II.) Currents

A) Electric current, i, is the rate at which charge is transported through a given surface

$$i = \frac{dq}{dt}$$

1) SI units: $[i] = 1\ \text{Ampere} = 1\ \frac{\text{Coulomb}}{\text{Second}}$

2) Convention: Current is in the direction of positive charge flow.

3) Current obeys scalar addition

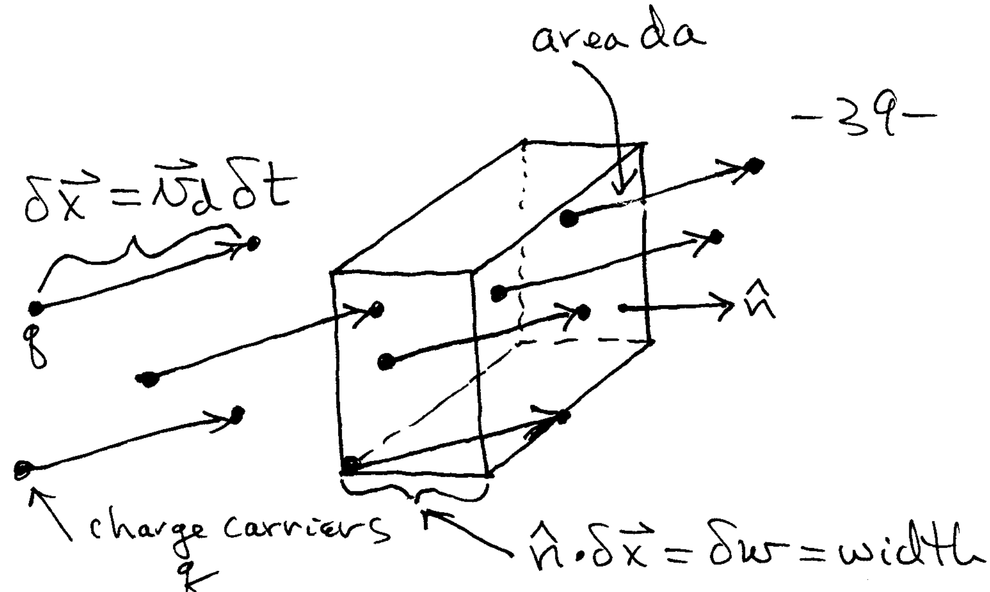
$$\Rightarrow \quad i_1 = i_2 + i_3$$

B) Conducting medium: free electron gas model electrons move at constant drift velocity $\vec{v}_d$ when an $\vec{E}$-field is established in conducting wire

1) Suppose conducting medium has one type of charge carrier q with $N = \frac{\#\text{ of charged carriers}}{\text{Volume}}$

each has drift velocity $\vec{v}_d$ in an $\vec{E}$-field

II B) 1)



In time δt, each charge carrier moves $\delta\vec{x} = \vec{v}_d\,\delta t$.
Charge δQ passes through area da in time δt

$$\delta Q = (qN)(\delta\vec{x}\cdot\hat{n}\,da)$$

$$= qN\,\vec{v}_d\cdot d\vec{A}\,\delta t$$ where $d\vec{A} = da\,\hat{n}$

Hence, di = current transported through da

$$di = \frac{\delta Q}{\delta t} = Nq\,\vec{v}_d\cdot d\vec{A}$$

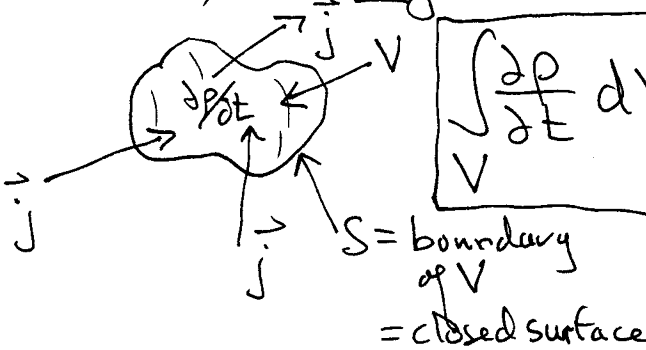
$$\boxed{di = \vec{j}\cdot d\vec{A}}$$

With the current density $\vec{j} \equiv Nq\,\vec{v}_d$

2) Current flowing through an arbitrarily shaped surface, "S", of macroscopic size

$$i = \int_S \vec{j}\cdot d\vec{A}$$

II B3) Charge Conservation = Continuity Equation



$$\boxed{\int_V \frac{\partial \rho}{\partial t}\, dV + \oint_S \vec{j} \cdot d\vec{A} = 0}$$

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c) Ohm's Law: Linear, isotropic conducting medium

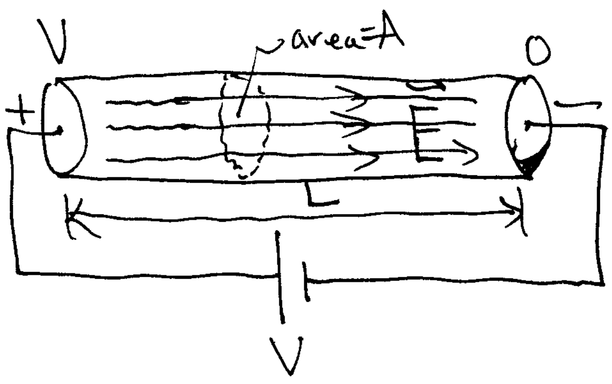
$$\vec{j} = \sigma \vec{E}$$

σ = conductivity of material

1) SI units: $1\ \text{ohm}\ (\Omega) \equiv 1\ \frac{\text{Volt}}{\text{Ampere}}$

$$[\sigma] = \Omega^{-1} m^{-1}$$

2) Wire of conductivity σ, length L, cross sectional area A



$$V = +\int_0^L \vec{E} \cdot d\vec{s}$$

$$= EL$$

$$\Rightarrow E = \frac{V}{L}$$

II.C.) 2) $i = \int_A \vec{j} \cdot d\vec{A} = jA = \sigma E A = \underbrace{\left(\sigma \frac{A}{L}\right)}_{\equiv R^{-1}} V$

$\Rightarrow$ $\boxed{V = iR \quad \text{Macroscopic Ohm's Law}}$

a) SI units of R: $[R] = \text{ohm}$

b) $\rho \equiv \frac{1}{\sigma}$ = resistivity of material

$[\rho] = \Omega \cdot m$

$R = \frac{\rho L}{A}$

c) Circuit Symbol for <u>Resistor</u>

D) Work done by $\vec{E}$-field in moving dQ through potential difference V

$$dW = dQ\,V = i\,dt\,V$$

Rate of energy transfer = Power P

$$\boxed{P = \frac{dW}{dt} = iV}$$

(Units: 1 Amp·Volt = 1 watt)

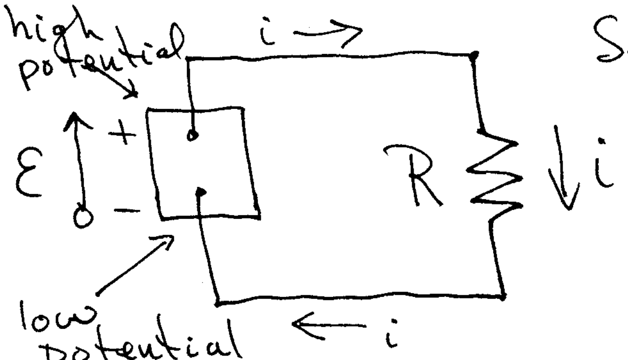
1) For Ohmic Device $V = iR$

$$\Rightarrow P = iV = i^2 R = \frac{V^2}{R}$$

II.E.) Electromotive Force (emf) $\mathcal{E}$

$$\mathcal{E} = \frac{dW}{dq}$$

1) SI units: $[\mathcal{E}] = 1 \frac{\text{Joule}}{\text{Coulomb}} = 1\,\text{volt}.$

2) 

Seat of emf does work

$$dW = \mathcal{E}\,dq$$
$$= \mathcal{E}\,i\,dt$$

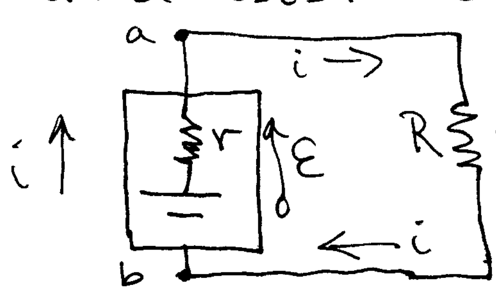
Conservation of energy: $\mathcal{E}\,i\,dt = i^2 R\,dt$

$$\Rightarrow \quad i = \mathcal{E}/R$$

3) Kirchhoff's Second Rule (Loop Rule):
(Conservation of energy)

The algebraic sum of the changes in potential encountered in a complete traversal of any closed circuit is zero.

4) Internal resistance of seat of emf



Loop Rule:

$$V_b + \mathcal{E} - ir - iR = V_b$$

$$\Rightarrow \boxed{i = \frac{\mathcal{E}}{R+r}}$$

II E.4.) Potential Difference V_{ab} : traverse part of circuit by any path that includes the two points a & b

a) Start at b and go CCW to a

$$V_b + iR = V_a$$

$$\Rightarrow V_a - V_b \equiv V_{ab} = iR$$

$$= \mathcal{E}\frac{R}{R+r}$$

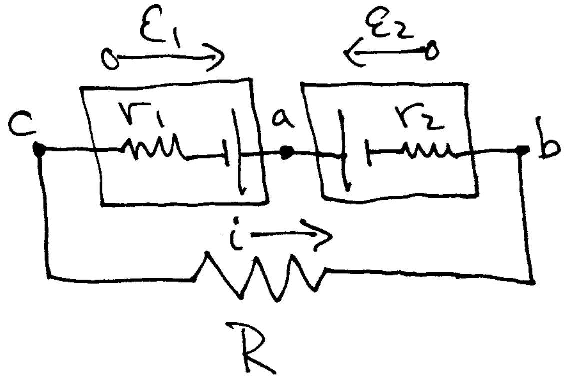
b) Start at a and go CCW to b

$$V_a + ir - \mathcal{E} = V_b$$

$$\Rightarrow V_{ab} = \mathcal{E} - ir = \mathcal{E}\frac{R}{R+r}$$

the same as above.

5.) Example



a) $i = ?$ Loop Rule: Start at "a" go CW back to "a"

$$-\mathcal{E}_2 + ir_2 + iR + ir_1 + \mathcal{E}_1 = 0$$

$$\Rightarrow \boxed{i = \frac{\mathcal{E}_2 - \mathcal{E}_1}{R + r_1 + r_2}}$$

II E5) a) The sign of i tells its direction of flow

b) $V_{ab} = ?$ Start at "b" go CCW to "a"

$$V_b - i r_2 + \mathcal{E}_2 = V_a$$

$$\Rightarrow \boxed{V_{ab} = -i r_2 + \mathcal{E}_2 = \frac{\mathcal{E}_1 r_2 + \mathcal{E}_2 (R + r_1)}{R + r_1 + r_2}}$$

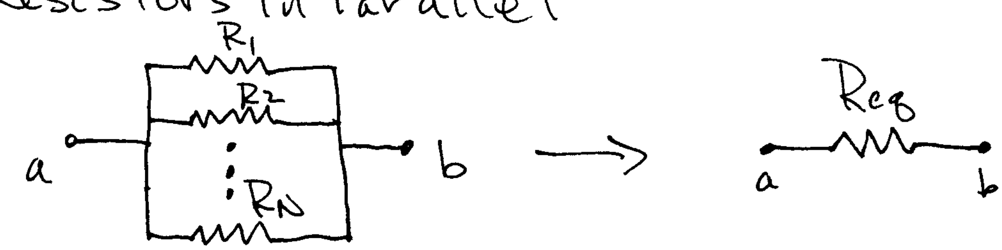
c) $V_{ac} = ?$ Start at "c" go CW to "a"

$$V_c + i r_1 + \mathcal{E}_1 = V_a$$

$$\Rightarrow \boxed{V_{ac} = \mathcal{E}_1 + i r_1 = \frac{\mathcal{E}_1 (R + r_2) + \mathcal{E}_2 r_1}{R + r_1 + r_2}}$$

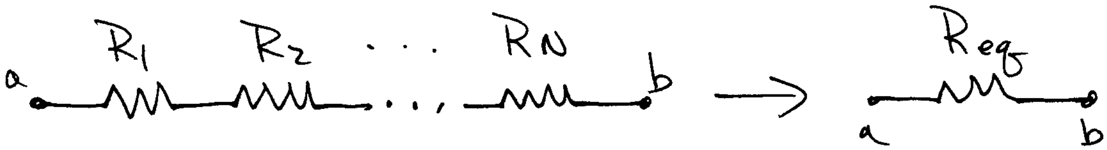
F) Equivalent Resistance

1) Resistors in Parallel



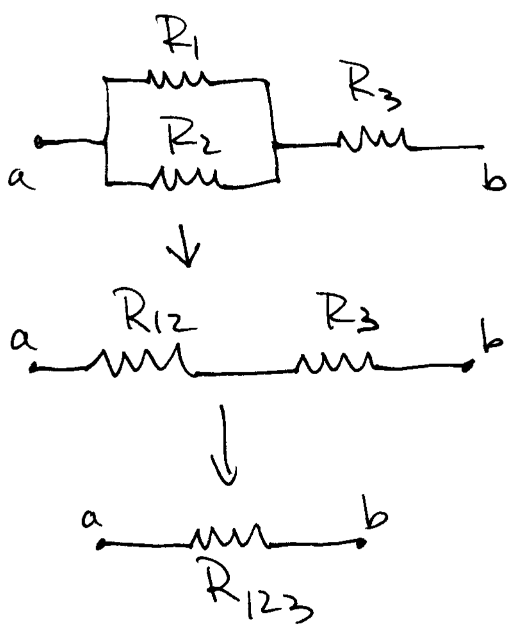
II F 1) $$\frac{1}{R_{eq}} = \sum_{i=1}^{N} \frac{1}{R_i} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}$$

2) Resistors in Series



$$R_{eq} = \sum_{i=1}^{N} R_i = R_1 + R_2 + \cdots + R_N$$

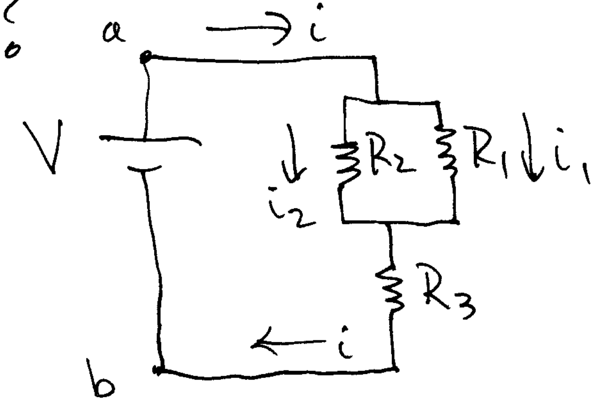
3) Example



$$R_{12} = \frac{R_1 R_2}{R_1 + R_2}$$

$$R_{123} = R_{12} + R_3 = \frac{R_1 R_2}{R_1 + R_2} + R_3$$

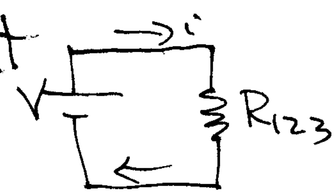
$i_1 = ?$



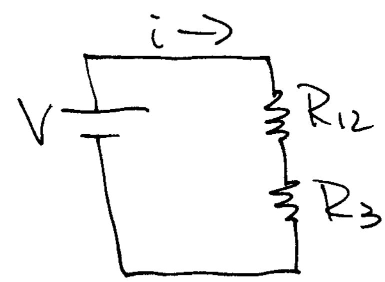
Steps:

a) $$i = \frac{V}{R_{123}}$$

Equivalent Circuit



II F3) b) Equivalent Circuit



$$V_{12} = i R_{12}$$

c) Since R_1 & R_2 are in ∥, V_{12} is the voltage that appears across both — so
$(V_1 = V_2 = V_{12})$

$$i_1 = \frac{V_{12}}{R_1}$$

$$= \frac{V R_{12}}{R_1 R_{123}}$$

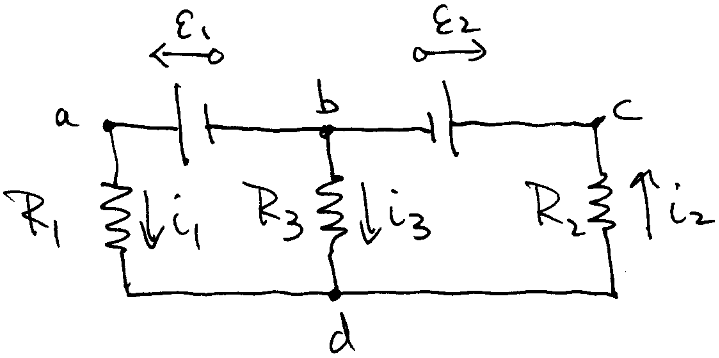
G) <u>Multiloop Circuits</u>: Made up of <u>junctions</u> and <u>branches</u>

Junction = 3 or more wire segments meet
Branch = Circuit path that starts at one junction and ends at another.

1) <u>Kirchhoff's First Rule</u> (Junction Rule)
(Conservation of Charge)

At any junction, the sum of the currents leaving the junction equals the sum of the currents entering the junction.

- II G 2) Example:



a) 2 Junctions b & d
b) 3 Branches: Left Branch bad
Central Branch bd
Right Branch bcd

Find i_1, i_2, i_3:

1) Apply Junction Rule to d $\Rightarrow$

$$\boxed{i_1 + i_3 = i_2}$$

2) Apply Loop Rule: CCW b-a-d-b

$$\boxed{\mathcal{E}_1 - i_1 R_1 + i_3 R_3 = 0}$$

3) Apply Loop Rule: CCW b-d-c-b

$$\boxed{-i_3 R_3 - i_2 R_2 - \mathcal{E}_2 = 0}$$

$\Rightarrow$ 3 equations for 3 unknowns i_1, i_2, i_3
(Use either junction to apply junction rule, get same equation, Use any 2 independent loops, get same information)

- II G 2) Solve 3 simultaneous equations

$$i_3 = -\frac{R_2\mathcal{E}_1 + R_1\mathcal{E}_2}{R_1R_2 + R_2R_3 + R_1R_3}$$

(negative result ⇒ i_3 flows from $d \to b$ opposite how we drew)

$$i_2 = \frac{\mathcal{E}_1 R_3 - \mathcal{E}_2(R_1+R_3)}{R_1R_2 + R_2R_3 + R_1R_3}$$

$$i_1 = \frac{\mathcal{E}_1(R_2+R_3) - \mathcal{E}_2 R_3}{R_1R_2 + R_2R_3 + R_1R_3}$$

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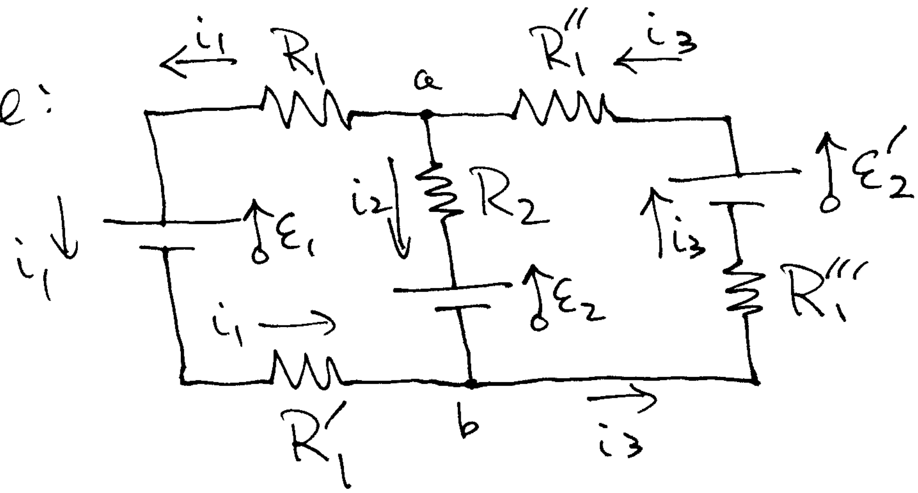
3) Example:



1) Apply Junction Rule at b

$$i_1 + i_2 = i_3$$

2) Apply Loop Rule: CCW a-b-a (Left Loop)

$$-i_1R_1 - \mathcal{E}_1 - i_1R_1' + \mathcal{E}_2 + i_2R_2 = 0$$

(if $R_1' = R_1 \Rightarrow 2i_1R_1 - i_2R_2 = \mathcal{E}_2 - \mathcal{E}_1$)

II G3) 3) Apply Loop Rule: CW a-b-a (Right Loop)

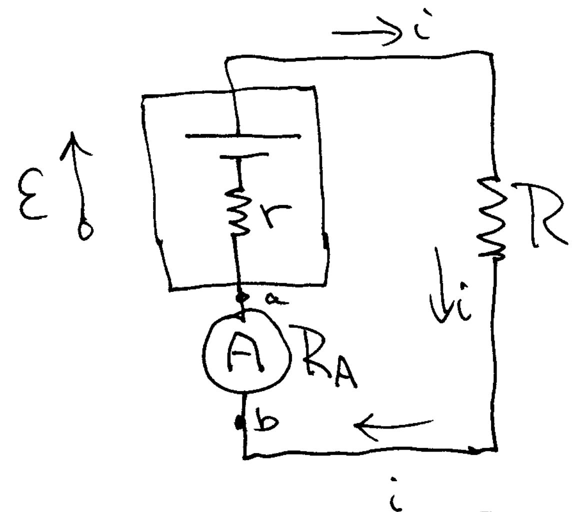
$$i_3 R_1'' - \mathcal{E}_2' + i_3 R_1''' + \mathcal{E}_2 + i_2 R_2 = 0$$

$$\left(\text{if } R_1'' = R_1''' = R_1 \text{ and } \mathcal{E}_2' = \mathcal{E}_2 \Rightarrow \quad i_2 R_2 + 2 i_3 R_1 = 0\right)$$

Hence we have 3 equations for the 3 unknowns i_1, i_2, i_3 — Solve simultaneously:
(See class notes for solution and $V_a - i_2 R_2 - \mathcal{E}_2 = V_b \Rightarrow V_{ab} = \mathcal{E}_2 + i_2 R_2$)

H) Measuring Instruments:

1) Ammeter measures current



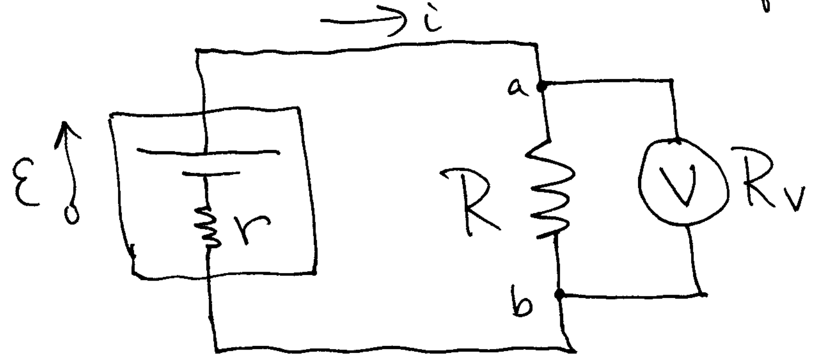
Loop Rule

$$0 = \mathcal{E} - iR - iR_A - ir$$

$$\Rightarrow \quad i = \frac{\mathcal{E}}{r + R + R_A}$$

Ideal Ammeter $R_A = 0$
Realistic Ammeter $R_A \ll r + R$ to give small error

II H 2) Voltmeter measure potential difference



Loop Rule

$$\mathcal{E} - iR_{eq} - ir = 0$$

$$i = \frac{\mathcal{E}}{r + R_{eq}}$$

$$R_{eq} = \frac{R R_V}{R + R_V}$$

Potential Difference measured

$$V_a - iR_{eq} = V_b$$

$$\Rightarrow \quad V_{ab} = iR_{eq} = \mathcal{E}\,\frac{R_{eq}}{R_{eq} + r}$$

Without Voltmeter this would be $V^0_{ab} = \mathcal{E}\frac{R}{R+r}$

So ideal Voltmeter $R_V = \infty$ (Then $R_{eq} = R$)

Realistic Voltmeter $R_V \gg R$ to give small error

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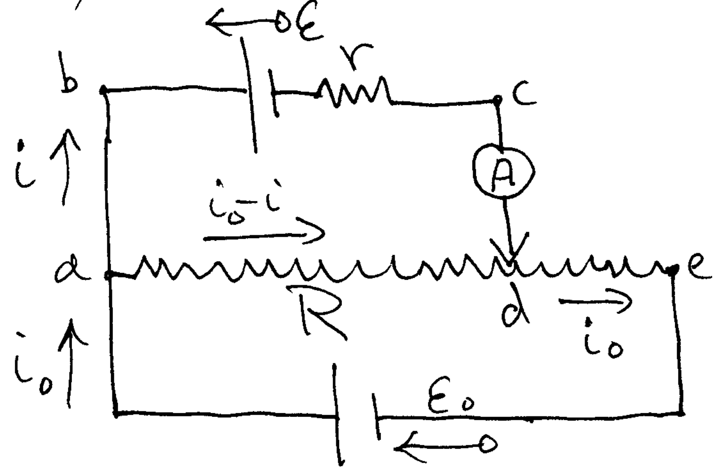
3) Potentiometer: Measures unknown emf $\mathcal{E}_x$ by comparing to standard emf $\mathcal{E}_S$



R = resistance between a-d

II H3) <u>Step 1</u>: Place $\mathcal{E}_s$ where $\mathcal{E}$ is and adjust R until $i = 0$ - potentiometer is "balanced"

call this value of $R = R_s$ at balance

Loop Rule CW a-b-c-d-a

$$-\mathcal{E}_s - ir + (i_0 - i)R_s = 0$$

But $i = 0$ in balance $\Rightarrow$

$$\mathcal{E}_s = i_0 R_s .$$

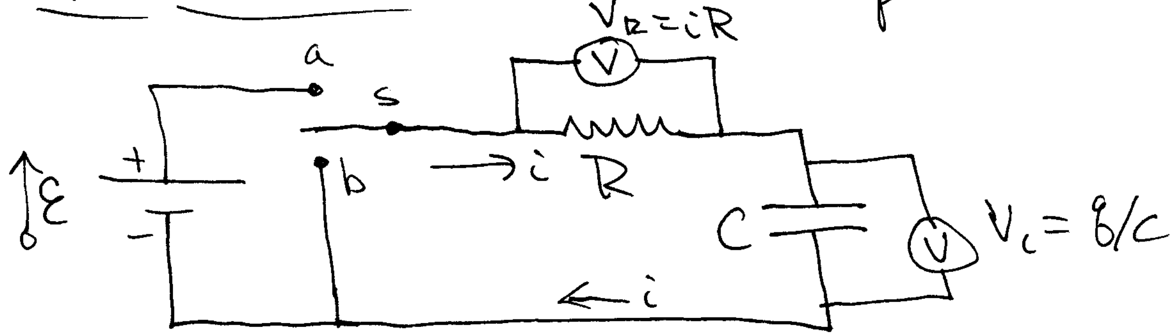
<u>Step 2</u>: Substitute $\mathcal{E}_x$ for $\mathcal{E}_s$ and adjust R to R_x for balance $i = 0$ again

Now loop Rule gives $\mathcal{E}_x = i_0 R_x$ (same i_0)

$$\Rightarrow \quad \boxed{\mathcal{E}_x = \mathcal{E}_s \frac{R_x}{R_s}}$$

(independent of $\mathcal{E}_0$ — $i = 0$ "null instrument")

II.I) RC-circuits: Current will depend on time



a) Throw switch to a — charge capacitor
Loop Rule: CW start at a end at a
with $V_R = iR$ and $V_c = q/C$

$$\mathcal{E} = iR + q/C$$

$$= R\frac{dq}{dt} + q/C$$

This is DE for $q = q(t)$:

$$\int_0^{q(t)} \frac{dq}{\mathcal{E}C - q} = \frac{1}{RC}\int_0^t dt$$

$$-\ln\left[\mathcal{E}C - q\right]\Big|_0^{q(t)} = \frac{1}{RC}\, t$$

$$\Rightarrow \quad \ln\left[\frac{\mathcal{E}C - q(t)}{\mathcal{E}C}\right] = -\frac{t}{RC}$$

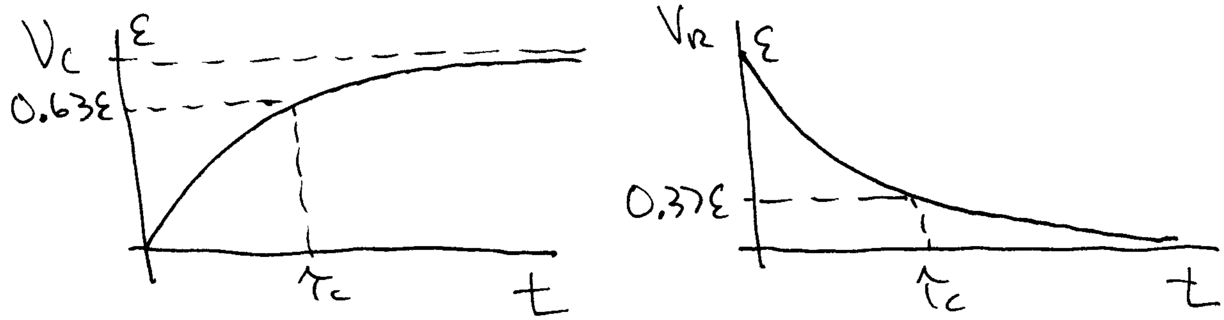
$$\Rightarrow \quad \frac{\mathcal{E}C - q(t)}{\mathcal{E}C} = e^{-t/RC}$$

IIa) $\Rightarrow$

$$q(t) = \mathcal{E}C\left[1 - e^{-t/\tau_C}\right]$$

with $\tau_C \equiv RC$ = Capacitive Time Constant

Hence $i = i(t) = \dfrac{dq(t)}{dt} = \dfrac{\mathcal{E}}{R}\, e^{-t/\tau_C}$



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b) Throw switch from a to b: Discharge Capacitor

Loop Rule:

$$iR + q/C = 0$$

$$\Rightarrow R\frac{dq}{dt} + \frac{1}{C}q = 0$$

$$\Rightarrow \int_{q_0}^{q(t)} \frac{dq}{q} = -\frac{1}{RC}\int_0^t dt$$

$$\Rightarrow \boxed{q(t) = q_0\, e^{-t/\tau_C}}$$

II.I.b) q_0 = initial charge on capacitor
for $t \gg \tau_c$ in part a $q_0 \approx \mathcal{E} C$.

$$q(t) = \mathcal{E} C e^{-t/\tau_c}$$

and

$$i(t) = \frac{dq(t)}{dt} = -\frac{\mathcal{E}}{R} e^{-t/\tau_c}$$