IV C3E) Toroid: $\boxed{B(r) = \frac{\mu_0 i N}{2\pi r}}$

V.) Electromagnetic Induction

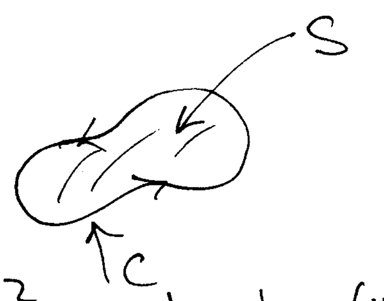
A) Faraday's Law: (emf form)

The induced emf in a circuit is equal to the negative of the rate at which the magnetic flux through the circuit is changing with time!

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

1) Magnetic flux through open surface S bounded by

$$\Phi_B = \int_S \vec{B} \cdot d\vec{A}$$



a) SI unit: $[\Phi_B] = 1\,T\cdot m^2 \equiv 1\,\text{weber (Wb)}$

V. A) 2) Lenz' Law: The induced current in a closed conducting loop appears in such a direction that it opposes the change that produced it. (this is the minus sign in Faraday's Law)

3) emf $\mathcal{E} = \oint_C \vec{f} \cdot d\vec{s}$ where $\vec{f}$ is the total force per unit charge acting on the charges in a circuit. In general,

$\vec{f} = \vec{f}_s + \vec{E}$ where f_s is the force per unit charge the localized sources of emf (i.e. battery, solar cell, generator etc.) exert on charges in the circuit.

In electrostatics the $\vec{E}$-field is conservative, hence $\oint_C \vec{E} \cdot d\vec{s} = 0$, so for DC circuits:

$$\mathcal{E} = \oint_C \vec{f}_s \cdot d\vec{s} = \int_{\text{source}} \vec{f}_s \cdot d\vec{s} = \frac{dW}{dq}$$

Faraday discovered that time varying magnetic flux produced a non-conservative electric field in a closed circuit

$$\mathcal{E} = \oint_C \vec{E} \cdot d\vec{s} = -\frac{d}{dt}\Phi_B$$

V. B) Faraday's Law: (induced electric field form)

$$\boxed{\oint_C \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{A}}$$

1) Lenz' Law (Re-visited): The induced electric field has direction so as to oppose the change that produces it.

2) The closed curve C is any arbitrary closed curve, not just physical circuits, S is any open surface with C as it's boundary.

3) Time varying magnetic fields produce electric fields!

4) The induced electric field is <u>not</u> conservative $\oint_C \vec{E}\cdot d\vec{s} \neq 0$

5) Induced electric field lines form closed loops

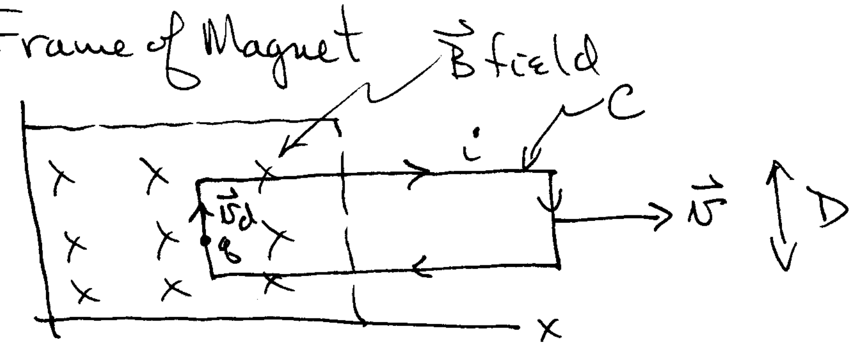
6) Faraday's law of electromagnetic induction, $\oint_C \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{A}$, is one of the fundamental laws of electromagnetism (it is one of the 4 Maxwell equations)

V. C) Examples:

1) Motional emf: Relative motion of $\vec{B}$-field and circuit produce emf

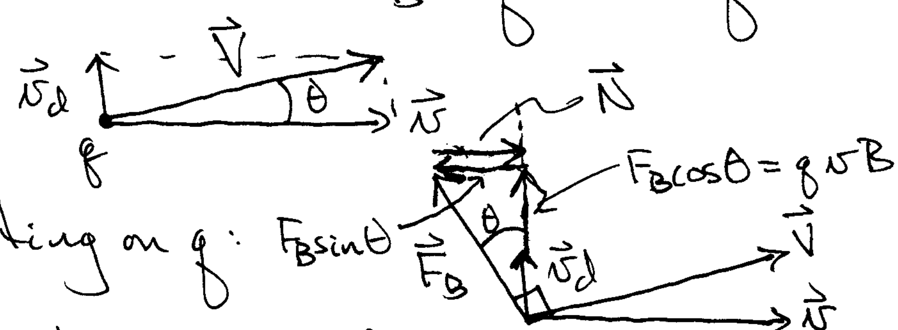
a) Rest Frame of Magnet

Observer S at rest wrt magnet



Lorentz force $\vec{F}_B = q\vec{V}\times\vec{B} = q\vec{v}\times\vec{B} + q\vec{v}_d\times\vec{B}$

Forces acting on q:



Wire exerts normal force on charges to keep them inside wire

$$\vec{N} = -q\vec{v}_d\times\vec{B} = F_B\sin\theta\,\hat{i} = qv_dB\,\hat{i}$$

Only $F_B\cos\theta$ is component of force along wire

So

$$\mathcal{E} = \oint_C \vec{f}\cdot d\vec{s} = \oint_C \vec{v}\times\vec{B}\cdot d\vec{s}$$

$$= vBD$$ (top & bottom paths give 0 and right side $\vec{B}=0$)

only from left side of C

VC. 1)(a) Now the work done on the charges in the circuit is supplied by the external agent pulling the loop with force $\vec{N}$. In time dt the charge moves

$$d\vec{s} = \vec{V}dt = (\vec{v} + \vec{v}_d)\,dt$$

So the work done by the force $\vec{N}$ is

$$dW = \vec{N}\cdot d\vec{s} = N v\,dt$$

But

$$N = F_B \sin\theta = qVB\sin\theta = qVB\left(\frac{v_d}{V}\right) = q v_d B$$

So

$$dW = q v_d B\, v\,dt = qvB\underbrace{(v_d\,dt)}$$

This is just the distance the charge moves up the left side of the wire.

Hence

$$W = \oint_C dW = qvBD$$

(again top & bottom paths give 0, and $B=0$ for Right hand side)

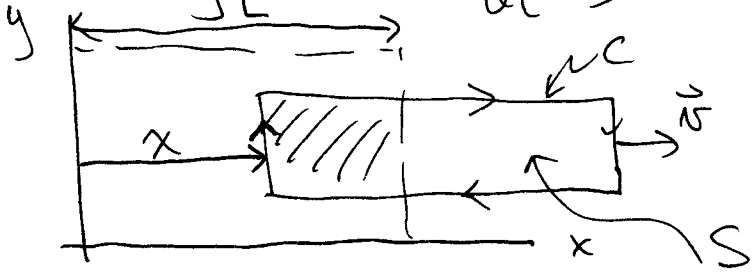
The work done per unit charge is just equal to the emf above

$$\mathcal{E} = \frac{dW}{dq} = vBD$$

as we found above.

The emf $\mathcal{E}$ is produced by the magnetic force $q\vec{v}\times\vec{B}$, the Work W performed is done by the pulling force, $\vec{N}$, yet $\mathcal{E} = dW/dq$.

V C.1a) This is the same result we obtain by applying the emf form of Faraday's Law directly $\mathcal{E} = -\frac{d}{dt}\Phi_B$



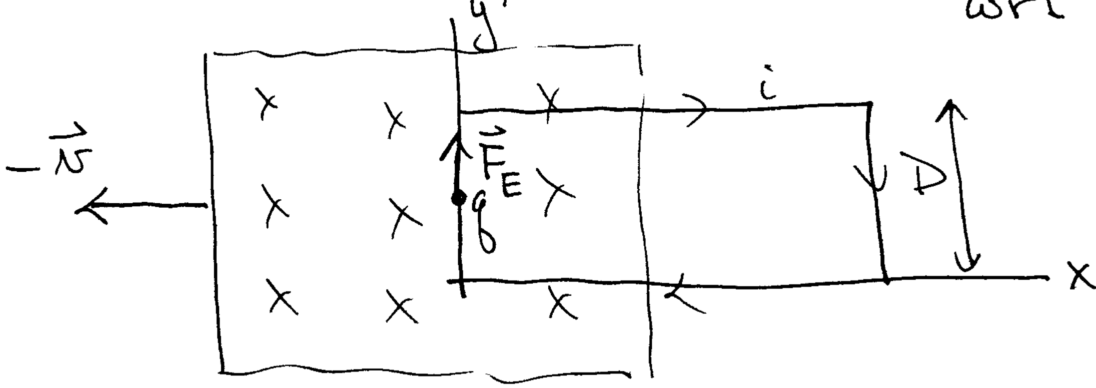
$$\Phi_B = \int_S \vec{B}\cdot d\vec{A} = BD(L-x) \quad \left(\begin{matrix}\text{shaded portion has}\\ B\neq 0 \text{ only}\end{matrix}\right)$$

So
$$\frac{d\Phi_B}{dt} = BD\left(-\frac{dx}{dt}\right) = -BDv$$

Faraday's Law: $\mathcal{E} = -\frac{d\Phi_B}{dt} = +BDv$ ✓
agrees with our force analysis.

b.) Rest frame of wire loop: magnet moves past with $-\vec{v}$

Observer S′ at rest wrt loop



Current i still flows in loop, the same emf will occur in circuit (for $v \ll c$) since the relative motion in the 2 cases

IV C 1) b) is the same. But there is no magnetic force in this frame $\vec{v}' \times \vec{B} = 0$ since $\vec{v}' = 0$, the wire is at rest. Can only conclude the current flows due to an induced $\vec{E}$-field, $\vec{E}'$, in the wire

$$\mathcal{E}' = \oint_C \vec{E}' \cdot d\vec{s} = E' D$$

$$\left(\begin{matrix}\text{same current} \\ \text{so } \mathcal{E}' = \mathcal{E}\end{matrix}\right) \quad = \mathcal{E} = vBD \qquad (v \ll c)$$

$$\Rightarrow \quad E' = vB$$ or in terms of vectors

$$\underline{\vec{E}' = \vec{v} \times \vec{B}}$$: Observer S' sees $\vec{E}'$ related to Observer S's $\vec{B}$-field.

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c.) Observer S: stationary wrt magnet Says the force on the charges arises from magnetic field with

$$\mathcal{E} = \oint_C (\vec{v} \times \vec{B}) \cdot d\vec{s}$$

Observer S': stationary wrt loop attributes current to electric field $\vec{E}'$ with

$$\mathcal{E}' = \oint_C \vec{E}' \cdot d\vec{s}$$

V G 1(c) Third observer S'': magnet and loop moving
Force on charges both electric and magnetic

$$\vec{F}/q = \vec{E}'' + \vec{v}'' \times \vec{B}''$$

Conclusion: 1) $\vec{E}$ and $\vec{B}$ are not independent of eachother and have no separate unique existence, they depend on the inertial frame!

2) Still we find Faraday's Law holds in all inertial frames - it is covariant - that is form invariant

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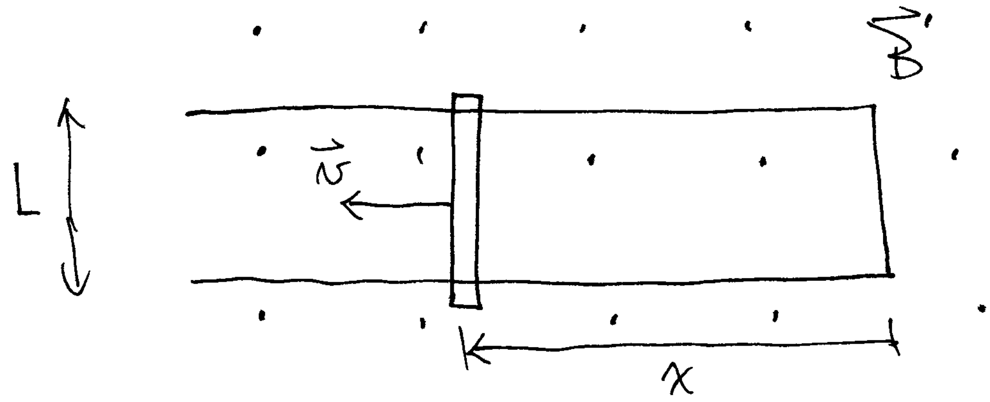
2) A conducting rod of length L is pulled along conducting rails at velocity $\vec{v}$ with a uniform $\vec{B}$ field $\perp$ to it.



a) Induced emf in the rod: $\phi_B = BLx$

(magnitude) $\mathcal{E} = \frac{d\phi_B}{dt} = BL\frac{dx}{dt}$

$= BLv$ (minus sign: Lenz law current flows CW)

V C.2)b) Resistance of rod is R and rails negligible
What is current?

$i = \mathcal{E}/R$ (Lenz' Law: flows CW)

c) What rate is internal energy being created in rod?

$P = i^2 R$

d) What is the force that must be applied by an external agent to the rod to maintain its motion?

Current in rod feels force

$\vec{F} = i\vec{L}\times\vec{B}$

$\Rightarrow F = iLB$ to the right.

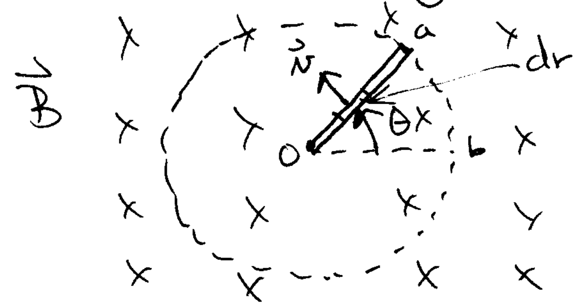
So agent must pull to the left with force $F = iLB$.

e) At what rate does this force do work on the rod?

$P = Fv = iLBv = i\mathcal{E}$

$= i^2 R$. External agent's work eventually appears as Joule heating.

V C. 3) A copper rod of length R rotates at angular frequency ω in a uniform $\vec{B}$-field. Find the emf $\mathcal{E}$ developed between the 2 ends of the rod.



A motional emf $d\mathcal{E}$ develops across element dr of the rod as it moves with velocity $\vec{v}$.

Charge separation produces an $\vec{E}$-field across dr at steady state $\vec{v} \times \vec{B} = -\vec{E} \Rightarrow E = vB = \omega r B$

The potential across dr is $E\,dr$ – these add up like batteries in series

$$\mathcal{E} = \int_0^R E\,dr = \omega B \int_0^R r\,dr = \tfrac{1}{2}\omega B R^2$$

$$\boxed{\mathcal{E} = \tfrac{1}{2}\omega B R^2}$$

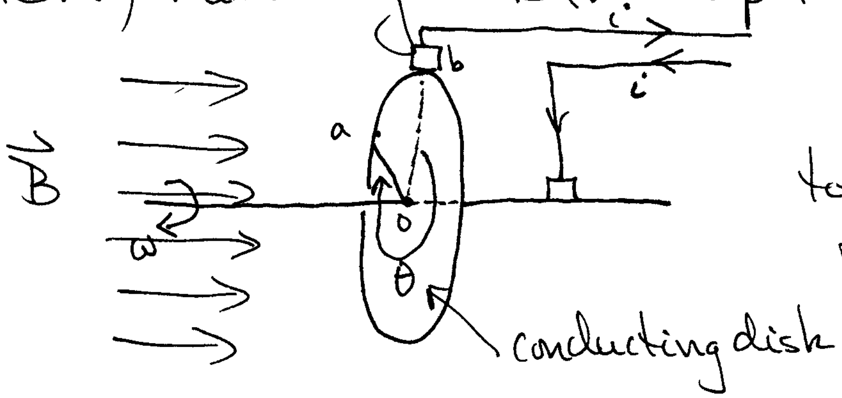
As well, directly from Faraday's Law we find the flux through aOb

$$\phi_B = \int_S \vec{B}\cdot d\vec{A} = B\left(\tfrac{1}{2}R^2\theta\right)$$

$$\frac{d\phi_B}{dt} = \tfrac{1}{2}BR^2\frac{d\theta}{dt} = \tfrac{1}{2}BR^2\omega$$

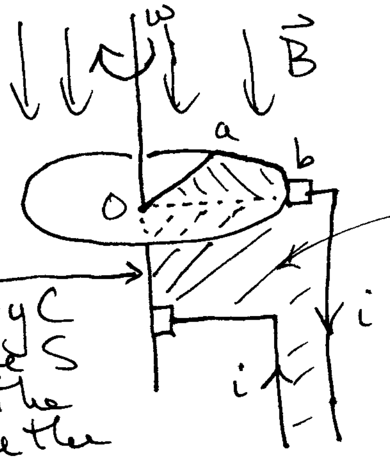
So the magnitude of $\mathcal{E} = \frac{d\phi_B}{dt} = \frac{1}{2}BR^2\omega$ ✓

V.C.4.) Faraday Disk (Homopolar Generator)



Use Faraday's law to find the emf produced by the spinning rotor

View from side



Open Surface S is formed by these 2 planar surfaces meeting at right angles

The boundary C of the surface S is given by the circuit outside the disk and Oab curve on disk

The flux $\Phi_B = \int_S \vec{B}\cdot d\vec{A} = \frac{1}{2}BR^2(2\pi - \theta) + 0$

$\vec{B}$ Through area O a b O on disk

for rest of circuit for which either $\vec{B}=0$ or $\vec{B}\cdot d\vec{A}=0$

Hence the emf by Faraday's Law is

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = \frac{1}{2}\omega BR^2 = \pi BR^2\nu$$

(where $\nu = \frac{\omega}{2\pi}$)

V C.4) The torque that must be provided to keep the rotor spinning when current i is produced is found from

$$\text{Power} = \mathcal{E} i \quad \text{also Power} = \tau\omega$$

$$\Rightarrow \quad \tau = \frac{\mathcal{E} i}{\omega} = \frac{1}{2} B R^2 i$$

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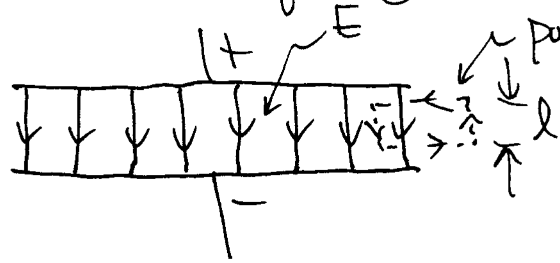
5) $\vec{E}$ - must fringe for a ||-plate capacitor



Assume no fringing
There is no $\vec{B}$-field in this problem so
$\phi_B = 0$; $\frac{d\phi_B}{dt} = 0$

$$\Rightarrow \quad \mathcal{E} = 0 \quad \text{by Faraday's law}$$

but $\quad \mathcal{E} = \oint_C \vec{E} \cdot d\vec{s} = E\ell = 0$

$$\Rightarrow E = 0 \quad \underline{\text{contradiction!}}$$

$\vec{E}$ must fringe so that $\mathcal{E} = \oint_C \vec{E} \cdot d\vec{s} = 0$