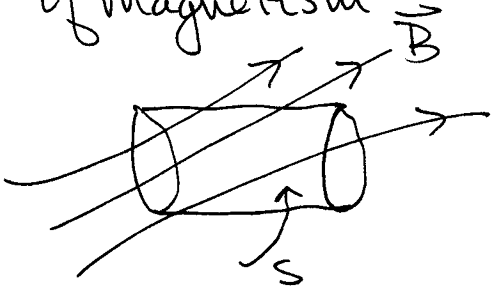
4) No Magnetic Charges: experimental result

Lines of $\vec{B}$ are continuous

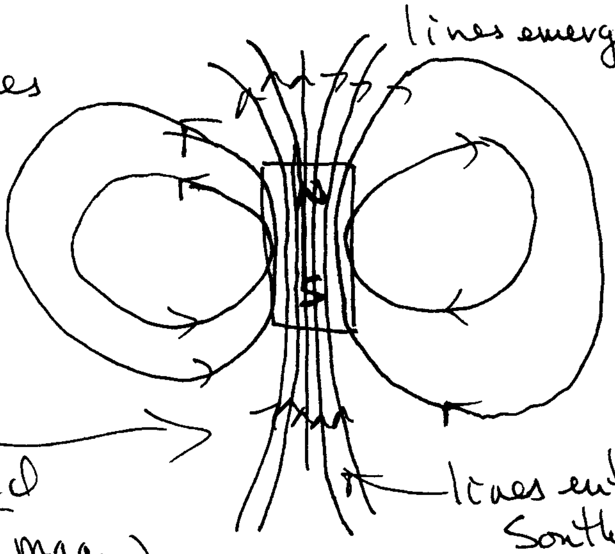
III A4) Hence Flux of $\vec{B}$ through any closed surface is zero

Fundamental Law of Magnetism

$$\boxed{\oint_S \vec{B}\cdot d\vec{A} = 0}$$



5) $\vec{B}$-lines



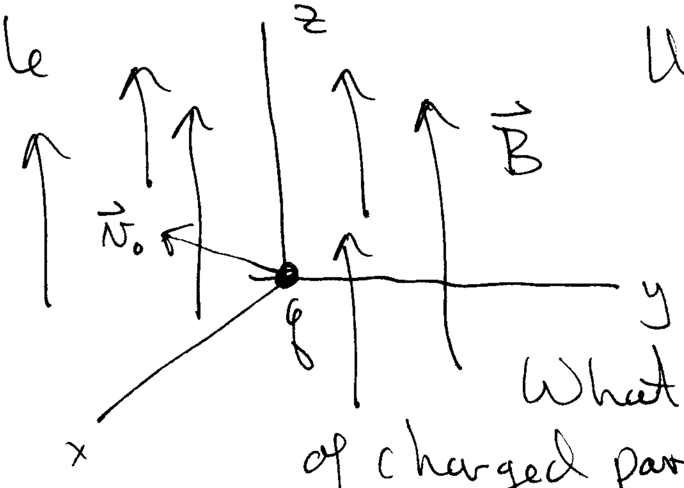
lines emerge from North pole

Opposite poles attract
Like poles repel

$\vec{B}$-lines form closed loops (no mag. charges)

lines enter South pole

B) Example



Uniform $\vec{B}$-field in z-direction
$\vec{B} = B\hat{k}$

What is trajectory of charged particle?

III B) Particle has charge q, mass m and at $t=0$ passes through origin with velocity $\vec{v}_0$

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

So initial conditions are $\left.\begin{array}{l}\vec{r}(0)=0\\ \frac{d\vec{r}}{dt}(0)=\vec{v}_0\end{array}\right\}$ six conditions

$$\Rightarrow \quad \begin{array}{ll} x(0)=0 & \frac{dx}{dt}(0)=v_{0x} \\ y(0)=0 & \frac{dy}{dt}(0)=v_{0y} \\ z(0)=0 & \frac{dz}{dt}(0)=v_{0z} \end{array}$$

Newton's 2nd Law: $\vec{F} = m\vec{a}$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x(t)}{dt^2}\hat{i} + \frac{d^2y(t)}{dt^2}\hat{j} + \frac{d^2z(t)}{dt^2}\hat{k}$$

and

$$\vec{F} = q\vec{v}\times\vec{B} = q\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \\ 0 & 0 & B \end{vmatrix}$$

$$= qB\left(\frac{dy}{dt}\hat{i} - \frac{dx}{dt}\hat{j}\right)$$

III B) Newton's Law ⇒

$$\boxed{\begin{aligned} m\frac{d^2x(t)}{dt^2} &= qB\frac{dy(t)}{dt} \\ m\frac{d^2y(t)}{dt^2} &= -qB\frac{dx(t)}{dt} \\ m\frac{d^2z(t)}{dt^2} &= 0 \end{aligned}}$$

(see class notes)

Solve ⇒

$$x(t) = \frac{v_{oy}}{\omega} + \frac{v_{ox}}{\omega}\sin\omega t - \frac{v_{oy}}{\omega}\cos\omega t$$

$$y(t) = -\frac{v_{ox}}{\omega} + \frac{v_{oy}}{\omega}\sin\omega t + \frac{v_{ox}}{\omega}\cos\omega t$$

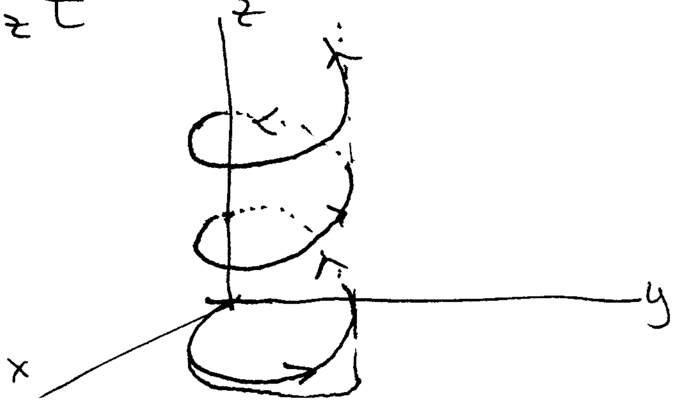
$$z(t) = v_{oz}t$$

where $\omega \equiv \frac{qB}{m}$

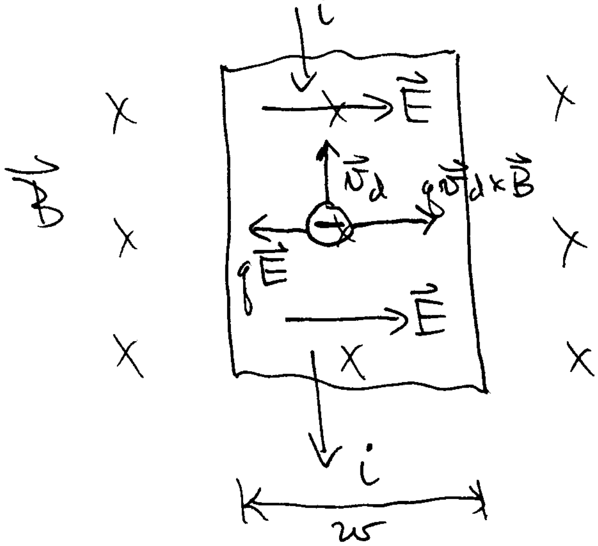
Trajectory is a helix:

$$\left(x(t) - \frac{v_{oy}}{\omega}\right)^2 + \left(y(t) + \frac{v_{ox}}{\omega}\right)^2 = \left[\frac{v_{ox}^2 + v_{oy}^2}{\omega^2}\right]$$

$$z(t) = v_{oz}t$$



III C) Hall Effect:



current carrying strip of thickness t

Potential difference across strip = Hall Voltage = V = Ew

At equilibrium

$$0 = q\vec{E} + q\vec{v}_d \times \vec{B}$$

$$\Rightarrow E = v_d B$$

Recall $j = nev_d = \frac{i}{A} = \frac{i}{wt}$

So

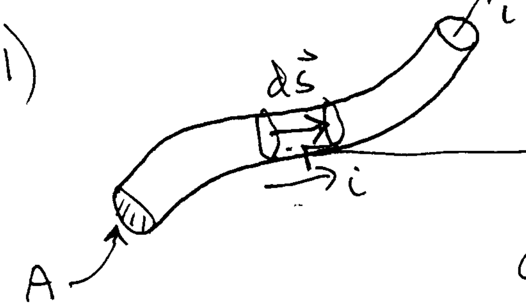
$$E = \frac{V}{w} = v_d B = \frac{j}{ne} B = \frac{iB}{newt}$$

$$\Rightarrow \boxed{n = \frac{iB}{etV}}$$ density of charge carriars

Measure Hall Voltage → if positive charge carriars are +'ve
if negative charge carriarsare -'ve.

III D) Magnetic Force on a Current

1)



charge $dq = nA|\vec{ds}|q$

$$d\vec{F} = dq\, \vec{v}_d \times \vec{B}$$

$$= nAq\,|\vec{ds}|\,\vec{v}_d \times \vec{B}$$

But $\vec{ds} \parallel \vec{v}_d \Rightarrow$

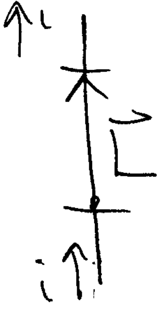
$$d\vec{F} = nA|\vec{v}_d|q\; \vec{ds} \times \vec{B}$$

Recall $i = nq|\vec{v}_d|A \Rightarrow$

$$\boxed{d\vec{F} = i\, \vec{ds} \times \vec{B}}$$ the

magnetic force exerted on wire segment $\vec{ds}$

2) Straight wire segment of length $\vec{L}$ in uniform $\vec{B}$



$$\vec{F} = \int d\vec{F} = i\left(\int_0^L \vec{ds}\right) \times \vec{B}$$

$$= i\vec{L} \times \vec{B}$$

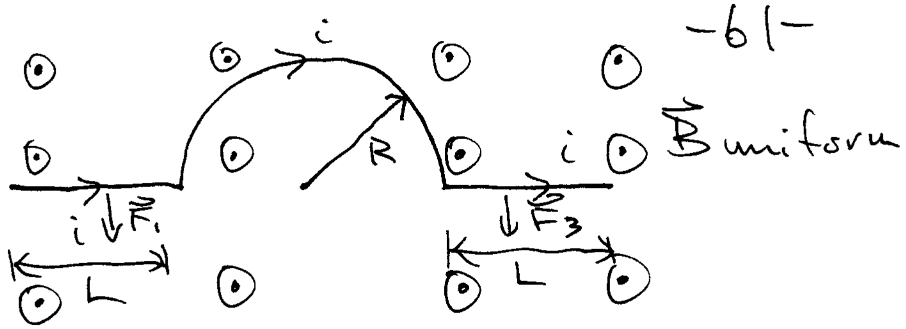
a sideways force.

---

3) Closed Loop in a constant $\vec{B}$-field

$$\vec{F} = \oint_C d\vec{F} = i\left(\oint_C \vec{ds}\right) \times \vec{B} = 0.$$

III D) 4.) Example:



$F_1 = F_3 = iLB$ pointing down

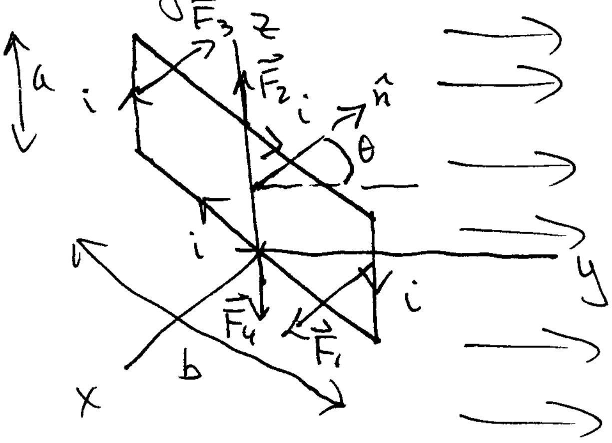
$$F_2 = \int_{\theta=0}^{\pi} dF \sin\theta = iBR \int_{\theta=0}^{\pi} \sin\theta \, d\theta$$

$$= 2iBR$$ pointing down

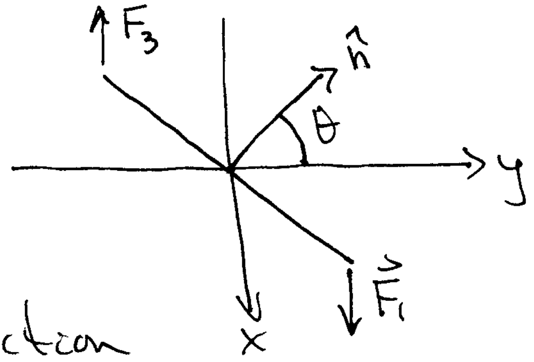
Total force $F = (2L+2R)iB$ pointing down

E) Torque Due to Magnetic Field

1) Rectangular Current loop in uniform $\vec{B}$-field



$\vec{B} = B\hat{j}$



$\vec{F}_2 = ibB\cos\theta \, \hat{k} = -\vec{F}_4$

no torque since same line of action

III.E.1) $\vec{F}_1 = iaB\,\hat{i} = -\vec{F}_3$ produce torque since they act as a couple.

$$\vec{\tau} = \frac{b}{2} F_1 \sin\theta(-\hat{k}) + \frac{b}{2} F_3 \sin\theta(-\hat{k})$$

$$\vec{\tau} = iabB\sin\theta(-\hat{k})$$

$$\vec{\tau} = iAB\sin\theta(-\hat{k})$$ $A = ab =$ area of loop

For N-turns of wire

$$\vec{\tau} = NiAB\sin\theta(-\hat{k})$$ this result is true for any shaped planar area A.

The normal $\hat{n}$ to loop points in direction determined by right hand rule applied to direction of current flow about perimeter

Define Magnetic Dipole Moment

$$\boxed{\vec{\mu} = NiA\,\hat{n}}$$

Then

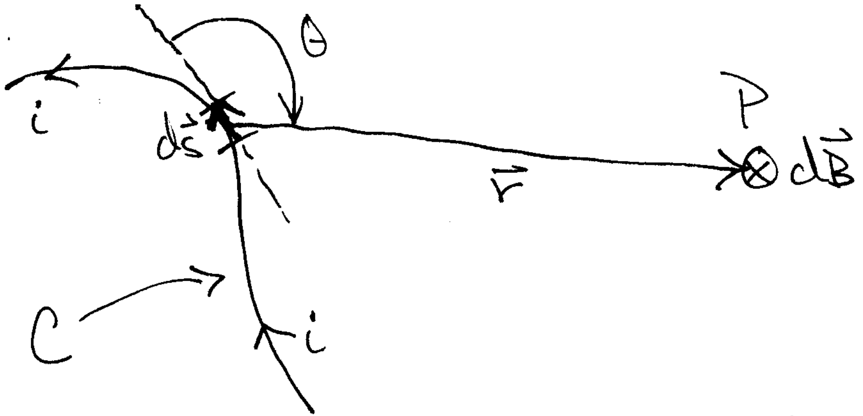
$$\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}}$$

III. E.2.) Potential energy of magnetic dipole in magnetic field $\vec{B}$

$$U(\theta) = -\vec{\mu}\cdot\vec{B} \quad .$$

IV) Magnetostatics: (steady currents)

A) Biot-Savart Law



$$d\vec{B} = \frac{\mu_0}{4\pi}\,\frac{i\,d\vec{s}\times\vec{r}}{r^3}$$

Magnetic Field at point P due to $i\,d\vec{s}$ current element

(SI: $\mu_0 \equiv 4\pi\times10^{-7}\,\frac{T\cdot m}{A}$

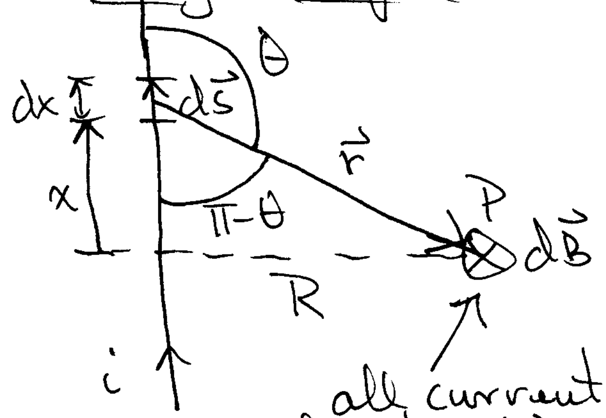
μ_0 = permeability of free space)

$$\vec{B} = \int_C d\vec{B} = \frac{\mu_0}{4\pi}\int_C \frac{i\,d\vec{s}\times\vec{r}}{r^3}$$

Magnetic Field at point P due to current carrying wire C.

IV B) Examples:

1) Long Straight Wire



all current elements $i d\vec{s}$ produce $d\vec{B}$'s in the same direction — into page — So just add up magnitudes — scalar integral

Magnitude of $d\vec{B}$ at P

$$dB = \frac{\mu_0 i}{4\pi}\,\frac{ds\sin\theta}{r^2}$$

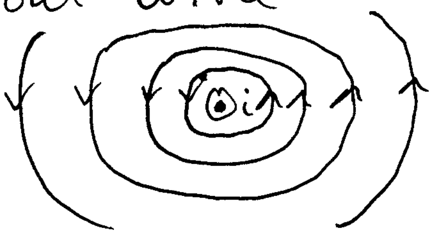
$$= \frac{\mu_0 i}{4\pi}\,\frac{dx\sin\theta}{r^2}$$

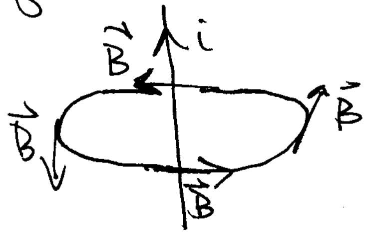
$$B = \int_{x=-\infty}^{+\infty} dB = \frac{\mu_0 i}{4\pi}\int_{x=-\infty}^{+\infty}\frac{\sin\theta}{r^2}\,dx$$

where $r = \sqrt{R^2+x^2}$, $\sin\theta = \sin(\pi-\theta) = \dfrac{R}{\sqrt{R^2+x^2}}$

$$\Rightarrow \boxed{B = \frac{\mu_0 i}{2\pi R} \text{ and points into page}}$$

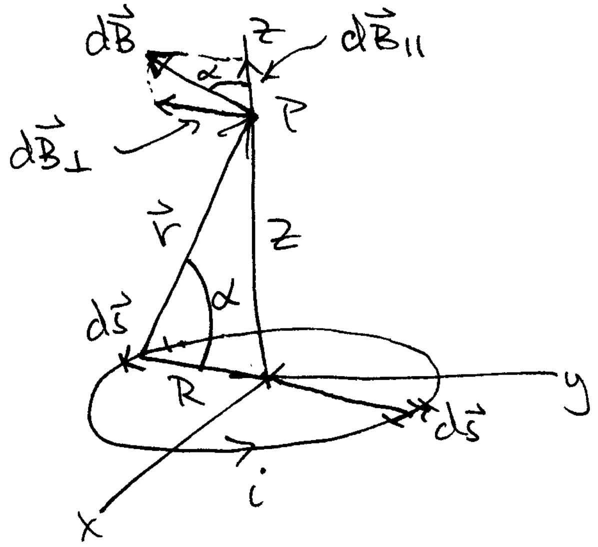
By symmetry, lines of $\vec{B}$ form circles about wire



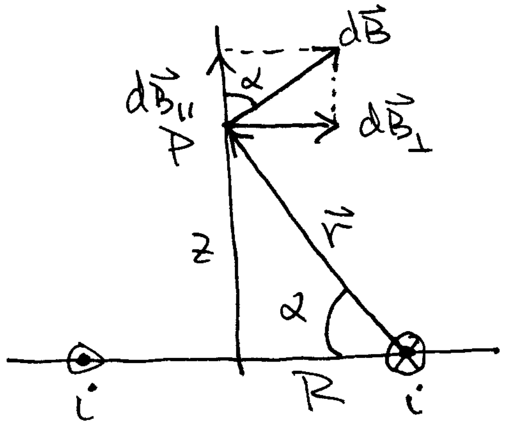


(Right Hand Rule)

IV B2) Circular Current Loop



Side View



By symmetry only $d\vec{B}_{\parallel}$ will contribute to the total $\vec{B}$ at P the $d\vec{B}_\perp$ contributions cancell out.

$$\vec{B} = B\hat{k}$$

$$B = \int_C dB_{\parallel} = \frac{\mu_0 i}{4\pi}\int_C \frac{\sin\theta\, ds}{r^2}\ \underline{\cos\alpha}$$

for $dB_{\parallel}$

$\theta = 90°$! so $\sin\theta = 1$

and $r = \sqrt{R^2+z^2}$; $\cos\alpha = \frac{R}{r} = \frac{R}{\sqrt{R^2+z^2}}$

Hence $$B = \frac{\mu_0 i}{4\pi}\left(\frac{1}{R^2+z^2}\right)\left(\frac{R}{\sqrt{R^2+z^2}}\right)\underbrace{\int_C ds}_{=2\pi R}$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 i R^2}{2(R^2+z^2)^{3/2}}\hat{k}}$$

IV B 2) a) Far from loop, $z >> R$, $B \approx \frac{\mu_0 i R^2}{2z^3}$

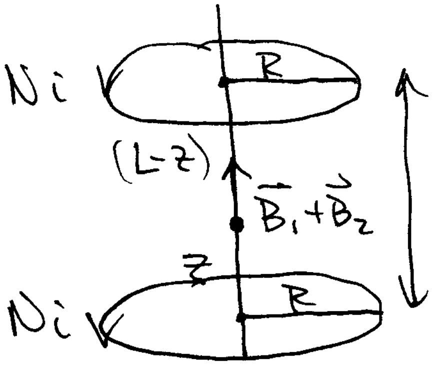
For N-turns of wire, $B = \frac{\mu_0 N i R^2}{2z^3}$

Dipole moment of coil $\mu = NiA = Ni\pi R^2$

$\Rightarrow$ $B = \frac{\mu_0}{2\pi}\frac{\mu}{z^3}$ ← dipole moment.

---

b) Helmholtz Coils:

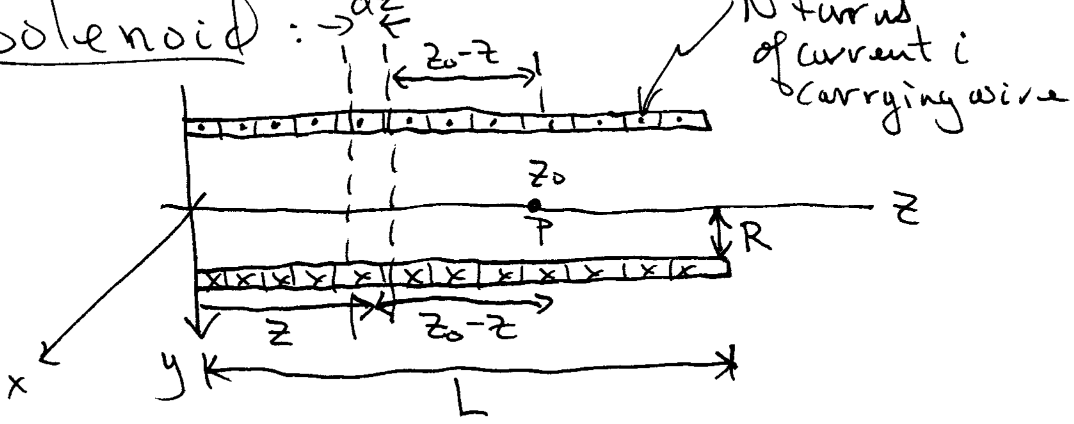


$$B = B_1 + B_2$$

$$\vec{B} = \frac{\mu_0 N i A}{2\pi}\hat{k} \times \left[\frac{1}{(R^2+z^2)^{3/2}} + \frac{1}{(R^2+(L-z)^2)^{3/2}}\right]$$

---

3) Solenoid:



Infinitesimal slice of solenoid acts like current loop $di = Ni\frac{dz}{L}$

IV B3) $\vec{B}$ is in the z-direction with magnitude

$$B(z_0) = \frac{\mu_0 N i R^2}{2L}\int_{z=0}^{L}\frac{dz}{[R^2+(z_0-z)^2]^{3/2}}$$

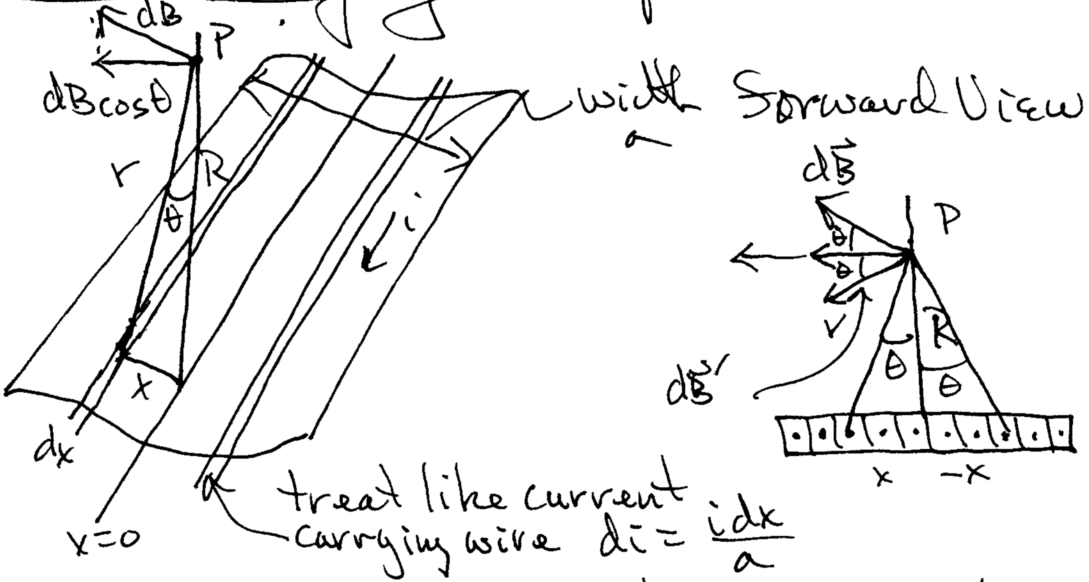
$$= \frac{\mu_0 N i}{L}\,\frac{1}{2}\left[\frac{1}{\sqrt{1+\left(\frac{R}{L-z_0}\right)^2}}+\frac{1}{\sqrt{1+\left(\frac{R}{z_0}\right)^2}}\right]$$

a) For $R \ll L$ and $z_0 = \frac{L}{2}$

$$B(z_0=\tfrac{L}{2}) \simeq \frac{\mu_0 N i}{L}\left[1-\frac{2R^2}{L^2}\right]$$

---

4) Current Carrying Strip



By symmetry only horizontal component of $d\vec{B}$ contributes to total $\vec{B}$ at P, vertical components cancel

IV B4) So again the horizontal magnitude is

$$B = \int dB \cos\theta$$

$$= \frac{\mu_0}{2\pi}\int \frac{di}{r}\cos\theta$$

But $r\cos\theta = R$
and $x = R\tan\theta$

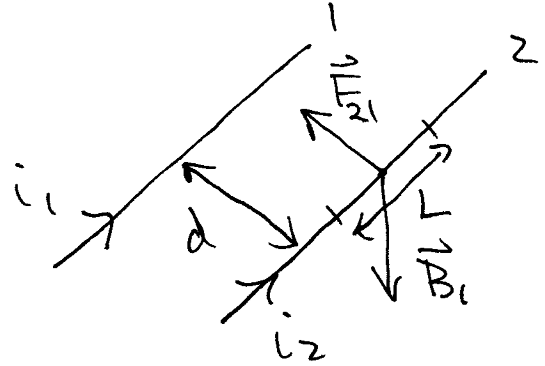
$$B = \frac{\mu_0}{2\pi R}\frac{i}{a}\int_{-a/2}^{+a/2}\frac{dx}{\sec^2\theta} = \frac{\mu_0 i}{2\pi a}2\alpha$$

where $\tan\alpha = \frac{a}{2R}$

So
$$\boxed{B = \frac{\mu_0 i}{\pi a}\tan^{-1}\frac{a}{2R}}$$

---

5) 2-Parallel Conductors:



$$B_1 = \frac{\mu_0 i_1}{2\pi d}$$

$$\Rightarrow F_{21} = i_2 L B_1$$
$$= \frac{\mu_0 L i_1 i_2}{2\pi d}$$

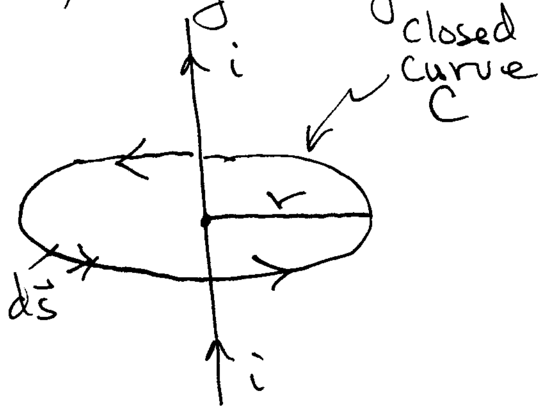
Likewise $F_{12} = \frac{\mu_0 L i_1 i_2}{d}$ pointing towards wire 2.
Attraction

\- IV B 5) This force law is used to define the unit of current – ampere

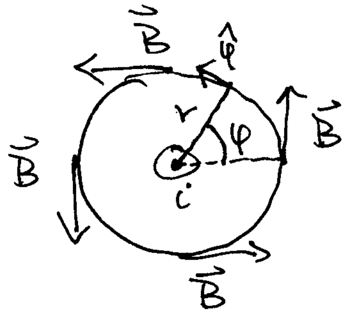
"Given 2 long parallel wires of negligible circular cross section separated in vacuum by a distance of 1 meter, the <u>ampere</u> is <u>defined</u> as the current in <u>each</u> <u>wire</u> that would produce a force of 2×10^{-7} newtons/meter of length."

C) Ampere's Law

1) Long Straight Wire



$$\vec{B} = \frac{\mu_0 i}{2\pi r}\,\hat{\varphi}$$ at points on circle



$d\vec{s}$ = element of path along circle

$= r\,d\varphi\,\hat{\varphi}$

$$\Rightarrow \quad \oint_C \vec{B}\cdot d\vec{s} = \oint_c \left(\frac{\mu_0 i}{2\pi r}\,\hat{\varphi}\right)\cdot\left(r\,d\varphi\,\hat{\varphi}\right)$$

- IV C 1) $\oint_C \vec{B}\cdot d\vec{s} = \int_{\varphi=0}^{2\pi} \frac{\mu_0 i}{2\pi}\, d\varphi = \mu_0 i$

$$= \mu_0 i$$

2) Ampere found this is generally true

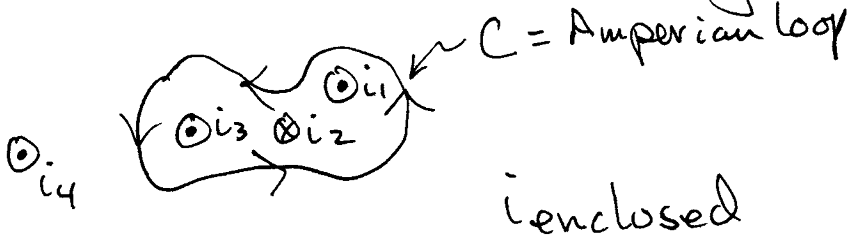
Ampere's Law

$$\boxed{\oint_C \vec{B}\cdot d\vec{s} = \mu_0 i_{enclosed}}$$

a) C is any closed curve called "Amperian loop"

b) $i_{enclosed}$ = net current that flows through surface bounded by C.

c)



$i_{enclosed} = i_1 + i_3 - i_2$

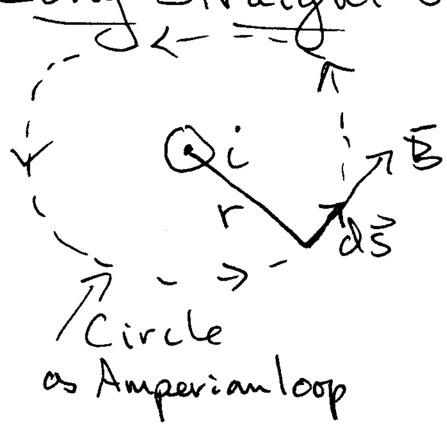
Right Hand Rule: fingers in direction of C, thumb points in "positive" current direction.

d) Ampere's Law replaces Biot-Savart law as one of the fundamental laws

IV C.2d) of magnetostatics.

3) Examples:

a) Long Straight Wire: By symmetry: $\vec{B} = B(r)\hat{\varphi}$



Ampere's Law

$$\oint_C \vec{B}\cdot d\vec{s} = \mu_0 i_{\text{enclosed}} = \mu_0 i$$

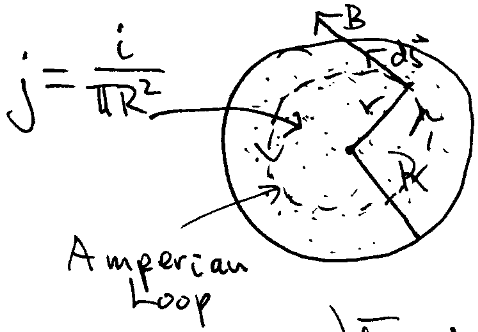
$$\Vert$$

$$B(r)\oint_C r\,d\varphi = B(r)\,2\pi r$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 i}{2\pi r}\hat{\varphi}}$$

---

b) Long cylindrical wire of radius R with i uniformly spread over cross section



By symmetry $\vec{B} = B(r)\hat{\varphi}$

Ampere's Law $\oint_C \vec{B}\cdot d\vec{s} = \mu_0 i_{\text{enclosed}}$

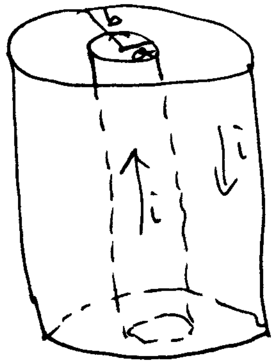
a) For $r \leq R$: $B(r)\,(2\pi r) = \mu_0\left(\frac{i}{\pi R^2}\right)(\pi r^2)$

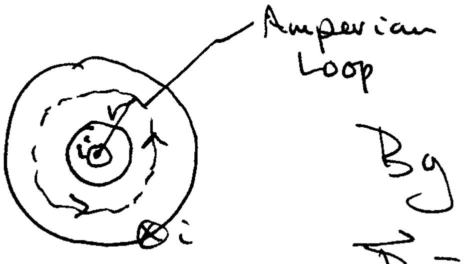
$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 i}{2\pi}\frac{r}{R^2}\hat{\varphi}}$$

- IV.C3b) b) For $r \geq R$: $\vec{B} = \frac{\mu_0 i}{2\pi r}\hat{\varphi}$.

---

c) Coaxial Cable





By symmetry
$\vec{B} = B(r)\,\hat{\varphi}$

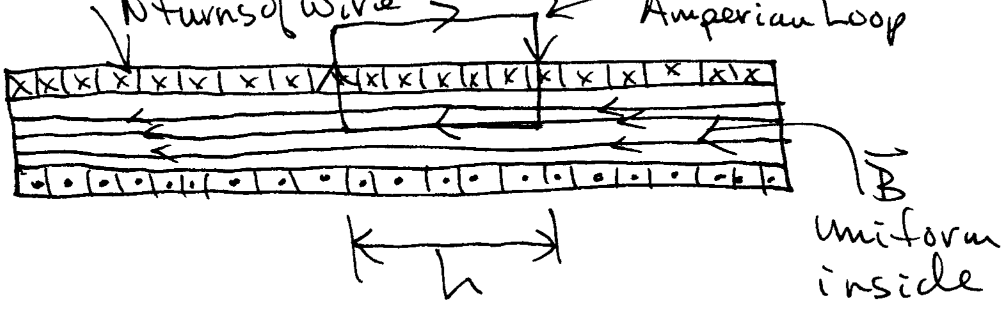
Ampere's Law:

$$\oint_C \vec{B}\cdot d\vec{s} = 2\pi r B(r) = \mu_0 i_{enclosed}$$

$$= \mu_0 \begin{cases} i & a<r<b \\ 0 & r>b \end{cases}$$

$$\Rightarrow \quad B(r) = \begin{cases} \frac{\mu_0 i}{2\pi r} & a<r<b \\ 0 & r>b \end{cases}$$

D) Solenoid : Assume ideal solenoid



Assume $\vec{B}=0$ outside to a good approximation (true for infinitely long solenoid)

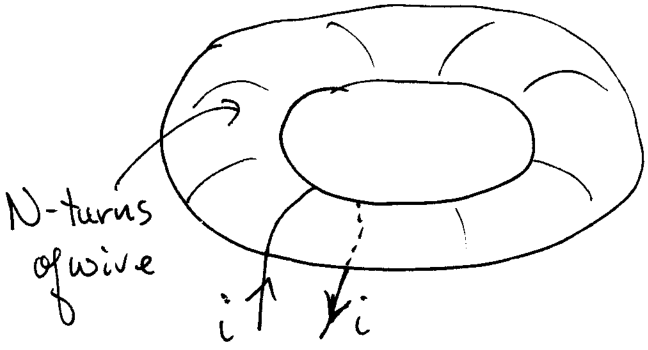
III C.3.D) Ampere's Law:

$$\oint_C \vec{B}\cdot d\vec{s} = Bh + \underset{\substack{\uparrow\\ \vec{B}\perp d\vec{s}\\ \Rightarrow \vec{B}\cdot d\vec{s}=0\\ \text{here}}}{0} + \underset{\substack{\uparrow\\ \vec{B}=0\\ \text{outside}}}{0} + \underset{\substack{\uparrow\\ \vec{B}\perp d\vec{s}\\ \Rightarrow \vec{B}\cdot d\vec{s}=0\\ \text{here}}}{0}$$

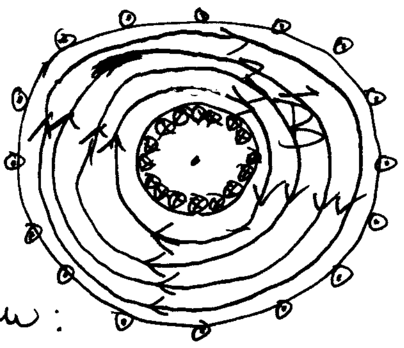
$$= \mu_0 \, i_{\text{enclosed}} = \mu_0 \, i \, \frac{N}{L}\cdot h$$

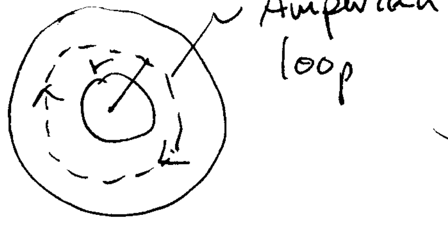
$$\Rightarrow \quad \boxed{B = \mu_0 i \underbrace{\frac{N}{L}}_{=n=\frac{\#\text{ of turns}}{\text{unit length}}} = \mu_0 i n}$$

E.) Toroid:



$\vec{B}$-lines form concentric circles inside toroid





Ampere's Law:

$$\oint_C \vec{B}\cdot d\vec{s} = B(r)\,2\pi r$$

$$= \mu_0 \, i_{\text{enclosed}}$$

$$= \mu_0 \, i N$$

IV C3E) Toroid: $\boxed{B(r) = \frac{\mu_0 i N}{2\pi r}}$

V.) Electromagnetic Induction

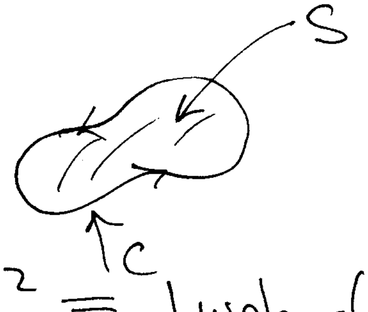
A) Faraday's Law: (emf form)

The induced emf in a circuit is equal to the negative of the rate at which the magnetic flux through the circuit is changing with time!

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

1) Magnetic flux through open surface S bounded by

$$\Phi_B = \int_S \vec{B} \cdot d\vec{A}$$



a) SI unit: $[\Phi_B] = 1\,T \cdot m^2 \equiv 1$ weber (Wb)

- V. A) 2) Lenz' Law: The induced current in a closed conducting loop appears in such a direction that it opposes the change that produced it. (this is the minus sign in Faraday's Law)

3) emf $\mathcal{E} = \oint_C \vec{f} \cdot d\vec{s}$ where $\vec{f}$ is the total force per unit charge acting on the charges in a circuit. In general, $\vec{f} = \vec{f_s} + \vec{E}$ where f_s is the force per unit charge the localized sources of emf (ie. battery, solar cell, generator etc.) exert on charges in the circuit.

In electrostatics the $\vec{E}$-field is conservative, hence $\oint_C \vec{E} \cdot d\vec{s} = 0$, so for DC circuits:

$$\mathcal{E} = \oint_C \vec{f_s} \cdot d\vec{s} = \int_{\text{source}} \vec{f_s} \cdot d\vec{s} = \frac{dW}{dq}$$

Faraday discovered that time varying magnetic flux produced a non-conservative electric field in a closed circuit

$$\mathcal{E} = \oint_C \vec{E} \cdot d\vec{s} = -\frac{d}{dt}\Phi_B$$

V. B) Faraday's Law: (induced electric field form)

$$\boxed{\oint_C \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{A}}$$

1) Lenz' Law (Re-visited): The induced electric field has direction so as to oppose the change that produces it.

2) The closed curve C is any arbitrary closed curve, not just physical circuits, S is any open surface with C as it's boundary.

3) Time varying magnetic fields produce electric fields!

4) The induced electric field is <u>not</u> conservative $\oint_C \vec{E}\cdot d\vec{s} \neq 0$

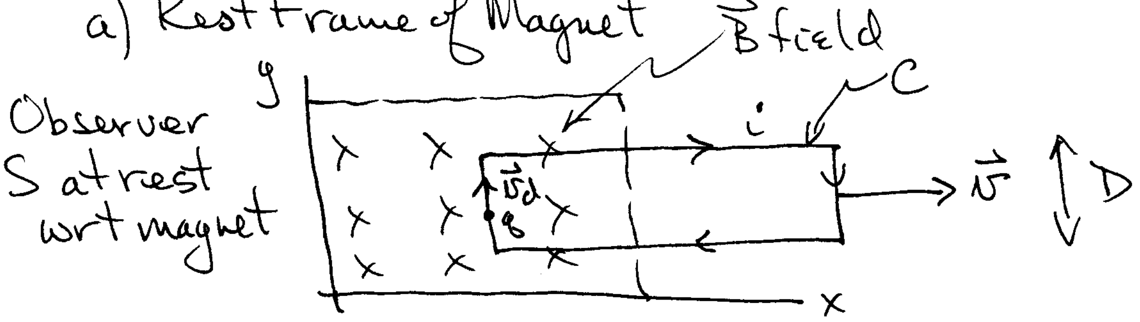
5) Induced electric field lines form closed loops

6) Faraday's law of electromagnetic induction, $\oint_C \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{A}$, is one of the fundamental laws of electromagnetism (it is one of the 4 Maxwell equations)

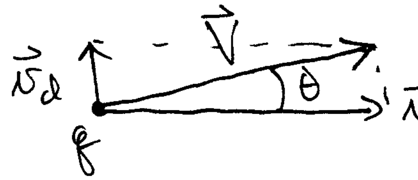
V. C) Examples:

1) Motional emf: Relative motion of $\vec{B}$-field and circuit produce emf

a) Rest Frame of Magnet



Lorentz force $\vec{F}_B = q\vec{V}\times\vec{B} = q\vec{v}\times\vec{B} + q\vec{v}_d\times\vec{B}$



Forces acting on q:

Wire exerts normal force on charges to keep them inside wire

$$\vec{N} = -q\vec{v}_d\times\vec{B} = F_B\sin\theta\,\hat{i} = qv_dB\,\hat{i}$$

Only $F_B\cos\theta$ is component of force along wire

So

$$\mathcal{E} = \oint_C \vec{f}\cdot d\vec{s} = \oint_C \vec{v}\times\vec{B}\cdot d\vec{s}$$

$$= vBD$$ (top & bottom paths give 0 and right side $\vec{B}=0$)

only from left side of C

V C. 1)a) Now the work done on the charges in the circuit is supplied by the external agent pulling the loop with force $\vec{N}$. In time dt the charge moves

$$d\vec{s} = \vec{V}dt = (\vec{v} + \vec{v}_d)\,dt$$

So the work done by the force $\vec{N}$ is

$$dW = \vec{N}\cdot d\vec{s} = N v\, dt$$

But

$$N = F_B \sin\theta = qVB\sin\theta = qVB\left(\frac{v_d}{V}\right) = q v_d B$$

So

$$dW = q v_d B\, v\, dt = q v B \underbrace{(v_d\, dt)}_{}$$

This is just the distance the charge moves up the left side of the wire.

Hence

$$W = \oint_C dW = q v B D$$

(again top & bottom paths give 0, and $\vec{B}=0$ for Right hand side)

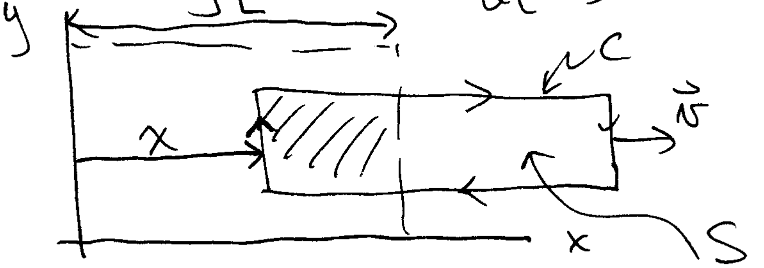
The work done per unit charge is just equal to the emf above

$$\mathcal{E} = \frac{dW}{dq} = vBD$$

as we found above.

The emf $\mathcal{E}$ is produced by the magnetic force $q\vec{v}\times\vec{B}$, the Work W performed is done by the pulling force, $\vec{N}$, yet $\mathcal{E} = dW/dq$.

V C.1a) This is the same result we obtain by applying the emf form of Faraday's Law directly $\mathcal{E} = -\frac{d}{dt}\Phi_B$



$$\Phi_B = \int_S \vec{B}\cdot d\vec{A} = BD(L-x)$$ (shaded portion has $B\neq 0$ only)

So $$\frac{d\Phi_B}{dt} = BD\left(-\frac{dx}{dt}\right) = -BDv$$

Faraday's Law: $\mathcal{E} = -\frac{d\Phi_B}{dt} = +BDv$ ✓
agrees with our force analysis.

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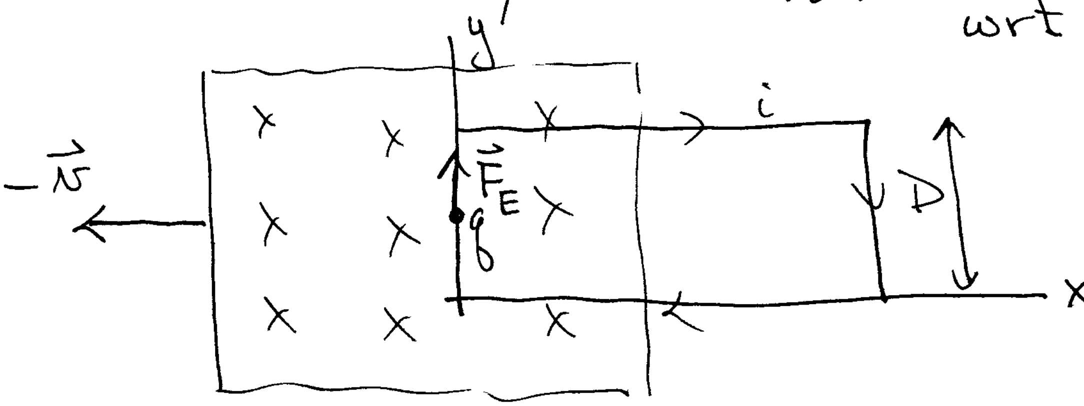
b.) Rest frame of wire loop: magnet moves past with $-\vec{v}$

Observer S' at rest wrt loop



Current i still flows in loop, the same emf will occur in circuit (for $v \ll c$) since the relative motion in the 2 cases

IVC1)b) is the same. But there is no magnetic force in this frame $\vec{v}' \times \vec{B} = 0$ since $\vec{v}' = 0$, the wire is at rest. Can only conclude the current flows due to an induced $\vec{E}$-field, $\vec{E}'$, in the wire

$$\mathcal{E}' = \oint_C \vec{E}' \cdot d\vec{s} = E' D$$

$$\left(\begin{array}{l}\text{same current}\\ \text{so } \mathcal{E}' = \mathcal{E}\end{array}\right) \quad = \mathcal{E} = vBD \qquad (v \ll c)$$

$$\Rightarrow \quad E' = vB$$ or in terms of vectors

$$\underline{\vec{E}' = \vec{v} \times \vec{B}}$$: Observer S' sees $\vec{E}'$ related to Observer S's $\vec{B}$-field.

---

c.) Observer S: stationary wrt magnet Says the force on the charges arises from magnetic field with

$$\mathcal{E} = \oint_C (\vec{v} \times \vec{B}) \cdot d\vec{s}$$

Observer S': stationary wrt loop attributes current to electric field $\vec{E}'$ with

$$\mathcal{E}' = \oint_C \vec{E}' \cdot d\vec{s}$$

V G(c) Third observer S'': magnet and loop moving
Force on charges both electric and magnetic

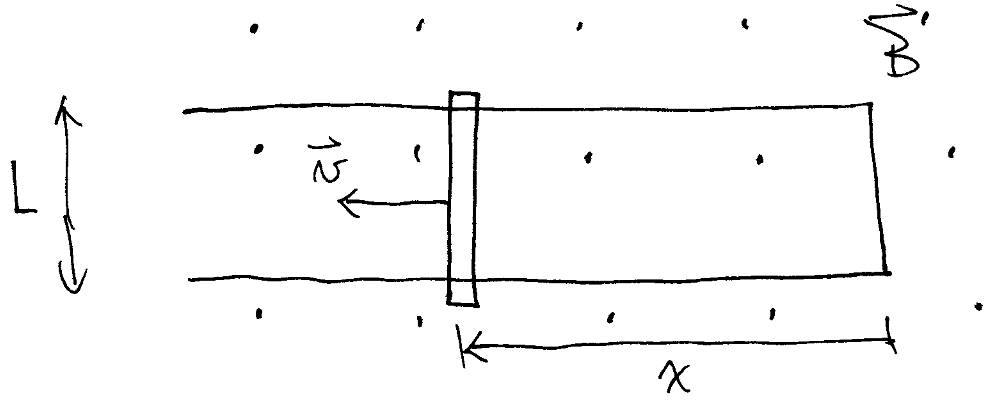
$$\vec{F}/q = \vec{E}'' + \vec{v}'' \times \vec{B}''$$

Conclusion: 1) $\vec{E}$ and $\vec{B}$ are not independent of eachother and have no separate unique existence, they depend on the inertial frame.

2) Still we find Faraday's Law holds in all inertial frames - it is covariant - that is form invariant

---

2) A conducting rod of length L is pulled along conducting rails at velocity $\vec{v}$ with a uniform $\vec{B}$ field ⊥ to it.



a) Induced emf in the rod: $\phi_B = BLx$

$$\text{(magnitude)} \quad \mathcal{E} = \frac{d\phi_B}{dt} = BL\frac{dx}{dt}$$

$$= BLv \quad \left(\begin{array}{l}\text{minus sign: Lenz law} \\ \text{current flows CW}\end{array}\right)$$

V C.2)b) Resistance of rod is R and Rails negligible
What is current?

$i = \mathcal{E}/R$ (Lenz' Law: flows CW)

c) What rate is internal energy being created in rod?

$P = i^2 R$

d) What is the force that must be applied by an external agent to the rod to maintain its motion?

Current in rod feels force

$$\vec{F} = i \vec{L} \times \vec{B}$$

$\Rightarrow F = iLB$ to the right.

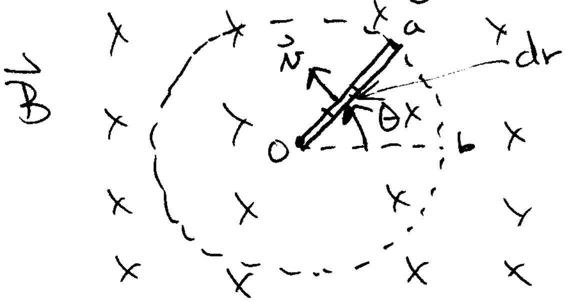
So agent must pull to the left with force $F = iLB$.

e) At what rate does this force do work on the rod?

$$P = Fv = iLBv = i\mathcal{E}$$

$= i^2 R$. External agent's work eventually appears as Joule heating.

V C.3) A copper rod of length R rotates at angular frequency ω in a uniform $\vec{B}$-field. Find the emf $\mathcal{E}$ developed between the 2 ends of the rod.



A motional emf $d\mathcal{E}$ develops across element dr of the rod as it moves with velocity $\vec{v}$.

Charge separation produced an $\vec{E}$-field across dr at steady state $\vec{v}\times\vec{B} = -\vec{E} \Rightarrow E = vB$
$= \omega r B$

The potential across dr is $E\,dr$ - these add up like batteries in series

$$\mathcal{E} = \int_0^R E\,dr = \omega B \int_0^R r\,dr = \tfrac{1}{2}\omega B R^2$$

$$\boxed{\mathcal{E} = \tfrac{1}{2}\omega B R^2}$$

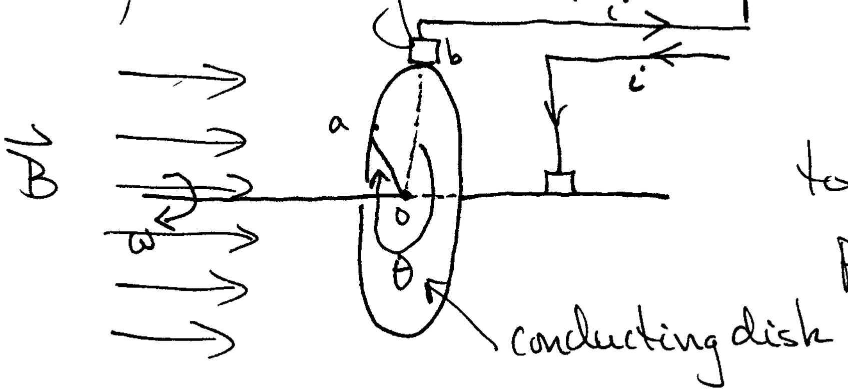
As well, directly from Faraday's Law we find the flux through aOb

$$\phi_B = \int_S \vec{B}\cdot d\vec{A} = B\left(\tfrac{1}{2}R^2\theta\right)$$

$$\frac{d\phi_B}{dt} = \tfrac{1}{2}BR^2\frac{d\theta}{dt} = \tfrac{1}{2}BR^2\omega$$

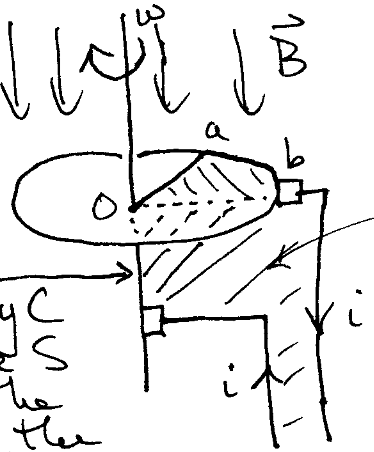
So the magnitude of $\mathcal{E} = \frac{d\phi_B}{dt} = \frac{1}{2}BR^2\omega$ ✓

V.C.4.) Faraday Disk (Homopolar Generator)



Use Faraday's law to find the emf produced by the spinning rotor

View from side



Open Surface S is formed by these 2 planar surfaces meeting at right angles

The boundary C of the surface S is given by the circuit outside the disk and Oab curve on disk

The flux $\Phi_B = \int_S \vec{B}\cdot d\vec{A} = \frac{1}{2}BR^2(2\pi-\theta) + 0$

$\vec{B}$ Through area OabO on disk

for rest of circuit for which either $\vec{B}=0$ or $\vec{B}\cdot d\vec{A}=0$

Hence the emf by Faraday's Law is

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = \frac{1}{2}\omega BR^2 = \pi BR^2\nu$$

(where $\nu = \frac{\omega}{2\pi}$)

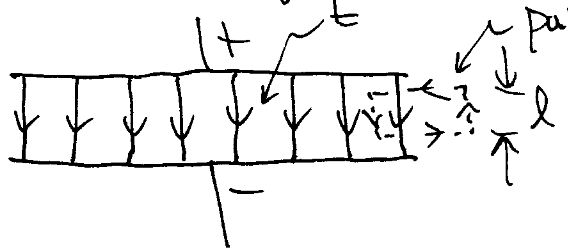
V C.4) The torque that must be provided to keep the rotor spinning when current i is produced is found from

$$\text{Power} = \mathcal{E} i \quad \text{also Power} = \tau \omega$$

$$\Rightarrow \quad \tau = \frac{\mathcal{E} i}{\omega} = \frac{1}{2} B R^2 i$$

---

5) $\vec{E}$ - must fringe for a $\parallel$-plate capacitor



Assume <u>no</u> fringing

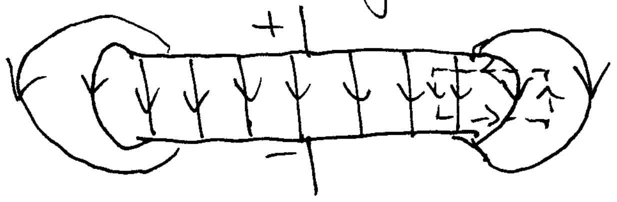
There is no $\vec{B}$-field in this problem so $\phi_B = 0$; $\frac{d\phi_B}{dt} = 0$

$$\Rightarrow \quad \mathcal{E} = 0 \quad \text{by Faraday's law}$$

but $\quad \mathcal{E} = \oint_C \vec{E} \cdot d\vec{s} = E l = 0$

$$\Rightarrow E = 0 \quad \underline{\text{contradiction!}}$$

$\vec{E}$ must fringe so that $\mathcal{E} = \oint_C \vec{E} \cdot d\vec{s} = 0$



VI) Magnetic Properties of Matter

A) Material placed in a magnetic field is either

1) Diamagnetic: when placed in an external magnetic field $\vec{B}_o$, it tends to lessen the resultant field $\vec{B}$. It is repelled from a region of greater magnetic field to lesser magnetic field.

2) Paramagnetic: when placed in an external magnetic field $\vec{B}_o$, it tends to increase the resultant $\vec{B}$-field. It is attracted toward a region of greater magnetic field from lesser magnetic field.

a) Ferromagnetic: when placed in an external magnetic field $\vec{B}_o$, it greatly increases ($\sim 10^2$) the resultant magnetic field $\vec{B}$.

B) Magnetic properties are due to atomic motion of electron clouds swirling about nucleus and electrons having intrinsic spin. This spin and orbital motion of the electrons results in atomic magnetic moments ("atomic currents")

1) No net magnetic moment results in diamagnetic properties — in an external $\vec{B}$-field a magnetic moment is induced in the opposite direction to $\vec{B}_o$.

VI) B2) Net magnetic moment results in paramagnetic material – in an external $\vec{B}$-field the magnetic moments tend to align along the direction of $\vec{B}_0$.

C) Magnetization, $\vec{M}$, is the dipole moment per unit volume at each macroscopic point of the material — characterizes all macroscopic magnetic properties of the material!

$$\vec{M} \equiv \lim_{\Delta V \to 0}^{\text{macro}} \frac{\langle \vec{\mu} \rangle}{\Delta V}$$

where $\langle \vec{\mu} \rangle$ is the average over the magnetic dipole moments of all the many atoms i the macroscopically small, but microscopically large, volume element ΔV of material.

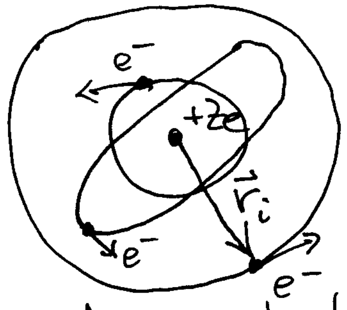
$$\langle \vec{\mu} \rangle = \underset{\substack{a \\ \text{atoms} \\ \text{in}\,\Delta V}}{\Sigma_i} \vec{\mu}_a .$$

If each $\vec{\mu}_a \simeq \vec{\mu}$, then $\vec{M} = N\vec{\mu}$ where $N = \frac{\#\text{ of atoms}}{\text{volume}}$.

1) SI units of $\vec{M}$ are $\frac{\text{Ampere}}{\text{meter}}$

VI D) Microscopic View of Magnetic Materials:
Use simplified planetary model of atom

1)



circular orbits

$$\vec{v}_i = \vec{\omega}_i \times \vec{r}_i$$

i^{th} electron at radius $\vec{r}_i$ with velocity $\vec{v}_i$ ⇒ atomic current

$$i = \frac{e v_i}{2\pi r_i} = \frac{e\omega_i}{2\pi} = \frac{dq}{dt}$$

and $\tau = \frac{2\pi}{\omega}$ = time to make one orbit.

This current has an associated magnetic dipole moment

magnitude : $\mu_i = iA = i\pi r_i^2$

$$= \frac{e\omega_i}{2\pi}\pi r_i^2$$

$$= \frac{1}{2}\left(\frac{e}{m}\right)\underbrace{(m\omega_i r_i^2)}_{=L_i}$$

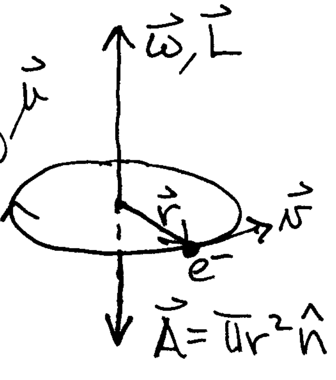
$$\mu_i = \frac{1}{2}\frac{e}{m}L_i$$

$\vec{L}_i$ = orbital ∡ momentum of i^{th} electron

$$\left(\vec{L} = \vec{r}\times\vec{p} = m\vec{r}\times\vec{v} = m\vec{r}\times(\vec{\omega}\times\vec{r}) = mr^2\vec{\omega}\right)$$

direction of $\vec{\mu}$ opposite $\vec{L}$

direction of positive charge flow! → i



$\vec{\mu} = i\vec{A}$

$$\boxed{\vec{\mu}_i = -\frac{e}{2m}\vec{L}_i}$$

VI.)D1) Total magnetic moment due to orbital motion of electrons

$$\vec{\mu}_L = -\frac{e}{2m}\underbrace{\sum_i \vec{L}_i}_{\equiv \vec{L}}$$

$$\boxed{\vec{\mu}_L = -\frac{e}{2m}\vec{L}}$$

a) Bohr magneton: $\mu_B \equiv \frac{e\hbar}{2m}\left(=9.27\times10^{-2}\,\frac{J}{T}\right)$

2) Electrons have intrinsic spin angular momentum S also which gives rise a spin magnetic dipole moment

$$\vec{\mu}_{s_i} = -\frac{eg}{2m}\vec{S}_i \qquad g \approx 2 \text{ for } e^- \quad \text{"g-factor"}$$

Total magnetic moment due to spin of electrons

$$\vec{\mu}_s = -\frac{eg}{2m}\underbrace{\sum_i \vec{S}_i}_{\equiv \vec{S}}$$

$$\boxed{\vec{\mu}_s = -\frac{eg}{2m}\vec{S}}$$

VI.) D2) Hence the total atomic magnetic moment is the vector sum of the orbital and spin moments

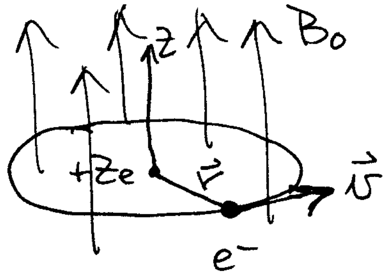
$$\vec{\mu} = -\frac{e}{2m}\left(\underbrace{\vec{L}+g\vec{S}}_{\neq\ \vec{J}}\right)$$

$\vec{J}$ = total $\measuredangle$ momentum of atom $= \vec{L}+\vec{S}$

3) So if $\vec{L}=0=\vec{S} \Rightarrow \vec{\mu}=0 \Rightarrow \vec{M}=0$ hence there is no net magnetization of the material. This is necessary for a diamagnetic material ~~&~~ we must still show opposite to $\vec{B}_0$ direction for the induced $\vec{\mu}$.

a) Lenz's Law: place atom in $\vec{B}$ field ⇒ flux through atomic orbits increases, Lenz's law implies atomic motion will change to oppose this increase in flux (i.e. e^- speed up or slow down), the atomic dipole moments will change (induced) to oppose the external $\vec{B}$-field ⇒ repulsion ⇒ diamagnetic material

- VI D3.b) Quantatative argument: Place atom in an external $\vec{B}$-field along the z-direction



Total force on electron

$$\vec{F} = \underbrace{-e\vec{v}\times\vec{B}_0}_{\text{magnetic force due to } \vec{B}_0} - \underbrace{\frac{Ze^2\hat{r}}{4\pi\epsilon_0 r^2}}_{\text{Coulomb force attraction of nucleus}}$$

circular motion ($\vec{v}=\vec{\omega}\times\vec{r}$) ⇒

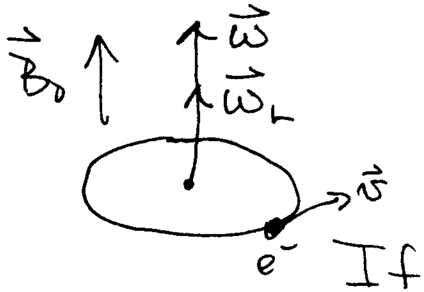
$$\vec{F} = \overset{\text{attractive force}}{-}\left[e\omega r B_0 + \frac{Ze^2}{4\pi\epsilon_0 r^2}\right]\hat{r}$$

Newton's 2nd law ⇒ $-mr\omega^2 = F = -\left[e\omega r B_0 + \frac{Ze^2}{4\pi\epsilon_0 r^2}\right]$

⇒ quadratic equation for ω

$$\omega \simeq \frac{e}{2m}B_0 \pm \underbrace{\frac{e}{2m}\sqrt{\frac{Zm}{\pi\epsilon_0 r^3}}}_{\equiv \omega_0}$$

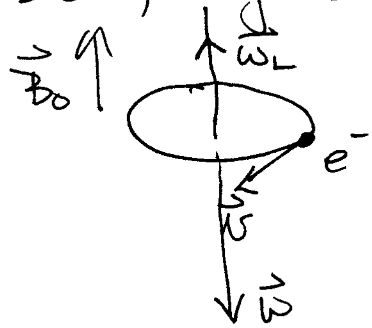
± Cases: 1) Positive Root:



$$\omega = \omega_0 + \underbrace{\frac{e}{2m}B_0}_{\equiv \omega_L = \text{Larmor frequency}}$$

If $\vec{B}_0 \parallel \vec{\omega}_0$ then ω increases to $\vec{\omega} = (\omega_0+\omega_L)\hat{k}$ ⇒ $\vec{L}$ increases since $\vec{L} = mr^2\vec{\omega}$ BUT $\vec{\mu}_i = -\frac{e}{2m}\vec{L}_i$ so $\vec{\mu}_i$ decreases ⇒ $\vec{\mu}_{\text{induced}}$ opposite $\vec{B}_0$

VI D3b2) Negative Root: $\vec{\omega} = -\omega_0 \hat{k} + \omega_L \hat{k}$



$\omega = |\vec{\omega}| = |\omega_0 - \omega_L|$ <u>decreases</u>

Since $\vec{\omega}_{induced} = \omega_L \hat{k}$

$\Rightarrow$ induced $\vec{L}_{induced} = mr^2 \vec{\omega}_{induced}$ is along $\vec{B}_0$

$\Rightarrow$ <u>induced magnetic moment opposite $\vec{B}_0$</u>

Both cases $\Rightarrow$

$$\boxed{\vec{\mu}_{induced} = -\frac{e}{2m}\vec{L}_{induced} = -\frac{e^2 r^2}{4m}\vec{B}_0}$$

Now Z electrons with different orbits - we must average for total induced moment — need QM result is

$$\boxed{\vec{\mu} = \langle \vec{\mu}_{induced} \rangle_{QM} = -\frac{Ze^2}{6m} r_0^2 \vec{B}_0}$$

with r_0^2 = mean square radius of atom

Hence

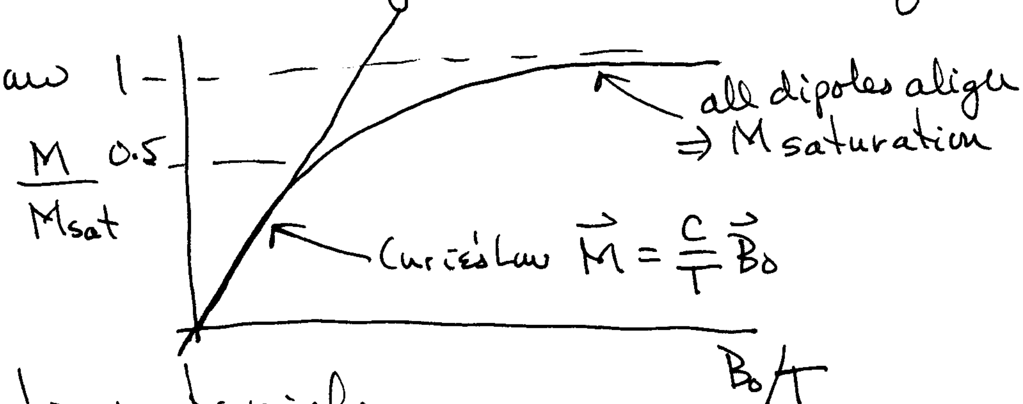
$$\boxed{\vec{M} = -\left[\frac{NZe^2 r_0^2}{6m}\right]\vec{B}_0}$$

$\vec{M}$ is opposite direction of $\vec{B}_0 \Rightarrow$ <u>decreases resultant field</u>

VI D 3b) So this is diamagnetic material.

4) Paramagnetic material: atomic magnetic moments do not sum to zero ⇒ molecules have a permanent magnetic dipole moment — these tend to align in external $\vec{B}$-field

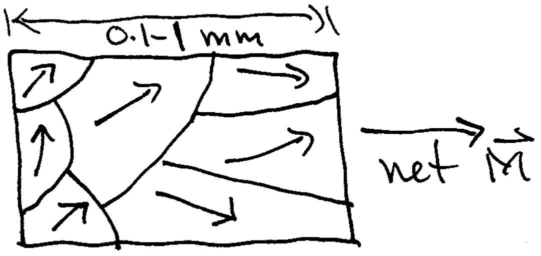
Curie's Law



Paramagnetic materials

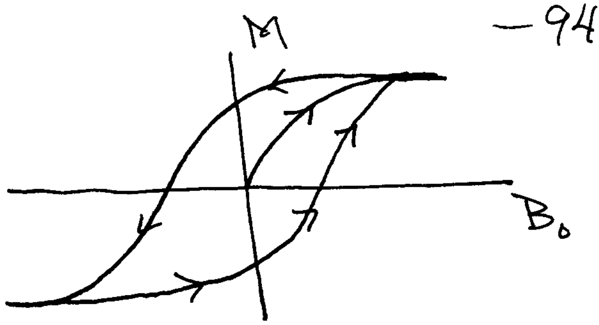
$$\vec{M} = \left[\frac{C}{T} - \frac{NZe^2 r_0^2}{6m}\right]\vec{B}_0 \approx \frac{C}{T}\vec{B}_0$$

5) Ferromagnetic material: Large paramagnetic effects — electron spin magnetic moment and strong interactions between neighboring atoms ⇒ alignment of dipoles even after external $\vec{B}$-field is removed



Macroscopic regions of aligned moments called domains

VII D5) Hysteresis Curve

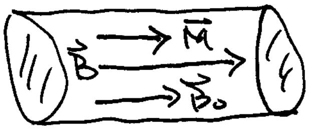


E) Macroscopic Fields & the Magnetization Current

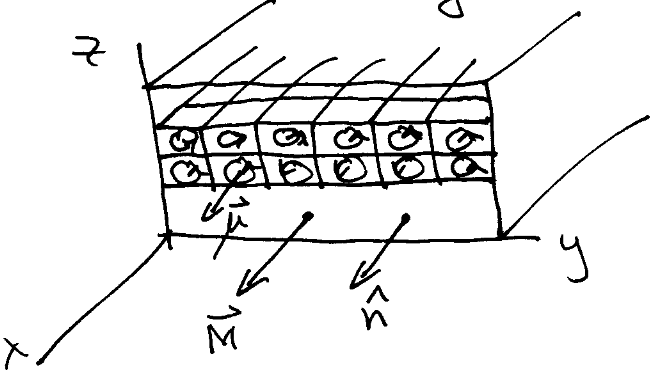
$$\vec{B} = \vec{B}_0 + \vec{B}_M$$

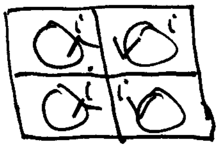
Total external due to Magnetization $\vec{M}$ of material

1) Consider solenoid containing magnetic material



$\vec{M}\Delta V = \vec{\mu}$ each magnetic moment $\vec{\mu}$ is viewed as caused by an atomic current loop





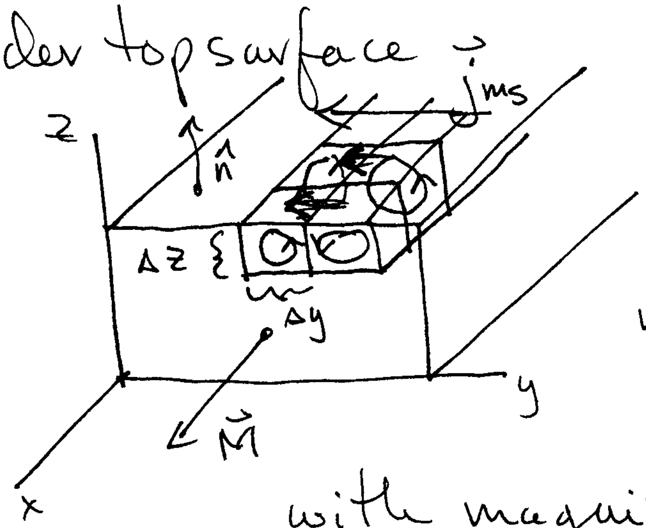
adjacent current loops flow in opposite direction — <u>no net flow</u>

So the current density flowing in a macroscopically thin layer of this surface will be zero

$$\boxed{\vec{j}_{M_s} = 0 \text{ when } \vec{M} \parallel \hat{n}}$$

But

- VII E) Consider top surface



A surface current density appears to flow across top surface in the direction $\vec{M}\times\hat{n}$

with magnitude

$$j_{MS}=\frac{i}{\Delta x}$$

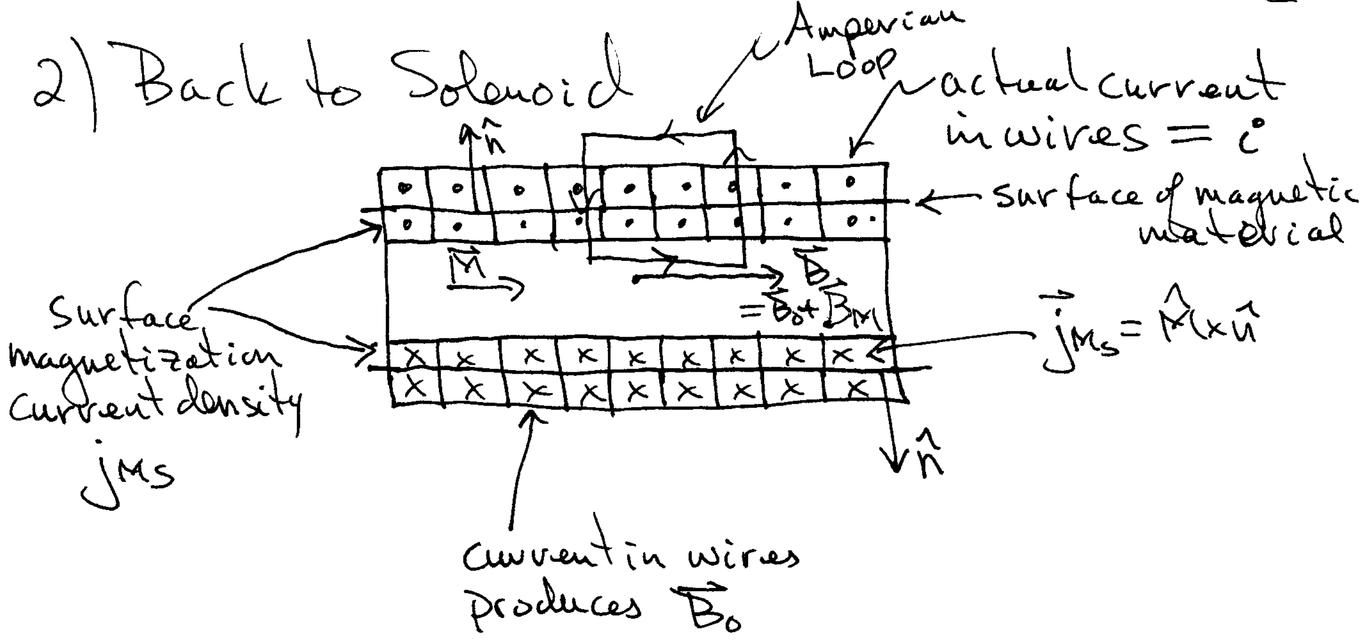
But $\mu = iA = i\Delta y\Delta z = M\Delta V = M\Delta x\Delta y\Delta z$

$$\Rightarrow M=\frac{i}{\Delta x}=j_{MS}$$

$$\Rightarrow \boxed{\vec{j}_{MS}=\vec{M}\times\hat{n}}$$

The magnetization current density

2) Back to Solenoid



VI)E2) Ampere's Law: $\oint_C \vec{B}\cdot d\vec{s} = \mu_0 i_{enclosed}$

$$\| $$

$$Bh = \mu_0\left[\frac{Nh}{L} i + j_{Ms} h\right]$$

$$= \mu_0 \frac{Nh}{L} i + \mu_0 M h$$

$$\Rightarrow \boxed{B = \underbrace{\mu_0 \frac{N}{L} i}_{=B_0} + \underbrace{\mu_0 M}_{=B_M}} \qquad \text{i.e. } \vec{B}_M = \mu_0 \vec{M}$$

So in general Ampere's law can be written

$$\oint_C \vec{B}\cdot d\vec{s} = \mu_0 i_{free} + \mu_0 \underbrace{\oint_C \vec{M}\cdot d\vec{s}}_{=i_M}$$

conduction current (for i_{free})

magnetization current (for i_M)

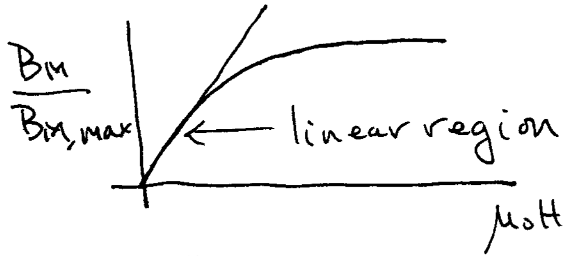
That is defining the magnetic intensity field

$$\vec{H} \equiv \frac{1}{\mu_0}(\vec{B} - \mu_0\vec{M}) \quad \left(= \frac{1}{\mu_0}\vec{B}_0\right)$$

Ampere's law can be written as

$$\boxed{\oint_C \vec{H}\cdot d\vec{s} = i_{\substack{conduction\\(free)}}}$$

VI) F) <u>Constitutive Equation</u>: For paramagnetic and diamagnetic materials that are isotropic and linear



$$\boxed{\vec{M} = \chi_M \vec{H}}$$ linear, isotropic magnetic material.

$\chi_M \equiv$ <u>magnetic susceptibility</u>

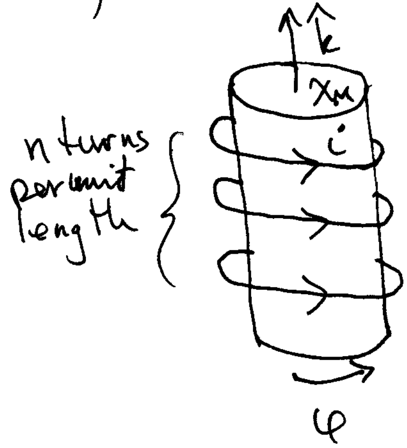
But $$\vec{B} = \vec{B}_0 + \mu_0 \vec{M} = \mu_0 \vec{H} + \mu_0 \vec{M}$$
$$= \mu_0 (1 + \chi_M) \vec{H} \equiv \boxed{\mu \vec{H} = \vec{B}}$$ for linear, isotropic magnetic material.

$\mu \equiv \mu_0 (1 + \chi_m) =$ <u>permeability</u> of material

(Other notation: $K_M = \mu_r \equiv \frac{\mu}{\mu_0}$ = relative permeability)

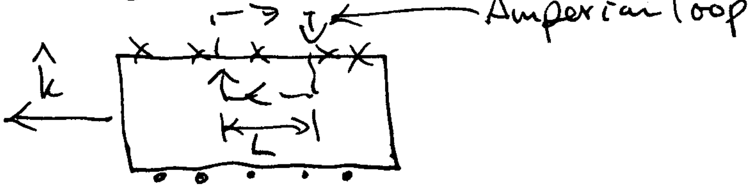
VI G) Examples

1) Solenoid filled with magnetic material



Ampere's law

$$\oint_C \vec{H}\cdot d\vec{s} = i_{free}$$

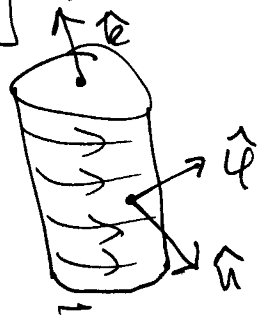


$$\Rightarrow HL = inL \Rightarrow \vec{H} = in\hat{k}$$

$$\Rightarrow \boxed{\begin{aligned}\vec{B} &= \mu_0(1+\chi_m)\vec{H} \\ &= \mu_0(1+\chi_m)\, in\,\hat{k}\end{aligned}}$$

a) If material is paramagnetic $\chi_m > 0 \Rightarrow B > B_0$
if diamagnetic $\chi_M < 0 \Rightarrow B < B_0$.

b) The magnetization current density runs around cylinder's surface



$$\vec{j}_{Ms} = \vec{M}\times\hat{n} = \chi_M \vec{H}\times\hat{n}$$

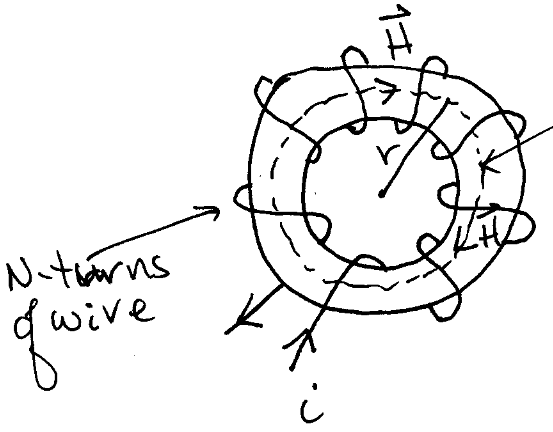
$$= \chi_M in\,\underbrace{\hat{k}\times\hat{n}}_{=\hat{\varphi}}$$

$$\boxed{\vec{j}_{Ms} = \chi_M in\,\hat{\varphi}}$$

$\chi_M > 0$ $\vec{j}_{Ms}$ is same direction as i

$\chi_M < 0$ $\vec{j}_{Ms}$ opposite i

-VII. G.) 2) Toroid with magnetic material



(Note: H=0=B outside toroidal Volume)

Ampere's Law

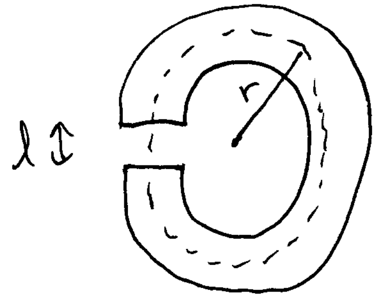
$$\oint_C \vec{H}\cdot d\vec{s} = i_{free}$$

$$H 2\pi r = Ni$$

$$\Rightarrow H = \frac{Ni}{2\pi r}$$

But $\vec{B} = \mu\vec{H} \Rightarrow B = \frac{\mu N i}{2\pi r} = (1+\chi_m)\frac{\mu_0 N i}{2\pi r}$

3) Torus with a gap



$B_{in} = B_{gap}$ ($\oint_S \vec{B}\cdot d\vec{A} = 0 = B_{in}A - B_{gap}A$ $\Rightarrow B_{in} = B_{gap}$)

So $B_{in} = B_{gap} \equiv B(r)$

But $H_{inside} = \frac{B}{\mu}$; $H_{gap} = \frac{B}{\mu_0}$

Now apply Ampere's Law

$$\oint_C \vec{H}\cdot d\vec{s} = i_{free} = Ni$$

$$\parallel$$

$$H_{in}(2\pi r - l) + H_{gap}l = \frac{B}{\mu}(2\pi r - l) + \frac{B}{\mu_0}l$$

VI.G) 3) $\Rightarrow$ $$\boxed{B = \frac{\mu\mu_0 N i}{\mu l + \mu_0(2\pi r - l)}}$$

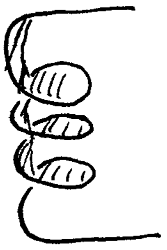
for ferromagnetic materials $\mu l >> 2\pi r$
($\mu = \mu(H)$)
$$B \simeq \frac{\mu_0 N i}{l} >> \frac{\mu_0 N i}{2\pi r}$$

VII) Inductance:

A) Self inductance: Faraday's law $\mathcal{E}_L = -\frac{d\phi_B}{dt}$
$$= -L\frac{di}{dt}$$
$$L = \frac{d\phi_B}{di} = \frac{\phi_B}{i} \text{ for linear media.}$$

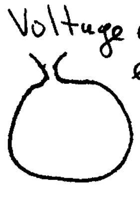
1) SI unit of inductance = henry $(H) \equiv 1 \frac{\text{Volt}\cdot\text{sec.}}{\text{Ampere}}$

2) Solenoid: a) flux linkages = total flux thru solenoid.



$$\phi_B = nl\,\phi_{B\text{coil}} = nl\,B\pi r^2$$
$$= \mu_0 n^2 l A i$$
$$\Rightarrow L = \mu_0 n^2 l A$$

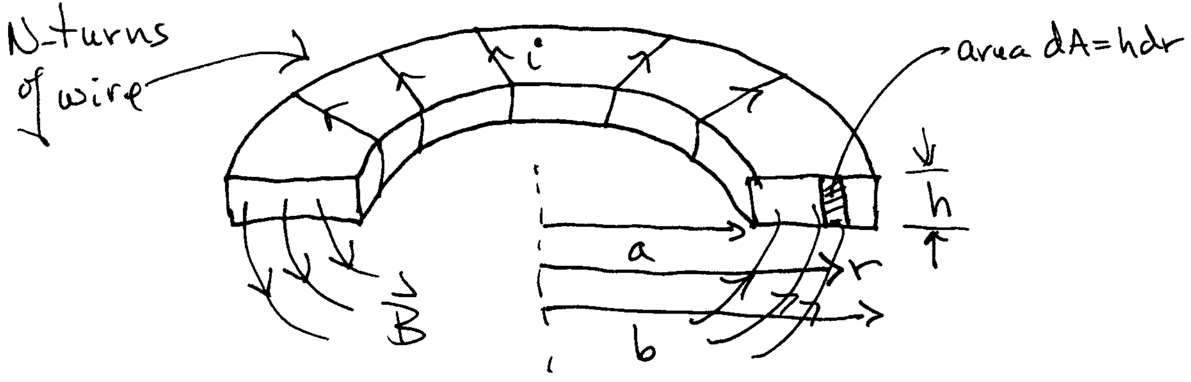
b)



$$\mathcal{E}_{coil} = -\frac{d\phi_{B\text{coil}}}{dt} = -\mu_0 n \pi r^2 \frac{di}{dt}$$

But there are nl-coils $\Rightarrow \mathcal{E}_L = nl\cdot\mathcal{E}_{coil}$
$$\Rightarrow L = \mu_0 n^2 l A$$

VII.A) 3) Toroid of rectangular cross section:



Total flux through coil of wire $\phi_B = N\int_{1\,\text{coil}} \vec{B}\cdot d\vec{A}$

$$\phi_B = N\int_{r=a}^{b} B(r)\,h\,dr$$

Use Ampere's law to find $B(r)$

$$\oint_c \vec{B}\cdot d\vec{s} = \mu_0\, i_{\text{enclosed}} = \mu_0 N i$$

$$\underset{B(r)2\pi r}{\parallel} \qquad \Rightarrow B(r) = \frac{\mu_0 N i}{2\pi r}$$

So $\phi_B = \dfrac{\mu_0 N^2 i h}{2\pi}\ln\left(\dfrac{b}{a}\right) i$

$$\Rightarrow \boxed{L = \frac{\mu_0 N^2 h}{2\pi}\ln\left(\frac{b}{a}\right)}$$

---

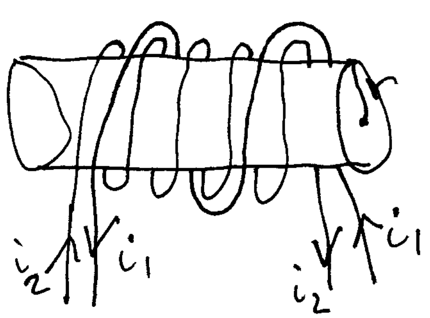
4) Fill inductors with magnetic material

$$\vec{B} = \frac{\mu}{\mu_0}\vec{B}_0 = (1+\chi_m)\vec{B}_0 \leftarrow \vec{B}\ \text{without magnetic material}$$

- VIII.A.4) ⇒ Solenoid: $L = \mu n^2 A l$

Toroid: $L = \frac{\mu N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)$

B) Mutual Inductance: Wrap 2 independent coils of wire around solenoid



coil 1: $\phi_{B_1} = \phi_{B11} + \phi_{B12}$

coil 2: $\phi_{B_2} = \phi_{B21} + \phi_{B22}$

ϕ_{B21} = flux thru coil 2 due to B_1 = $\mu_0 n_1 n_2 l A\, i_1$

ϕ_{B12} = flux thru coil 1 due to B_2 = $\mu_0 n_1 \mu_2 l A\, i_2$

ϕ_{B11} = $\overset{\text{coil 1}}{\text{self-linkages}}$ = $\mu_0 n_1^2 l A\, i_1$

ϕ_{B22} = $\underset{\text{coil 2}}{\text{self-linkages}}$ = $\mu_0 n_2^2 l A\, i_2$

So Faraday's law ⇒

$$\mathcal{E}_1 = -L_1 \frac{di_1}{dt} - M \frac{di_2}{dt}$$

$$\mathcal{E}_2 = -L_2 \frac{di_2}{dt} - M \frac{di_1}{dt}$$

$M = \sqrt{L_1 L_2}$

L_1 = self-inductance of coil 1 = $\frac{\phi_{B11}}{i_1} = \mu_0 n_1^2 l A$

L_2 = self-inductance of coil 2 = $\frac{\phi_{B22}}{i_2} = \mu_0 n_2^2 l A$

M = mutual inductance = $\frac{\phi_{B12}}{i_2} = \frac{\phi_{B21}}{i_1} = \mu_0 n_1 n_2 l A$

-VII. c.) Magnetic Energy: Work required to move charge through inductor

$$dW = -\mathcal{E}\,dq = dq\,\frac{d\phi_B}{dt}$$

But $dq = i\,dt$ & $\frac{d\phi_B}{dt} = \frac{d\phi_B}{di}\frac{di}{dt} = L\frac{di}{dt}$.

Hence $dW = i\,dt\,L\frac{di}{dt} = L\,i\,di$

This is stored in the magnetic field of the inductor — it is the increase in Potential Energy stored in the magnetic field:

$$dU_B = dW = L\,i\,di$$

$$\Rightarrow \quad \boxed{U_B = \tfrac{1}{2}Li^2 = \tfrac{1}{2}i\,\phi_B}$$

1) Applied to a solenoid: $L = \mu_0 n^2 A l \Rightarrow$

$$U_B = \tfrac{1}{2}\mu_0 n^2 i^2 (Al) \quad \text{but } B = \mu_0 n i$$

$$\Rightarrow \quad \boxed{U_B = \frac{1}{2\mu_0}B^2(Al) = \frac{1}{2\mu_0}\int_{\substack{\text{All}\\\text{Space}}} \vec{B}\cdot\vec{B}\,dV}$$

2) Energy density stored in magnetic field

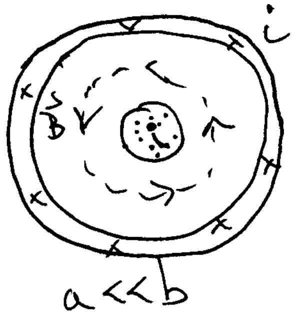
$$\boxed{u_B = \frac{U_B}{Al} = \frac{1}{2\mu_0}\vec{B}\cdot\vec{B}}\;; U_B = \int_{\substack{\text{All}\\\text{Space}}} u_B\,dV$$

VII.C) 3) If a magnetic material is present

$$\boxed{\begin{array}{c} U_B = \frac{1}{2}\int_{\substack{\text{All}\\\text{Space}}} \vec{B}\cdot\vec{H}\,dV \\ \\ u_B = \frac{1}{2}\vec{B}\cdot\vec{H} \end{array}}$$

---

4) Example: Coaxial Cable: What is the energy stored for a length l of cable.



inner conductor radius = a
outer conductor inner radius = b

Ampere's Law

$$\oint_C \vec{B}\cdot d\vec{s} = \mu_0 i_{enclosed}$$

$$\|$$

$$B\,2\pi r = \mu_0 i$$

$$B(r) = \frac{\mu_0 i}{2\pi r}$$

So energy stored in magnetic field $(a \le r \le b)$

$$u_B = \frac{1}{2\mu_0}B^2 = \frac{\mu_0 i^2}{8\pi^2 r^2}$$

Assume center conductor is small radius and outer conductor thin $\Rightarrow$

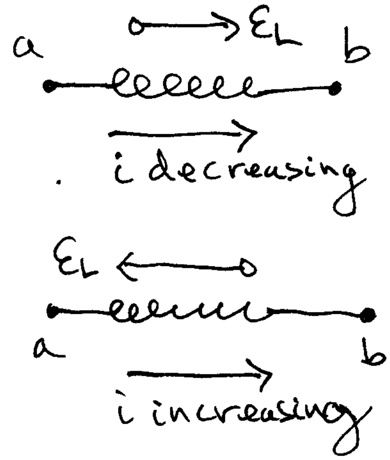
$$U_B = \frac{1}{2}\int_{r=a}^{b} \frac{1}{\mu_0}B^2\,dV = \frac{\mu_0 i^2 l}{4\pi}\ln\left(\frac{b}{a}\right)$$

VII.C.4) Recalling that $U_B = \frac{1}{2} L i^2$ we can determine the inductance of the cable from U_B above

$$\boxed{L = \frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a}\right)}$$ which agrees with $L = \frac{\Phi_B}{i}$

D) LR-Circuits

1) Potential Difference across inductor (Lenz' Law)



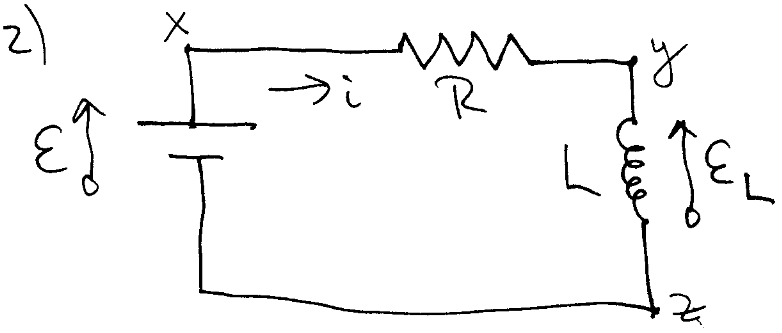
$V_b > V_a$

$V_a > V_b$

$V_b - V_a = -L \frac{di}{dt}$ in both cases

---

2)



Current Build Up

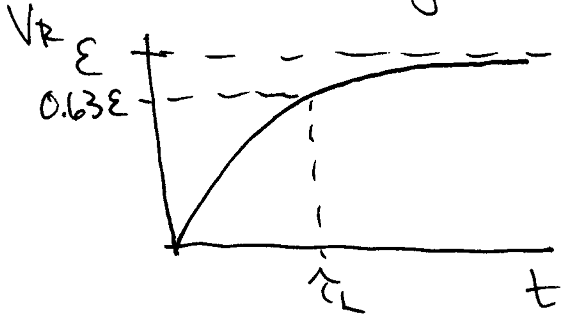
Loop Rule: $\mathcal{E} = iR + L \frac{di}{dt}$

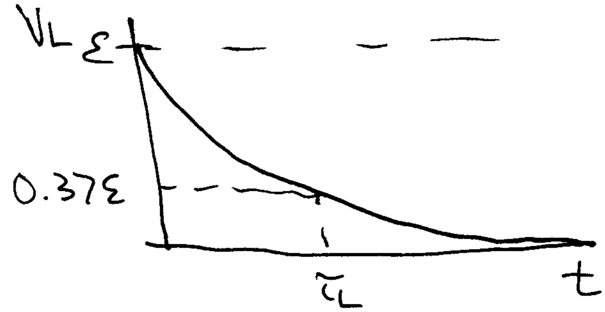
DE for i $\Rightarrow$ $i(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau_L}\right)$

-VII.D) 2) with $\tau_L \equiv L/R$ = inductive time constant

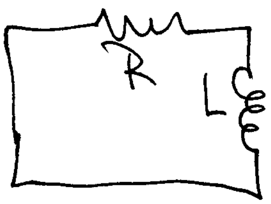
So $V_R = V_x - V_y = iR = \mathcal{E}(1-e^{-t/\tau_L})$

$V_L = V_y - V_z = -\mathcal{E}_L = L\frac{di}{dt} = \mathcal{E}e^{-t/\tau_L}$





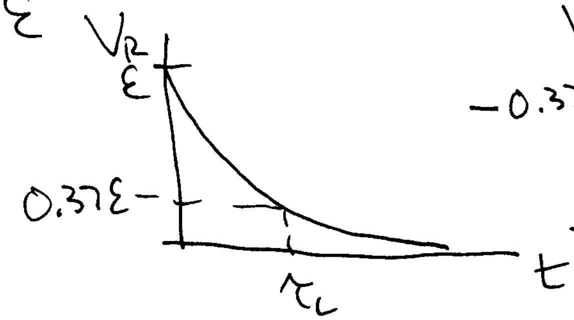
3) Current Decay

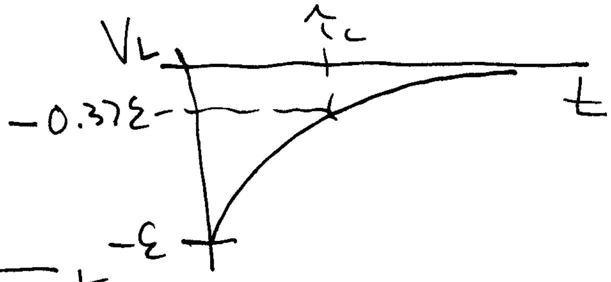


Loop Rule: $L\frac{di}{dt} + iR = 0$

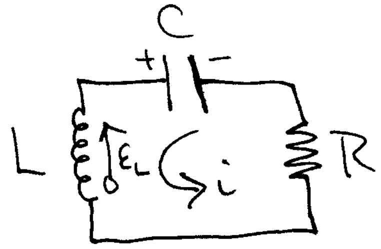
$\Rightarrow \boxed{i(t) = i_0 e^{-t/\tau_L}}$

Let $i_0 = \mathcal{E}$





VII. E) LCR-Circuit: Initially charged capacitor



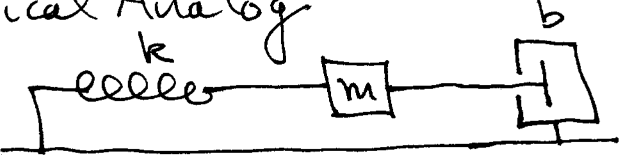
Loop Rule:

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=0$$

with initial conditions $q(0)=q_m$

$$\frac{dq(0)}{dt}=0.$$

(Mechanical Analog



$F=-kx-bv$

Newton's 2nd Law $\vec{F}=m\vec{a} \Rightarrow$

with i.c. $x(0)=x_m$ $\quad m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$)

$$\frac{dx(0)}{dt}=0$$

For $4\frac{L}{C}>R^2$ $\quad q(t)=q_0e^{-\frac{Rt}{2L}}\cos(\omega' t+\phi)$

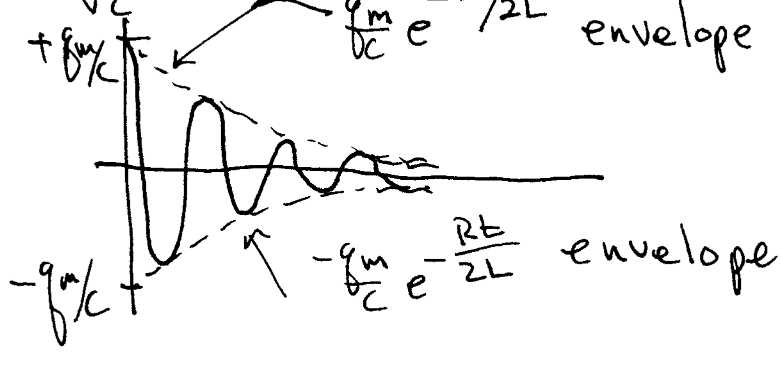
i.c. $\Rightarrow$ $\tan\phi=\frac{-1}{\sqrt{\frac{4L}{RC^2}-1}}$; $q_0=q_m\frac{1}{\sqrt{LC}\,\omega'}$

1) Energy Loss:

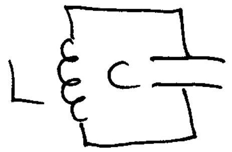
$$U=U_E+U_B=\frac{1}{2}\frac{q^2}{C}+\frac{1}{2}Li^2$$

$$\frac{dU}{dt}=-i^2R \quad \text{Joule heating}$$

2)



VII. E. 3) $R \to 0$

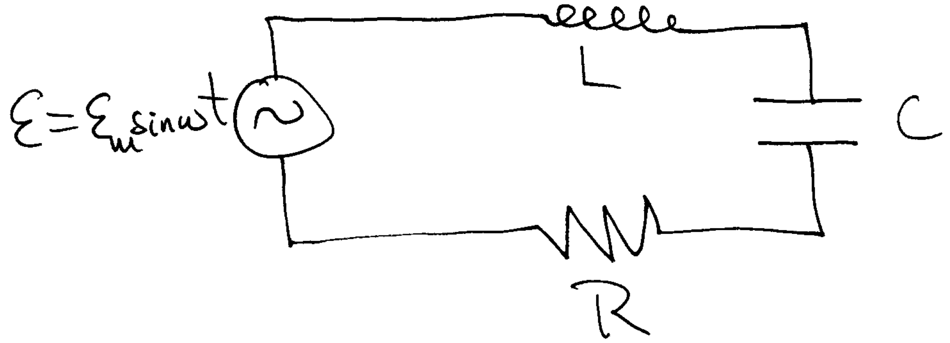


$$\frac{dU}{dt} = 0$$

$$U = \text{constant} = U_E + U_B = \frac{1}{2}\,q_m^2/C$$

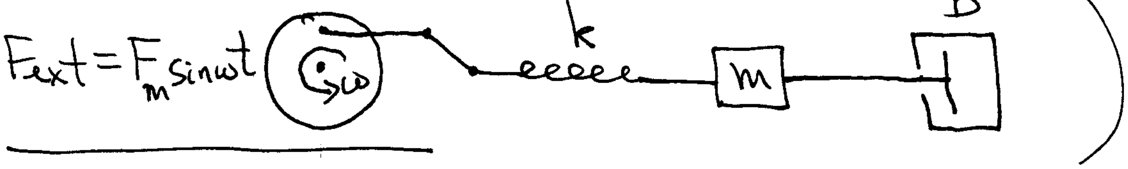
energy oscillates between being stored in electric field and the magnetic fields the current oscillates sinusoidally CW then CCW in circuit.

F.) Driven LCR-Circuit – Resonance



Loop Equation ⇒ $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = \mathcal{E}_m \sin\omega t$$

(Mechanical Analog



)

$$q(t) = q_c(t) + q_p(t)$$

$q_c(t)$: complimentary solution = transient solution above

$q_p(t)$: particular solution = steady state solution

VII. F) Ansatz: $q_p(t) = \frac{i_m}{\omega}\sin(\omega t - \delta)$

Conventional $\delta \equiv \phi + \frac{\pi}{2}$ so

$$i = \dot{q}_p(t) = i_m \sin(\omega t - \phi)$$

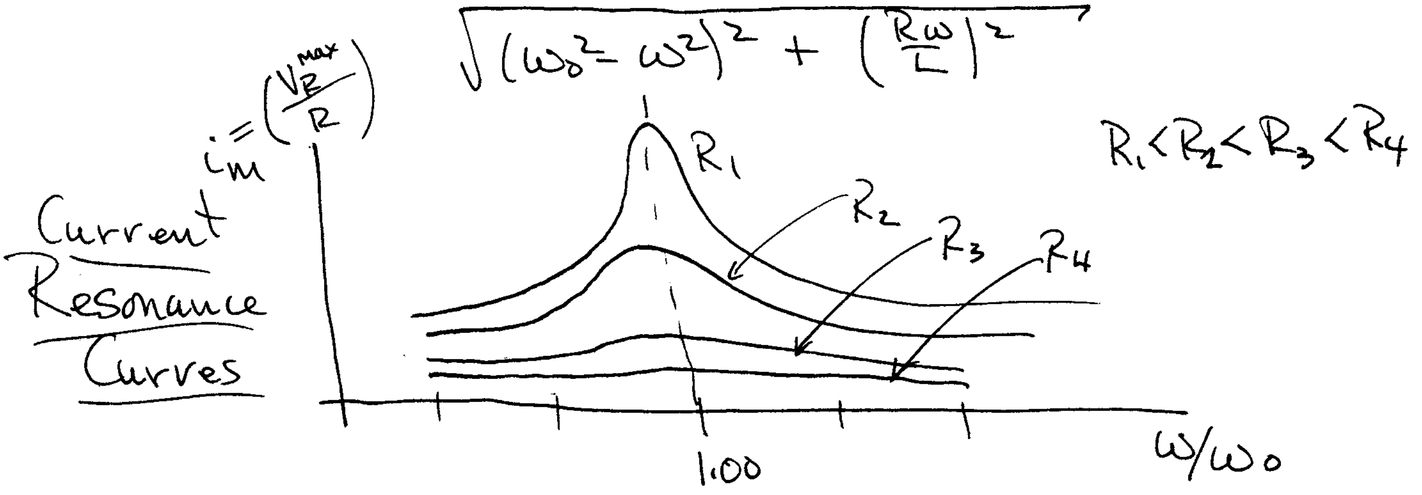
Loop equation $\Rightarrow$ $\tan\delta = \frac{\frac{R}{L}\omega}{\omega_0^2 - \omega^2}$

or 1) $\boxed{\tan\phi = \frac{\omega^2 - \omega_0^2}{\frac{R}{L}\omega}}$

where

$\omega_0^2 \equiv \frac{1}{LC}$ (natural frequency of LC circuit)

2) $$i_m = \frac{\mathcal{E}_m}{R}\sec\phi = \frac{\omega \cdot \mathcal{E}_m/L}{\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\frac{R\omega}{L}\right)^2}}$$



Current Resonance Curves

$\omega_R = \omega_0 = \frac{1}{\sqrt{LC}}$ (Current (V_R) Resonance)

VII. F. 3) Voltage drop across capacitor $= V_c = \frac{q_p}{C}$

$V_{C\max} = \frac{i_m}{\omega C}$ has resonance at charge q_p resonance at

frequency: $\omega_{q_R} = \sqrt{\omega_0^2 - \frac{1}{2}\left(\frac{R}{L}\right)^2} < \omega_0$

4) Voltage drop across inductor $= V_L = L\frac{di}{dt}$

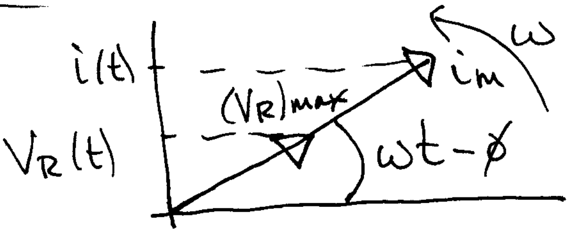
$V_{L\max} = L\omega i_m$ has resonance at

frequency: $\omega_{LR} = \frac{\omega_0}{\sqrt{1-\frac{1}{2}\left(\frac{R^2C}{L}\right)}} > \omega_0$

G) Phasor Diagrams: $i(t) = i_m\sin(\omega t - \phi)$ through circuit element

1) Resistive Load: $V_R = i_m R\sin(\omega t - \phi)$

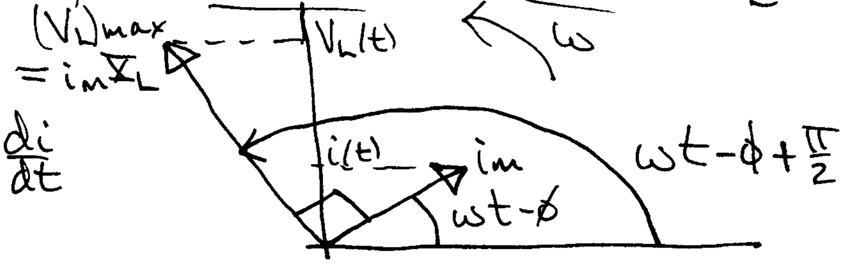
$V_R = iR$



V_R & i are in phase

2) Inductive Load: $V_L = i_m X_L \sin(\omega t - \phi + \frac{\pi}{2})$

$V_L = L\frac{di}{dt}$

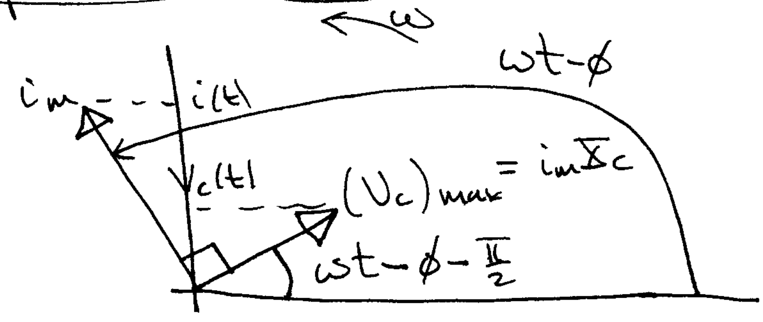


a) ELI

b) $X_L \equiv \omega L$ = inductive reactance

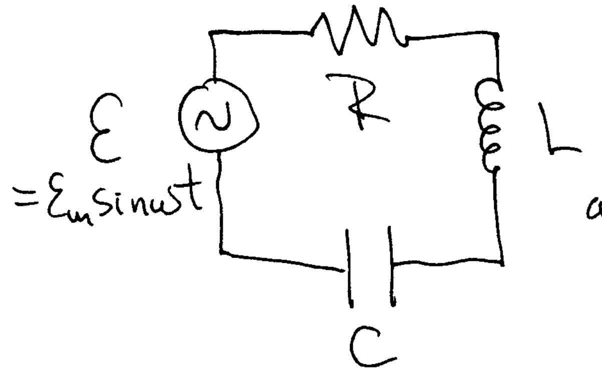
- VIII .G.) 3) Capacitive Load : $V_C = i_m X_C \sin(\omega t - \phi - \frac{\pi}{2})$

$V_C = \frac{q}{C}$
$= \frac{\int i\,dt}{C}$



a) ICE
b) $X_C \equiv \frac{1}{\omega C}$
= capacitive reactance

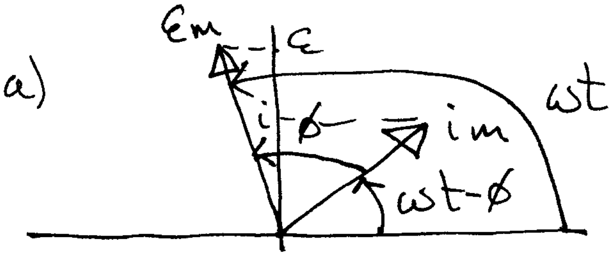
4) LCR-Circuit:



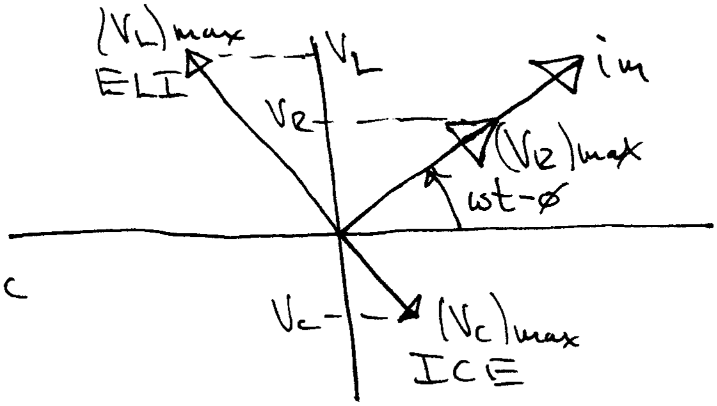
$\mathcal{E} = \mathcal{E}_m \sin\omega t$

Ansatz: $i = i_m \sin(\omega t - \phi)$

a)



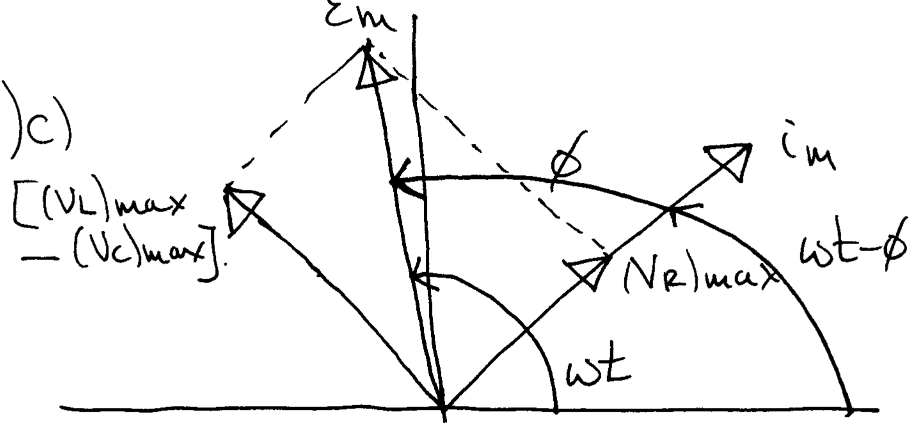
b) RLC Components:



loop Rule:
$\mathcal{E} = V_R + V_L + V_C$
(scalar sum)

c) Vector sum of $(V_R)_{max}$, $(V_L)_{max}$, $(V_C)_{max}$ phasors equals $\mathcal{E}_m$ phasor

VII. G4)c)



Since $V_{R,C,L}$ phasors are at right angles ⇒

$$\mathcal{E}_m = \sqrt{[(V_R)_{max}]^2 + [(V_L)_{max} - (V_C)_{max}]^2}$$

$$= i_m\sqrt{R^2 + (X_L - X_C)^2}$$

$$\boxed{\begin{array}{l} \Rightarrow \\ \quad i_m = \mathcal{E}_m / Z \\ Z \equiv \sqrt{R^2 + (X_L - X_C)^2} = \text{impedance} \end{array}}$$

$$\tan\phi = \frac{(V_L)_{max} - (V_C)_{max}}{(V_R)_{max}} = \boxed{\frac{X_L - X_C}{R} = \tan\phi}$$

1) Resonance for the current occurs at $\phi = 0$

$\Rightarrow X_L = X_C \Rightarrow \omega = \omega_0 = \frac{1}{\sqrt{LC}}$,

So $(V_L)_{max}$ and $(V_C)_{max}$ are equal and opposite to give i_m in phase with $\mathcal{E}_m$.

VII. H) Power in AC circuits:

1) Power dissipation in Resistor:

$$P = i^2 R = i_m^2 R \sin^2(\omega t - \phi)$$

Average power dissipation

$$\overline{P} \equiv \frac{1}{N\tau}\int_0^{N\tau} P(t)\,dt$$

$$= \frac{1}{2} i_m^2 R = i_{rms}^2 R$$

with root mean square current

$$i_{rms} = \frac{i_m}{\sqrt{2}} \equiv \sqrt{\overline{(i)^2}} \quad .$$

2) Power supplied to circuit by source of emf

$$P = \frac{dW}{dt} = \mathcal{E} i = \mathcal{E}_m i_m \sin\omega t \sin(\omega t - \phi)$$

Average power supplied

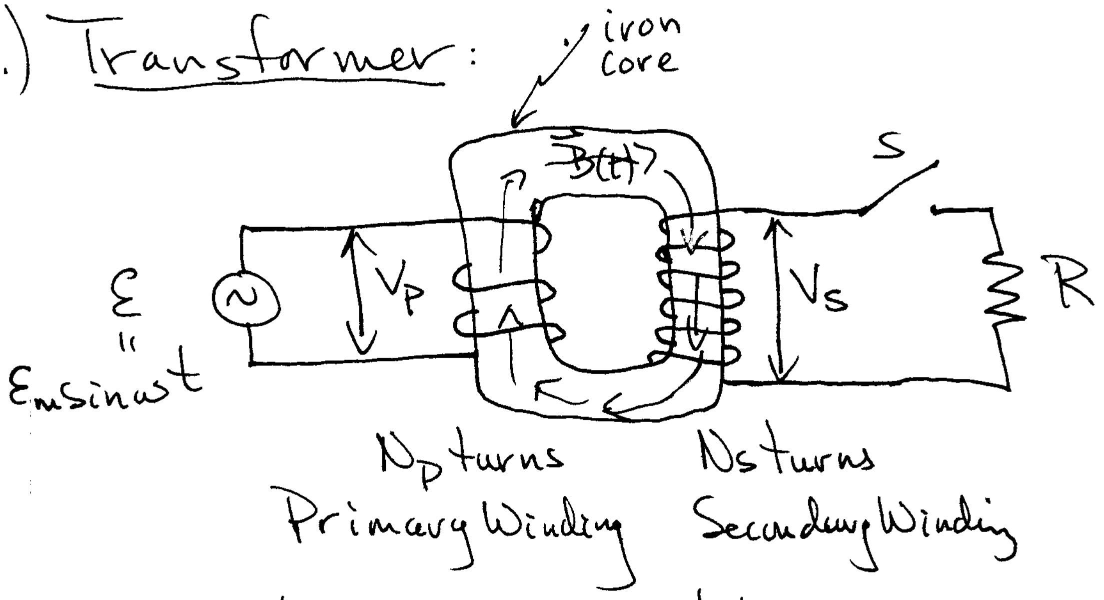
$$\boxed{\begin{aligned}\overline{P} &= \frac{1}{2}\mathcal{E}_m i_m \cos\phi \\ &= \mathcal{E}_{rms} i_{rms} \cos\phi\end{aligned}}$$

VII.H.2) $\cos\phi \equiv$ power factor

$$= \frac{R}{Z} = \frac{R}{\sqrt{R^2+(X_L-X_C)^2}}$$

If resistive load $X_L = X_C \Rightarrow \cos\phi = 1$
power delivered to circuit is maximum
$\bar{P} = \mathcal{E}_{rms} i_{rms}$ ($Z=R$ resistive load)

I.) <u>Transformer</u>:



1) $\left(\frac{d\phi_B}{dt}\right)_{\substack{\text{primary}\\ \text{per turn}}} = \left(\frac{d\phi_B}{dt}\right)_{\substack{\text{secondary}\\ \text{per turn}}}$

$\Rightarrow \frac{V_p}{N_p} = \frac{V_s}{N_s} \Rightarrow \boxed{V_s = V_p\left(\frac{N_s}{N_p}\right)}$

2) Energy Conservation: $i_p V_p = i_s V_s$ (Resistive load)

$\Rightarrow \boxed{i_s = i_p\left(\frac{N_p}{N_s}\right)}$

VIII.) Maxwell's Equations

A) Ampere's Law as modified by Maxwell to be consistent with charge conservation

$$\boxed{\frac{1}{\mu_0}\oint_C \vec{B}\cdot d\vec{s} = i + i_D = i + \epsilon_0 \frac{d\Phi_E}{dt}}$$

i_D = displacement current

$$\equiv \epsilon_0 \frac{d}{dt}\int_S \vec{E}\cdot d\vec{A} = \epsilon_0 \frac{d\Phi_E}{dt}$$

$\Phi_E = \int_S \vec{E}\cdot d\vec{A}$ = electric flux through open surface S bounded by C.

1) Time varying $\vec{E}$-field produces a $\vec{B}$-field

2) Charging capacitor: Gauss' law $q = \epsilon_0 E A$

$\Rightarrow$ Conduction current $= i = \frac{dq}{dt} = \epsilon_0 \frac{d(EA)}{dt}$

$= \epsilon_0 \frac{d\Phi_E}{dt} = i_D =$ displacement current

So appears as current is continuous

$\Rightarrow$

displacement current density

$= \vec{j}_D = \epsilon_0 \frac{d\vec{E}}{dt}$.

VIII) A3) Magnetic and Dielectric material present

Ampere's law

$$\frac{1}{\mu_0}\oint_C \vec{B}\cdot d\vec{s} = i + i_D + i_M$$

with

$$i_M = \oint_C \vec{M}\cdot d\vec{s}$$

and Gauss' law $\underbrace{\oint_S \underbrace{(\epsilon_0\vec{E} + \vec{P})}_{=\vec{D}}\cdot d\vec{A}}_{\equiv \phi_D = \oint_S \vec{D}\cdot d\vec{A}} = q_{free}$

Hence Maxwell's modification becomes

$$i_D = \frac{d}{dt}\phi_D = \frac{d}{dt}\int_S \vec{D}\cdot d\vec{A}$$

Recalling $\vec{H} = \frac{1}{\mu_0}(\vec{B} - \mu_0\vec{M})$

yields Ampere's law

$$\boxed{\oint_C \vec{H}\cdot d\vec{s} = i + \frac{d}{dt}\int_S \vec{D}\cdot d\vec{A}}$$

i = conduction current or free current
or transport current

IIII.) A4) When using Faraday's law in the past we ignored the displacement current in Ampere's law — estimate error — Apply an alternating $\vec{E}$-field to a conductor

$$\vec{E} = \vec{E}_0 \cos\omega t$$

Ohm's law $\Rightarrow \vec{j} = \sigma \vec{E} = \sigma \vec{E}_0 \cos\omega t$, with $\epsilon = 1$
$\vec{D} = \epsilon_0 \vec{E}$

$$\Rightarrow \frac{\partial \vec{D}}{\partial t} = -\omega \epsilon_0 \vec{E} \sin\omega t$$

Thus
$$\frac{|\vec{j}|}{|\vec{j}_D|} = \frac{|\sigma \vec{E}|}{\left|\frac{\partial \vec{D}}{\partial t}\right|} = \frac{\sigma}{\omega \epsilon_0}$$

For Cu $\frac{\sigma}{\epsilon_0} \approx 10^{19}$ and $\frac{|\vec{j}|}{|\vec{j}_D|} \approx \frac{10^{19}}{\omega}$; $\vec{j}_D$ is negligible compared to $\vec{j}$.

This justifies <u>quasi-static</u> regieme

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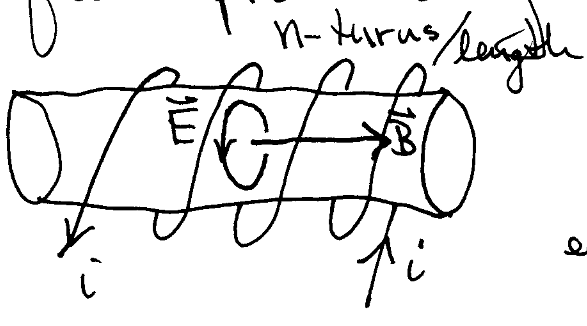
5) Displacement current produces a magnetic field in inductor — what is estimate of error in neglecting this relative to the magnetic field produced by conduction current?



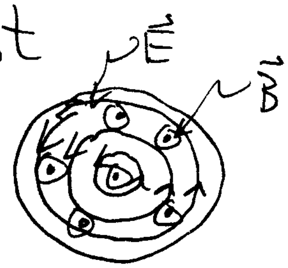
$\vec{E}$ has direction given by Lenz' law
ex. $\vec{B}\downarrow$, $\vec{E}$ causes $i \nearrow$

let $i = i_m \sin\omega t$

VIII. A)4) Ampere's law without displacement current

$$B \approx \mu_0 n i = \mu_0 n i_m \sin\omega t$$

induces $\vec{E}$-field by Faraday's law

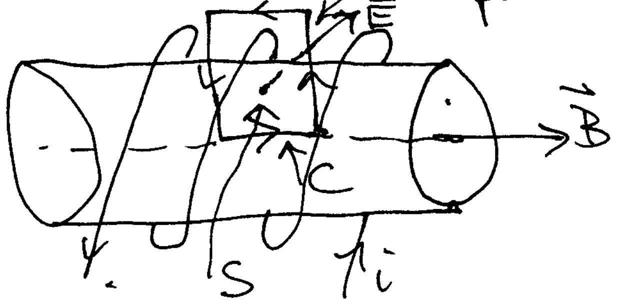


$$\oint_C \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\Phi_B$$

$$\Rightarrow \quad E = -\left(\tfrac{1}{2}\omega\mu_0 n i_m \cos\omega t\right) r$$

So with $D = \epsilon_0 E$ (vacuum) $\Rightarrow \boxed{D = -\tfrac{1}{2}\omega\epsilon_0\mu_0 n i_m r \cos\omega t}$

Ampere's law re-visited



$$\oint_C \vec{B}\cdot d\vec{s} = n l i + \int_S \frac{\partial\vec{D}}{\partial t}\cdot d\vec{A}$$

$$\text{Error} = \frac{\left|\int_S \frac{\partial\vec{D}}{\partial t}\cdot d\vec{A}\right|}{n i l} = \frac{1}{4}\omega^2\epsilon_0\mu_0 R^2$$

For $\frac{\omega}{2\pi} = 1\,\text{MHz}$, $R = 10\,\text{cm}$ $\Rightarrow$ Error $\approx 10^{-6}$.

VIII.) B) Maxwell's Equations (integral form in vacuum)

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0} q$$: Gauss' Law

$$\oint_S \vec{B}\cdot d\vec{A} = 0$$: No Magnetic Charge

$$\oint_C \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{A}$$: Faraday's Law

$$\oint_C \vec{B}\cdot d\vec{s} = \mu_0 i + \epsilon_0\mu_0 \frac{d}{dt}\int_S \vec{E}\cdot d\vec{A}$$: Ampere's Law as modified by Maxwell for charge conservation

1) Charge particles interact with the electromagnetic field according to the Lorentz force law

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

and their trajectories are determined by Newton's 2nd Law $\vec{F} = \frac{d\vec{p}}{dt}$.

2) All classical, macroscopic electromagnetic phenomena described by these laws.

VIII. c) Maxwell's Equations (integral form when dielectric and magnetic material is present)

$$\oint_S \vec{D}\cdot d\vec{A} = q_{free} \quad : \text{ Gauss' Law}$$

$$\oint_S \vec{B}\cdot d\vec{A} = 0 \quad : \text{ No Magnetic Charges}$$

$$\oint_C \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{A} \quad : \text{ Faraday's Law}$$

$$\oint_C \vec{H}\cdot d\vec{s} = i_{\text{conduction}} + \frac{d}{dt}\int_S \vec{D}\cdot d\vec{A} \quad : \text{ Ampere's Law as modified by Maxwell}$$

In addition we require the <u>constitutive equations</u>

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \qquad \text{with } \vec{P} = \vec{P}(\vec{E})$$

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \qquad \text{with } \vec{M} = \vec{M}(\vec{H}),$$

where for linear, isotropic materials

$$\vec{D} = \chi_e \epsilon_0 \vec{E} = \epsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

That is

$$\vec{P} = \chi_E \epsilon_0 \vec{E}$$

$$\vec{M} = \chi_M \vec{H}$$

VIII. C) In general we also consider Ohmic materials

$$\vec{j} = \sigma \vec{E} \quad : \text{Ohm's Law}$$

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1) For vacuum $\vec{D} = \epsilon_0 \vec{E}$; $\vec{B} = \mu_0 \vec{H}$.

D) Maxwell's Equation's (Differential form)

with $q_{free} = \int_V \rho dV$; $i_{conduction} = \int_S \vec{j} \cdot d\vec{A}$

1) $\vec{\nabla} \cdot \vec{D} = \rho$: Gauss' Law

$\vec{\nabla} \cdot \vec{B} = 0$: No Magnetic Charges

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$: Faraday's Law

$\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$: Ampere's Law

2) In vacuum $\vec{D} = \epsilon_0 \vec{E}$, $\vec{B} = \mu_0 \vec{H}$ =>

$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$: Gauss' Law

$\vec{\nabla} \cdot \vec{B} = 0$: No Magnetic Charges

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$: Faraday's Law

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$: Ampere's Law

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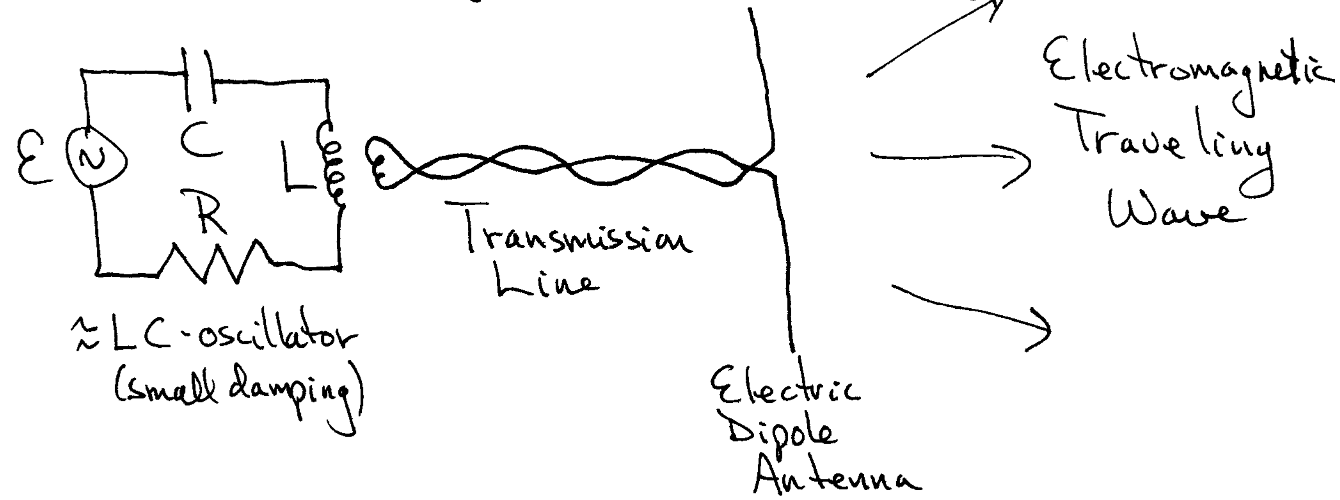
IX) Electromagnetic Radiation: Charges at rest or moving at constant velocity do not radiate.

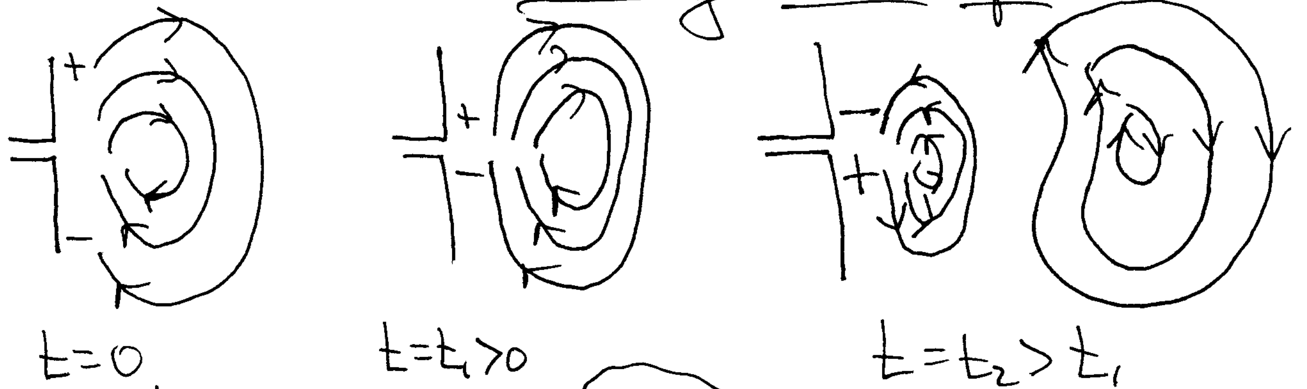
Accelerated charges radiate

A)

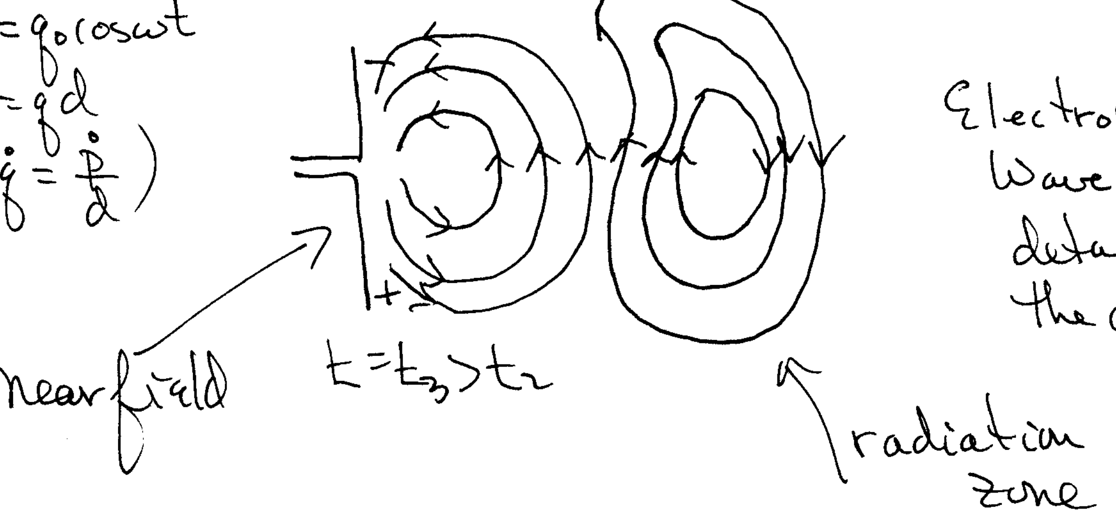


B) Antenna acts like oscillating electric dipole moment

($\vec{E}$-field shown only)

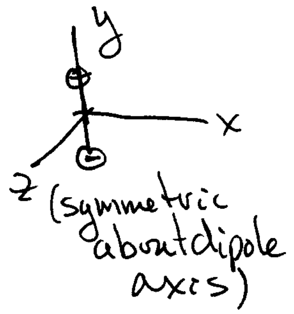


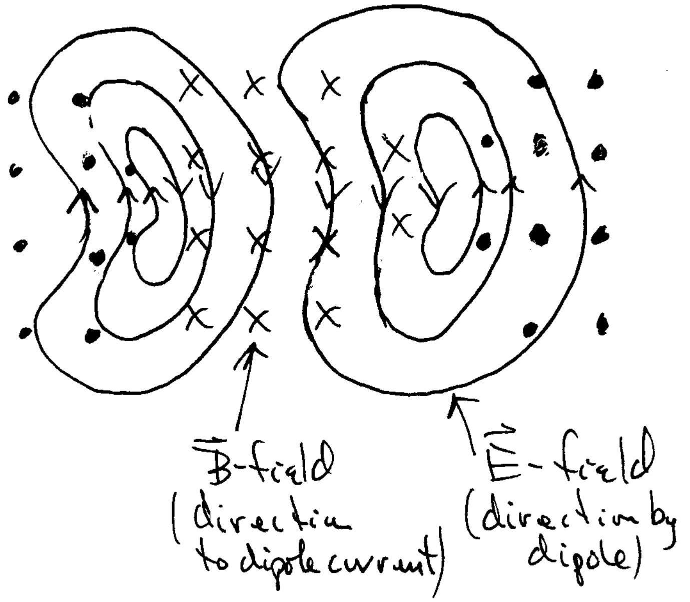
($q = q_0 \cos\omega t$
$p = qd$
$i = \dot{q} = \frac{\dot{p}}{d}$)



Electromagnetic Wave becomes detached from the charges

IX) C) Closer look at wave:



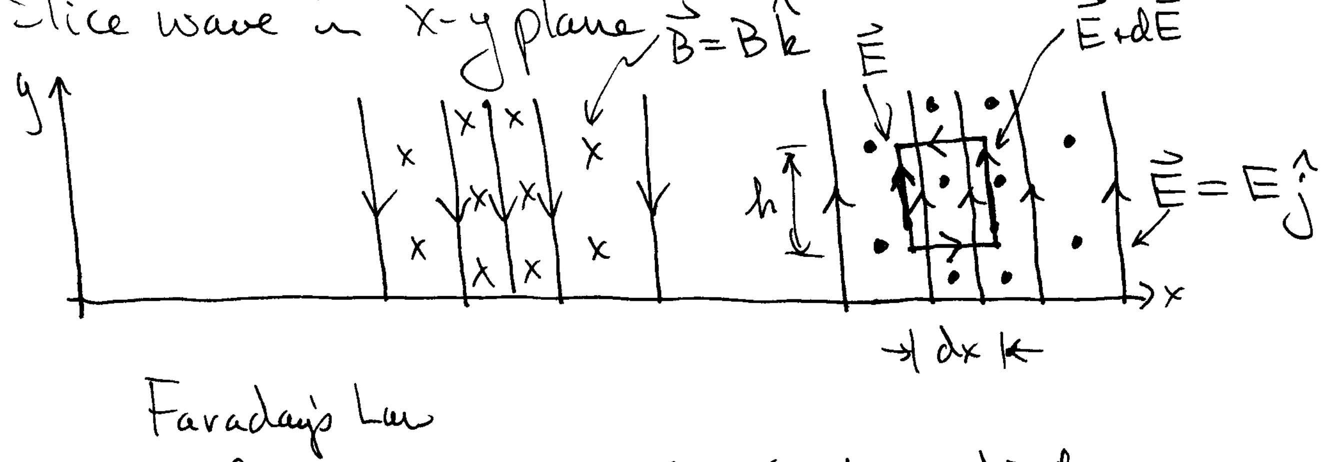


D) Maxwell's Equations applied to wave in Rad. zone

$$\oint_C \vec{E}\cdot d\vec{s} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$: Faraday's law

$$\oint_C \vec{B}\cdot d\vec{s} = \epsilon_0 \mu_0 \int_S \frac{\partial \vec{E}}{\partial t}\cdot d\vec{A}$$: Ampere's law (far from dipole in Radiation Zone)

1) Slice wave in x-y plane



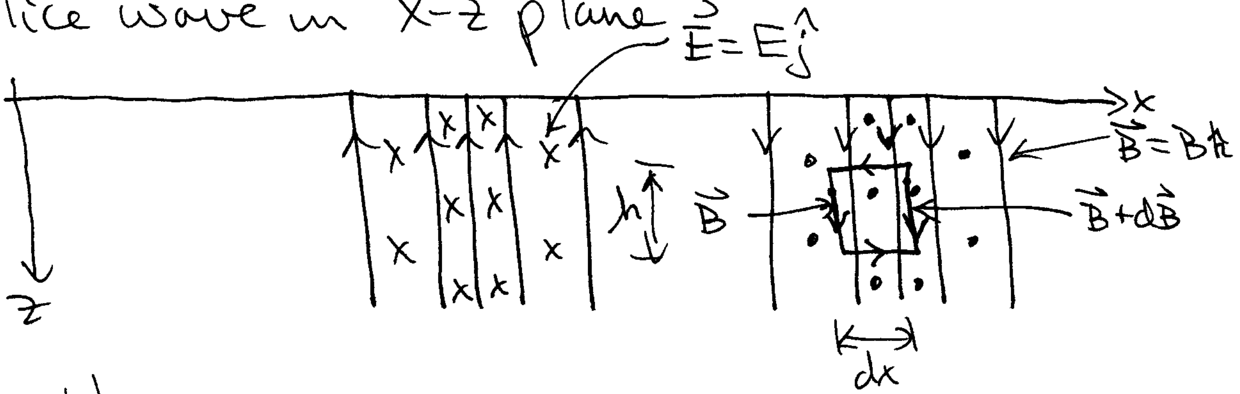
Faraday's Law

$$\oint_C \vec{E}\cdot d\vec{s} = (E+dE)h - (E)h = dEh$$

$$= -\frac{\partial B}{\partial t}(h\,dx)$$

IX. D)1) $\Rightarrow$ $$\boxed{\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}}$$

2) Slice wave in x-z plane



Ampere's Law:

$$\oint_C \vec{B}\cdot d\vec{s} = -(B+dB)h + Bh = -h\,dB$$
$$= \epsilon_0\mu_0 \frac{\partial E}{\partial t}(h\,dx)$$

$$\Rightarrow \quad \boxed{\frac{\partial B}{\partial x} = -\epsilon_0\mu_0\frac{\partial E}{\partial t}}$$

E) Wave Equation: uncouple the above by differentiating again $\Rightarrow$

$$\boxed{\begin{aligned} \left[\epsilon_0\mu_0 \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right] E(x,t) = 0 \\ \left[\epsilon_0\mu_0 \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right] B(x,t) = 0 \end{aligned}}$$

IX. E) General Solution to wave equation

$$\boxed{\begin{aligned} E(x,t) &= E(x-v t) \\ B(x,t) &= B(x-v t) \\ \Rightarrow \quad v &= \frac{1}{\sqrt{\epsilon_0\mu_0}} \equiv c \end{aligned}}$$

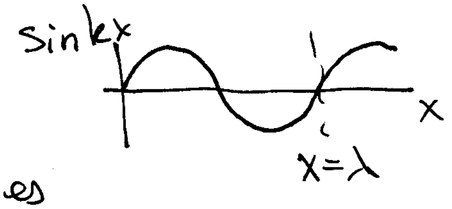
All electromagnetic waves travel at speed c!

and $$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t} - \frac{\partial^2}{\partial x^2}\right] \begin{matrix} E \\ \text{or} \\ B \end{matrix} = 0$$

F) Plane Waves: Simplest Solution

$$E(x,t) = E_m \sin(kx - \omega t)$$
$$B(x,t) = B_m \sin(kx - \omega t)$$

Wave Equation $\Rightarrow \frac{\omega}{k} = c$



k = wave number

$k\underset{\overset{\shortparallel}{\lambda}}{x} = 2\pi$ gives zeroes of sin kx

λ is called wave length $k\lambda = 2\pi$

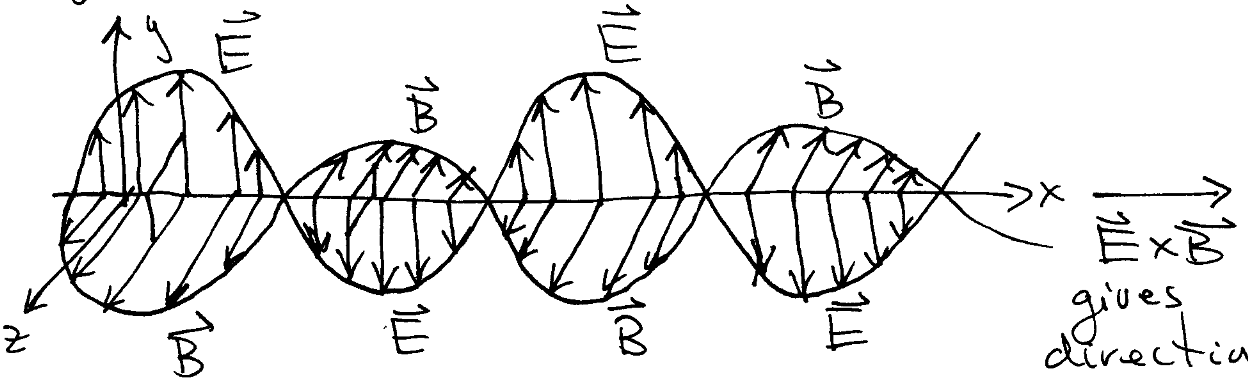
$$\Rightarrow \boxed{k = \frac{2\pi}{\lambda}}$$

So $$\boxed{\frac{\omega}{k} = c = \frac{\omega}{2\pi}\,\frac{2\pi}{k} = \nu\lambda}$$

IX. G) Linearly Polarized : $\vec{E} = E\hat{j}$
$\vec{B} = B\hat{k}$

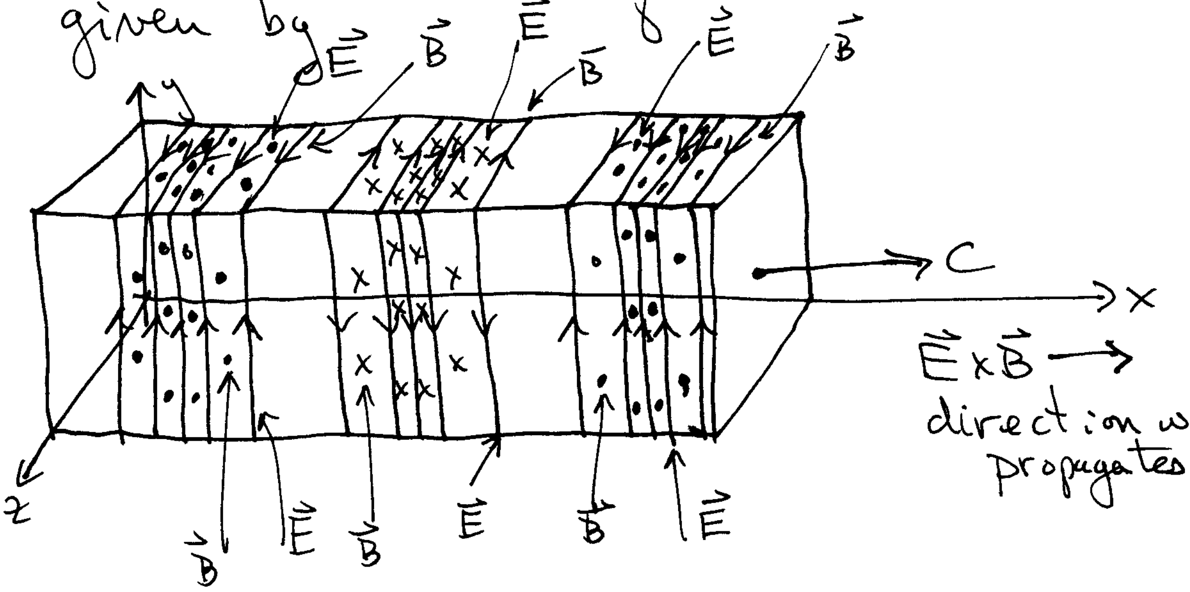
$\vec{E}$ oscillates along one direction only, as does $\vec{B}$.

For fixed t this looks like



$\vec{E}\times\vec{B}$ gives direction wave propagates.

Another representation of the wave is given by



$\vec{E}\times\vec{B}$ → direction wave propagates

IX. H) Maxwell's Equations: $\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$: Faraday's Law

$\frac{\partial B}{\partial x} = -\frac{1}{c^2}\frac{\partial E}{\partial t}$: Ampere's Law

⇒ 1) E & B must be in phase

2) $$\boxed{E = cB}$$

I) Energy Transport: The Poynting Vector

$$\vec{S} = \vec{E} \times \vec{H}$$

$$= \frac{1}{\mu_0}\vec{E} \times \vec{B} \quad \text{in vacuum}$$

$\vec{S}$ = rate of energy flow per unit area in the direction of $\vec{S}$

= electromagnetic power per unit area.

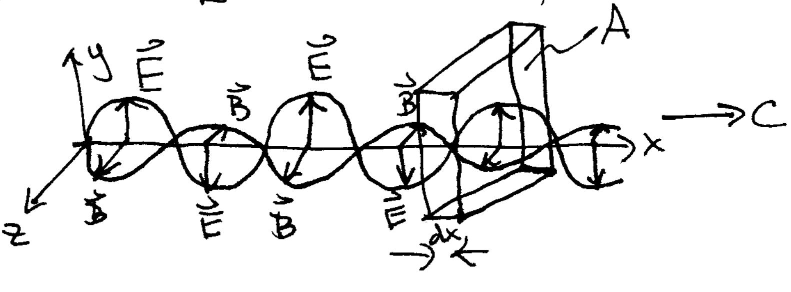
1) For above plane wave $\vec{S} = \frac{1}{\mu_0} EB\,\hat{i} \equiv S\,\hat{i}$

↑ gives direction of propagation

2) $dU = dU_E + dU_B = (u_E + u_B)(A\,dx)$

$u_E = \frac{1}{2}\epsilon_0 E^2$

$u_B = \frac{1}{2\mu_0} B^2$



IX.I.)2) $dU = \frac{EBA}{\mu_0}\frac{dx}{c}$ but $dt = \frac{dx}{c}$

This is amount of energy that passes through the box in time dt, the energy flow per unit time thru area A : $\frac{dU}{dt} = \frac{1}{\mu_0}EBA = S \cdot A$

So $S = \frac{dU}{dt\,A} = \frac{1}{\mu_0}EB$ is the energy flow per unit time per unit area.

3) Intensity $I = \overline{S}$ = time average of S

$$I = \overline{S} = \frac{1}{2\mu_0}E_m B_m$$

$$= \frac{1}{\mu_0}E_{rms} B_{rms}$$