III. E.2.) Potential energy of magnetic dipole in magnetic field $\vec{B}$

$$U(\theta) = -\vec{\mu}\cdot\vec{B} \quad .$$

IV) Magnetostatics: (steady currents)

A) Biot-Savart Law

$$d\vec{B} = \frac{\mu_0}{4\pi}\,\frac{i\,d\vec{s}\times\vec{r}}{r^3}$$

Magnetic Field at point P due to $i\,d\vec{s}$ current element

(SI: $\mu_0 \equiv 4\pi\times10^{-7}\,\frac{T\cdot m}{A}$

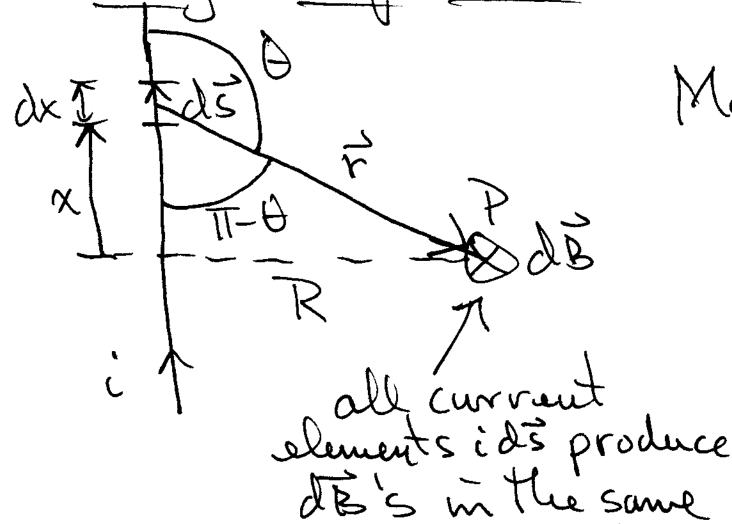
μ_0 = permeability of free space)

$$\vec{B} = \int_C d\vec{B} = \frac{\mu_0}{4\pi}\int_C \frac{i\,d\vec{s}\times\vec{r}}{r^3}$$

Magnetic Field at point P due to current carrying wire C.

IV B) Examples:

1) Long Straight Wire



Magnitude of $d\vec{B}$ at P

$$dB = \frac{\mu_0 i}{4\pi}\frac{ds \sin\theta}{r^2}$$

$$= \frac{\mu_0 i}{4\pi}\frac{dx \sin\theta}{r^2}$$

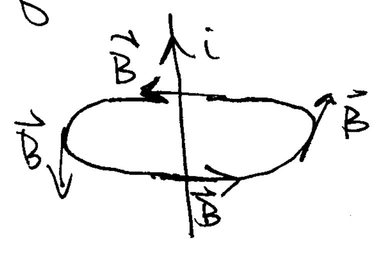
direction — into page — So just add up magnitudes — scalar integral

$$B = \int_{x=-\infty}^{+\infty} dB = \frac{\mu_0 i}{4\pi}\int_{x=-\infty}^{+\infty}\frac{\sin\theta}{r^2}\,dx$$

where $r = \sqrt{R^2+x^2}$, $\sin\theta = \sin(\pi-\theta)$
$= \frac{R}{\sqrt{R^2+x^2}}$

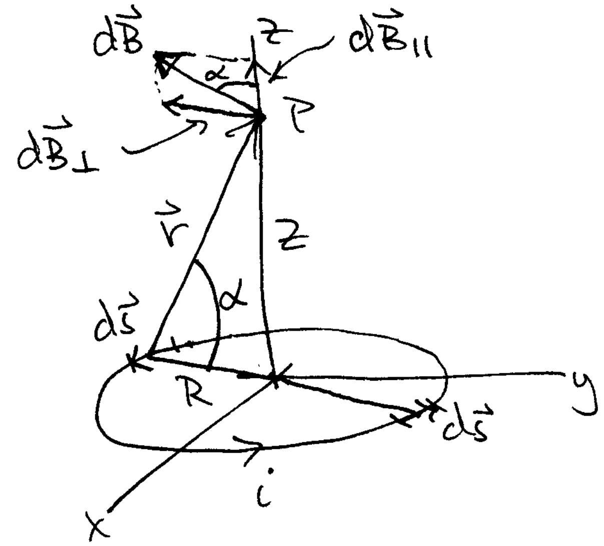
$$\Rightarrow \boxed{B = \frac{\mu_0 i}{2\pi R} \text{ and points into page}}$$

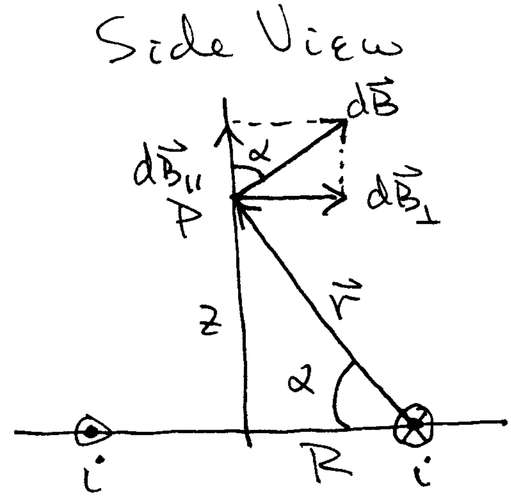
By symmetry, lines of $\vec{B}$ form circles about wire



(Right Hand Rule)

IV B2) Circular Current Loop





By symmetry only $d\vec{B}_{\parallel}$ will contribute to the total $\vec{B}$ at P the $d\vec{B}_{\perp}$ contributions cancel out.

$$\vec{B} = B\hat{k}$$

$$B = \int_C dB_{\parallel} = \frac{\mu_0 i}{4\pi}\int_C \frac{\sin\theta\, ds}{r^2} \underbrace{\cos\alpha}_{\text{for } dB_{\parallel}}$$

$\theta = 90°$! So $\sin\theta = 1$

and $r = \sqrt{R^2+z^2}$; $\cos\alpha = \dfrac{R}{r} = \dfrac{R}{\sqrt{R^2+z^2}}$

Hence $$B = \frac{\mu_0 i}{4\pi}\left(\frac{1}{R^2+z^2}\right)\left(\frac{R}{\sqrt{R^2+z^2}}\right)\underbrace{\int_C ds}_{=2\pi R}$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 i R^2}{2(R^2+z^2)^{3/2}}\hat{k}}$$

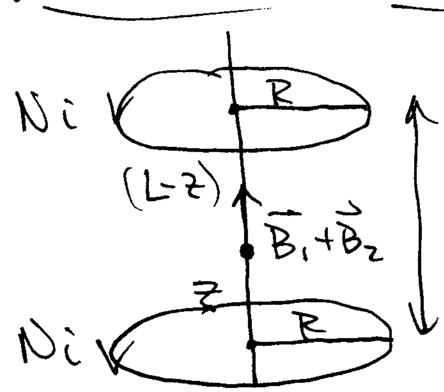
IV B 2) a) Far from loop, $z \gg R$, $B \approx \dfrac{\mu_0 i R^2}{2 z^3}$

For N-turns of wire, $B = \dfrac{\mu_0 N i R^2}{2 z^3}$

Dipole moment of coil $\mu = N i A = N i \pi R^2$

$$\Rightarrow \quad B = \frac{\mu_0}{2\pi} \frac{\mu}{z^3}$$ ← dipole moment.

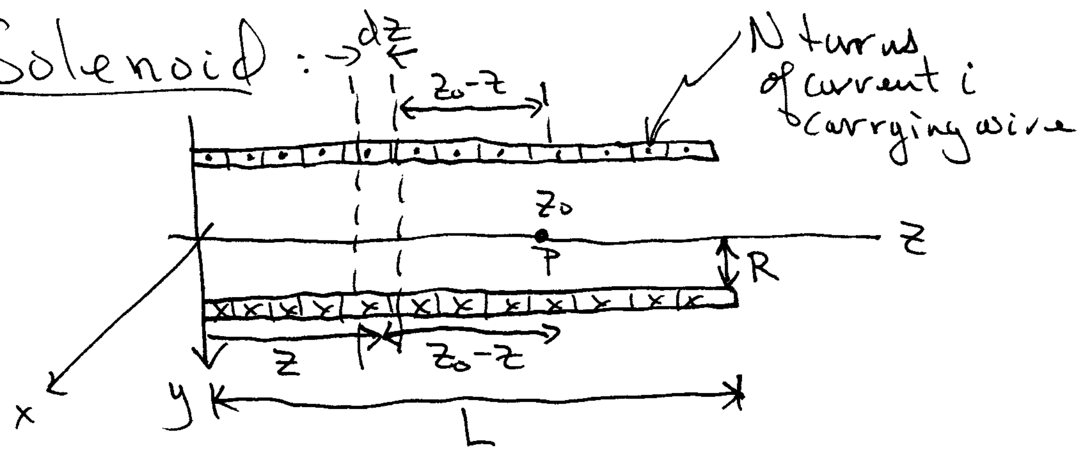
b) Helmholtz Coils:



$B = B_1 + B_2$

$$\vec{B} = \frac{\mu_0 N i A}{2\pi} \hat{k} \times \left[\frac{1}{(R^2+z^2)^{3/2}} + \frac{1}{(R^2+(L-z)^2)^{3/2}} \right]$$

3) Solenoid:



Infinitesimal slice of solenoid acts like current loop $\quad di = N i \dfrac{dz}{L}$

IV B3) $\vec{B}$ is in the z-direction with magnitude

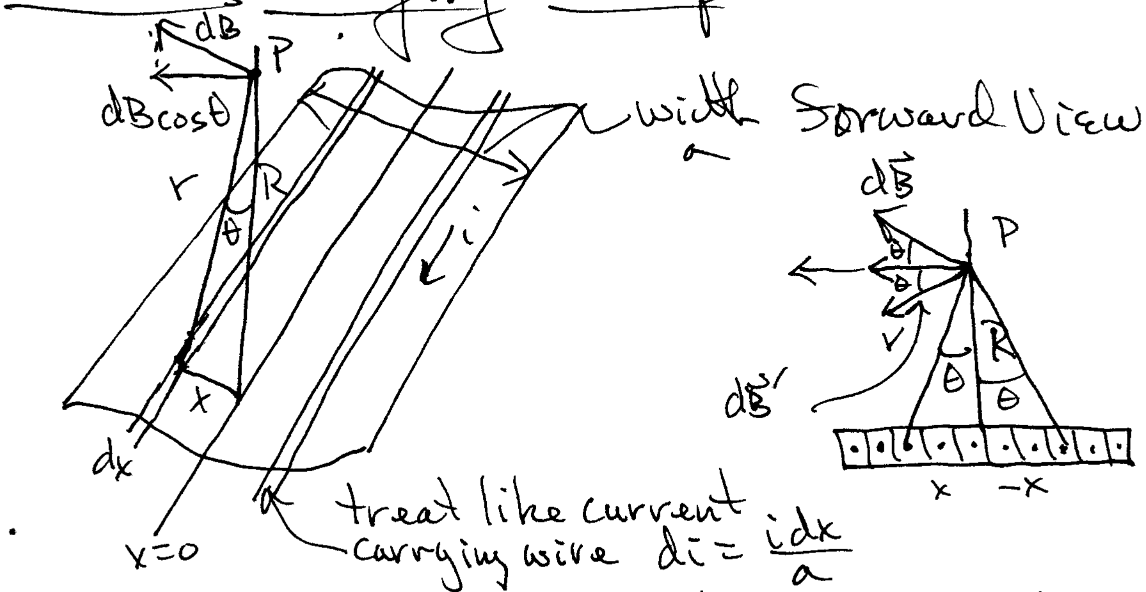
$$B(z_0) = \frac{\mu_0 N i R^2}{2L} \int_{z=0}^{L} \frac{dz}{[R^2 + (z_0 - z)^2]^{3/2}}$$

$$= \frac{\mu_0 N i}{L} \frac{1}{2}\left[\frac{1}{\sqrt{1+\left(\frac{R}{L-z_0}\right)^2}} + \frac{1}{\sqrt{1+\left(\frac{R}{z_0}\right)^2}}\right]$$

a) For $R \ll L$ and $z_0 = \frac{L}{2}$

$$B(z_0 = \tfrac{L}{2}) \simeq \frac{\mu_0 N i}{L}\left[1 - \frac{2R^2}{L^2}\right]$$

4) Current Carrying Strip



By symmetry only horizontal component of $d\vec{B}$ contributes to total $\vec{B}$ at P, vertical components cancel

IV B4) So again the horizontal magnitude is

$$B = \int dB \cos\theta$$

$$= \frac{\mu_0}{2\pi} \int \frac{di}{r} \cos\theta$$

But $r\cos\theta = R$
and $x = R\tan\theta$

$$B = \frac{\mu_0}{2\pi R} \frac{i}{a} \int_{-a/2}^{+a/2} \frac{dx}{\sec^2\theta} = \frac{\mu_0 i}{2\pi a} 2\alpha$$

where $\tan\alpha = \frac{a}{2R}$

So $$\boxed{B = \frac{\mu_0 i}{\pi a} \tan^{-1} \frac{a}{2R}}$$

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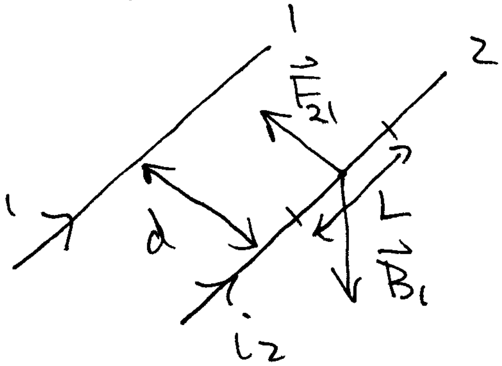
5) 2-Parallel Conductors:



$$B_1 = \frac{\mu_0 i_1}{2\pi d}$$

$$\Rightarrow F_{21} = i_2 L B_1$$

$$= \frac{\mu_0 L i_1 i_2}{2\pi d}$$

Likewise $F_{12} = \frac{\mu_0 L i_1 i_2}{d}$ pointing towards wire 2.

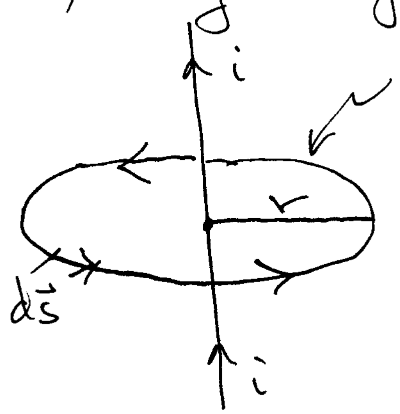
Attraction

IV B 5) This force law is used to define the unit of current — ampere

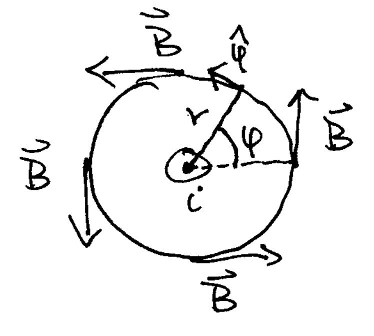
"Given 2 long parallel wires of negligible circular cross section separated in vacuum by a distance of 1 meter, the ampere is defined as the current in each wire that would produce a force of 2×10^{-7} newtons/meter of length."

C) Ampere's Law

1) Long Straight Wire



$$\vec{B} = \frac{\mu_0 i}{2\pi r} \hat{\varphi}$$ at points on circle



$d\vec{s}$ = element of path along circle

$= r\,d\varphi\,\hat{\varphi}$

$$\Rightarrow \oint_C \vec{B}\cdot d\vec{s} = \oint_C \left(\frac{\mu_0 i}{2\pi r}\hat{\varphi}\right)\cdot\left(r\,d\varphi\,\hat{\varphi}\right)$$

IV C 1) $\oint_C \vec{B}\cdot d\vec{s} = \int_{\varphi=0}^{2\pi} \frac{\mu_0 i}{2\pi}\, d\varphi = \mu_0 i$

$= \mu_0 i$

2) Ampere found this is generally true

Ampere's Law

$$\boxed{\oint_C \vec{B}\cdot d\vec{s} = \mu_0\, i_{enclosed}}$$

a) C is any closed curve called "Amperian loop"

b) $i_{enclosed}$ = net current that flows through surface bounded by C.

c) ← C = Amperian loop

$i_{enclosed} = i_1 + i_3 - i_2$

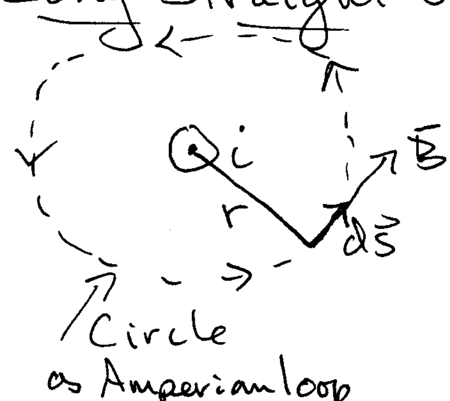
Right Hand Rule: fingers in direction of C, thumb points in "positive" current direction.

d) Ampere's Law replaces Biot-Savart law as one of the fundamental laws

IV C.2d) of magnetostatics.

3) Examples:

a) Long Straight Wire: By symmetry: $\vec{B} = B(r)\,\hat{\varphi}$



Ampere's Law

$$\oint_C \vec{B}\cdot d\vec{s} = \mu_0 i_{\text{enclosed}} = \mu_0 i$$

$$\| $$

$$B(r)\oint_C r\,d\varphi = B(r)\,2\pi r$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 i}{2\pi r}\,\hat{\varphi}}$$

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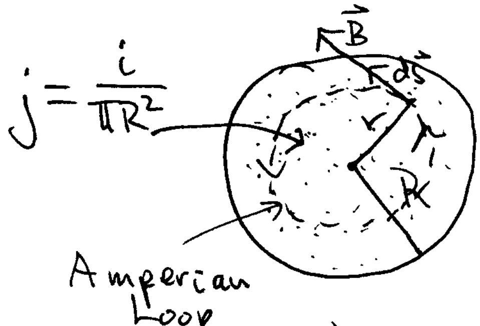
b) Long cylindrical wire of radius R with i uniformly spread over cross section



By symmetry $\vec{B} = B(r)\,\hat{\varphi}$

Ampere's Law $\oint_C \vec{B}\cdot d\vec{s} = \mu_0 i_{\text{enclosed}}$

a) For $r \leq R$: $B(r)\,(2\pi r) = \mu_0 \left(\frac{i}{\pi R^2}\right)(\pi r^2)$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 i}{2\pi}\,\frac{r}{R^2}\,\hat{\varphi}}$$

IV.C3b) b) For $r \geq R$: $\vec{B} = \frac{\mu_0 i}{2\pi r}\hat{\varphi}$.

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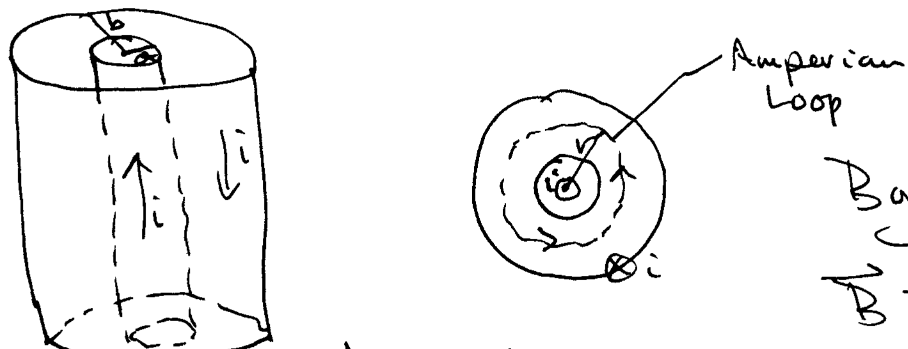
c) Coaxial Cable



By symmetry

$\vec{B} = B(r)\,\hat{\varphi}$

Ampere's Law:

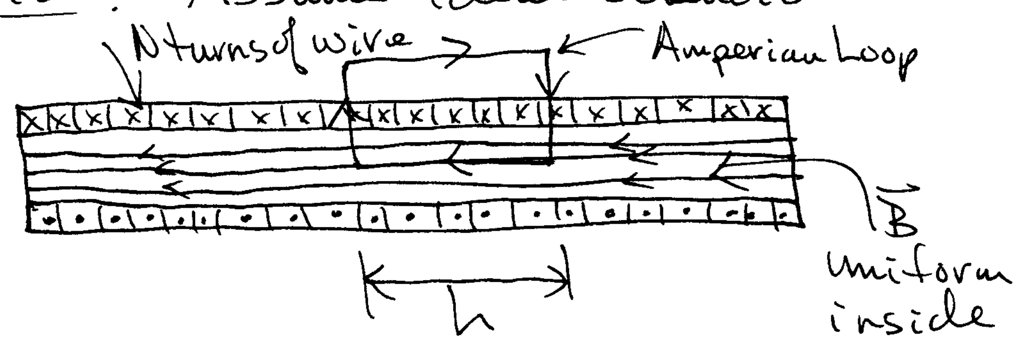
$$\oint_C \vec{B}\cdot d\vec{s} = 2\pi r B(r) = \mu_0 i_{enclosed}$$

$$= \mu_0 \begin{cases} i & a<r<b \\ 0 & r>b \end{cases}$$

$$\Rightarrow \quad B(r) = \begin{cases} \frac{\mu_0 i}{2\pi r} & a<r<b \\ 0 & r>b \end{cases}$$

D) Solenoid : Assume ideal solenoid

Assume $\vec{B}=0$ outside to a good approximation (true for infinitely long solenoid)



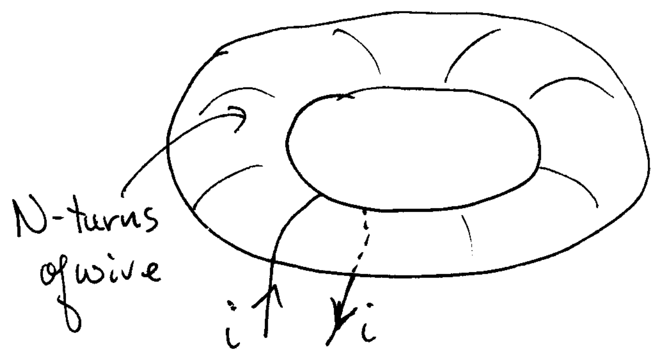
III C.3.D) Ampere's Law:

$$\oint_C \vec{B}\cdot d\vec{s} = Bh + \underset{\substack{\vec{B}\perp d\vec{s} \\ \Rightarrow \vec{B}\cdot d\vec{s}=0 \\ \text{here}}}{0} + \underset{\substack{\vec{B}=0 \\ \text{outside}}}{0} + \underset{\substack{\vec{B}\perp d\vec{s} \\ \Rightarrow \vec{B}\cdot d\vec{s}=0 \\ \text{here}}}{0}$$

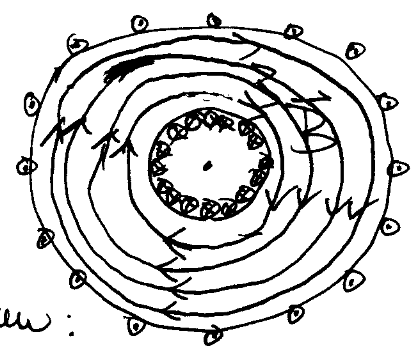
$$= \mu_0 i_{\text{enclosed}} = \mu_0 i \frac{N}{L}\cdot h$$

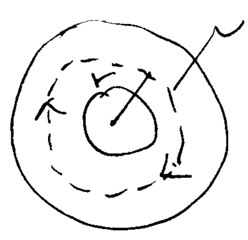
$$\Rightarrow \quad \boxed{B = \mu_0 i \underbrace{\frac{N}{L}}_{=n=\frac{\#\text{ of turns}}{\text{unit length}}} = \mu_0 i n}$$

E.) Toroid:



$\vec{B}$-lines form concentric circles inside toroid





Ampere's Law:

$$\oint_C \vec{B}\cdot d\vec{s} = B(r)\,2\pi r$$
$$= \mu_0 i_{\text{enclosed}}$$
$$= \mu_0 i N$$

IV C3E) Toroid: $\boxed{B(r) = \frac{\mu_0 i N}{2\pi r}}$

V.) Electromagnetic Induction

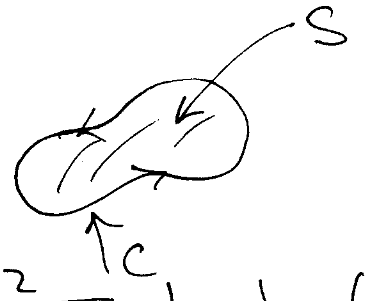
A) Faraday's Law: (emf form)

The induced emf in a circuit is equal to the negative of the rate at which the magnetic flux through the circuit is changing with time!

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

1) Magnetic flux through open surface S bounded by

$$\Phi_B = \int_S \vec{B} \cdot d\vec{A}$$



a) SI unit: $[\Phi_B] = 1\,T \cdot m^2 \equiv 1\,\text{weber (Wb)}$