VIII.) Maxwell's Equations

A) Ampere's Law as modified by Maxwell to be consistent with charge conservation

$$\boxed{\frac{1}{\mu_0}\oint_C \vec{B}\cdot d\vec{s} = i + i_D = i + \epsilon_0 \frac{d\Phi_E}{dt}}$$

i_D = displacement current

$$\equiv \epsilon_0 \frac{d}{dt}\int_S \vec{E}\cdot d\vec{A} = \epsilon_0 \frac{d\Phi_E}{dt}$$

$\Phi_E = \int_S \vec{E}\cdot d\vec{A}$ = electric flux through open surface S bounded by C.

1) Time varying $\vec{E}$-field produces a $\vec{B}$-field

2) Charging capacitor: Gauss' law $q = \epsilon_0 E A$

$\Rightarrow$ Conduction current $= i = \frac{dq}{dt} = \epsilon_0 \frac{d(EA)}{dt}$

$= \epsilon_0 \frac{d\Phi_E}{dt} = i_D$ = displacement current

So appears as current is continuous

$\Rightarrow$

displacement current density

$= \vec{j}_D = \epsilon_0 \frac{d\vec{E}}{dt}$.

VIII) A3) Magnetic and Dielectric material present

Ampere's law

$$\frac{1}{\mu_0}\oint_C \vec{B}\cdot d\vec{s} = i + i_D + i_M$$

with

$$i_M = \oint_C \vec{M}\cdot d\vec{s}$$

and Gauss' law $\underbrace{\oint_S (\underbrace{\epsilon_0\vec{E}+\vec{P}}_{=\vec{D}})\cdot d\vec{A}}_{\equiv \phi_D = \oint_S \vec{D}\cdot d\vec{A}} = q_{free}$

Hence Maxwell's modification becomes

$$i_D = \frac{d}{dt}\phi_D = \frac{d}{dt}\int_S \vec{D}\cdot d\vec{A}$$

Recalling $\vec{H} = \frac{1}{\mu_0}(\vec{B} - \mu_0\vec{M})$

yields Ampere's law

$$\boxed{\oint_C \vec{H}\cdot d\vec{s} = i + \frac{d}{dt}\int_S \vec{D}\cdot d\vec{A}}$$

i = conduction current or free current
or transport current

III.) A4) When using Faraday's law in the past we ignored the displacement current in Ampere's law — estimate error — Apply an alternating $\vec{E}$-field to a conductor

$$\vec{E} = \vec{E}_0 \cos\omega t$$

Ohm's law $\Rightarrow \vec{j} = \sigma \vec{E} = \sigma \vec{E}_0 \cos\omega t$, with $\epsilon = 1$, $\vec{D} = \epsilon_0 \vec{E}$

$$\Rightarrow \frac{\partial \vec{D}}{\partial t} = -\omega \epsilon_0 \vec{E} \sin\omega t$$

Thus
$$\frac{|\vec{j}|}{|\vec{j}_D|} = \frac{|\sigma \vec{E}|}{\left|\frac{\partial \vec{D}}{\partial t}\right|} = \frac{\sigma}{\omega \epsilon_0}$$

For Cu $\frac{\sigma}{\epsilon_0} \approx 10^{19}$ and $\frac{|\vec{j}|}{|\vec{j}_D|} \approx \frac{10^{19}}{\omega}$; $\vec{j}_D$ is negligible compared to $\vec{j}$.

This justifies quasi-static regieme

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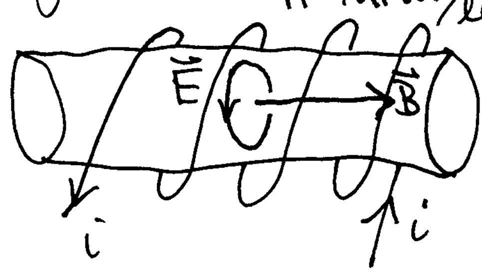
5) Displacement current produces a magnetic field in inductor — what is estimate of error in neglecting this relative to the magnetic field produced by conduction current?



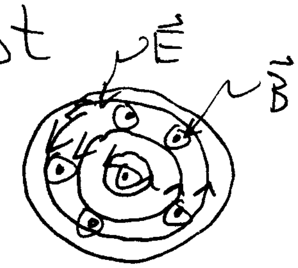
$\vec{E}$ has direction given by Lenz' law ex. $\vec{B}\downarrow$, $\vec{E}$ causes $i \nearrow$

let $i = i_m \sin\omega t$

VIII. A) 4) Ampère's law without displacement current

$$B \approx \mu_0 n i = \mu_0 n i_m \sin \omega t$$

induces $\vec{E}$-field by Faraday's law

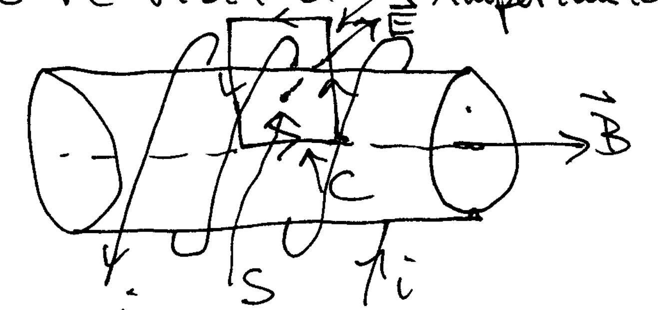


$$\oint_C \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \phi_B$$

$$\Rightarrow \quad E = -\left(\tfrac{1}{2} \omega \mu_0 n i_m \cos \omega t\right) r$$

So with $D = \epsilon_0 E$ (vacuum) $\Rightarrow$ $\boxed{D = -\tfrac{1}{2} \omega \epsilon_0 \mu_0 n i_m r \cos \omega t}$

Ampère's law re-visited



$$\oint_C \vec{B} \cdot d\vec{s} = n l i + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{A}$$

$$\text{Error} = \frac{\left| \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{A} \right|}{n i l} = \frac{1}{4} \omega^2 \epsilon_0 \mu_0 R^2$$

For $\frac{\omega}{2\pi} = 1\,\text{MHz}$, $R = 10\,\text{cm}$ $\Rightarrow$ Error $\simeq 10^{-6}$.

VIII.) B) Maxwell's Equations (integral form in vacuum)

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0} q$$: Gauss' Law

$$\oint_S \vec{B}\cdot d\vec{A} = 0$$: No Magnetic Charge

$$\oint_C \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{A}$$: Faraday's Law

$$\oint_C \vec{B}\cdot d\vec{s} = \mu_0 i + \epsilon_0 \mu_0 \frac{d}{dt}\int_S \vec{E}\cdot d\vec{A}$$: Ampere's Law as modified by Maxwell for charge conservation

1) Charge particles interact with the electromagnetic field according to the Lorentz force law

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

and their trajectories are determined by Newton's 2nd Law $\vec{F} = \frac{d\vec{p}}{dt}$.

2) All classical, macroscopic electromagnetic phenomena described by these laws.

VIII. c) Maxwell's Equations (integral form when dielectric and magnetic material is present)

$$\oint_S \vec{D}\cdot d\vec{A} = q_{free} \quad : \text{ Gauss' Law}$$

$$\oint_S \vec{B}\cdot d\vec{A} = 0 \quad : \text{ No Magnetic Charges}$$

$$\oint_C \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{A} \quad : \text{ Faraday's Law}$$

$$\oint_C \vec{H}\cdot d\vec{s} = i_{\text{conduction}} + \frac{d}{dt}\int_S \vec{D}\cdot d\vec{A} \quad : \text{ Ampere's Law as modified by Maxwell}$$

In addition we require the <u>constitutive equations</u>

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \qquad \text{with } \vec{P} = \vec{P}(\vec{E})$$

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \qquad \text{with } \vec{M} = \vec{M}(\vec{H}),$$

where for linear, isotropic materials

$$\vec{D} = \chi_e \epsilon_0 \vec{E} = \epsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

That is
$$\vec{P} = \chi_E \epsilon_0 \vec{E}$$

$$\vec{M} = \chi_M \vec{H}$$

VIII. C) In general we also consider Ohmic materials

$$\vec{j} = \sigma \vec{E} \quad : \text{Ohm's Law}$$

1) For vacuum $\vec{D} = \epsilon_0 \vec{E}$; $\vec{B} = \mu_0 \vec{H}$.

D) Maxwell's Equation's (Differential form)

with $q_{free} = \int_V \rho \, dV$; $i_{conduction} = \int_S \vec{j} \cdot d\vec{A}$

1) $\vec{\nabla} \cdot \vec{D} = \rho$: Gauss' Law

$\vec{\nabla} \cdot \vec{B} = 0$: No Magnetic Charges

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$: Faraday's Law

$\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$: Ampere's Law

2) In vacuum $\vec{D} = \epsilon_0 \vec{E}$, $\vec{B} = \mu_0 \vec{H}$ $\Rightarrow$

$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$: Gauss' Law

$\vec{\nabla} \cdot \vec{B} = 0$: No Magnetic Charges

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$: Faraday's Law

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$: Ampere's Law