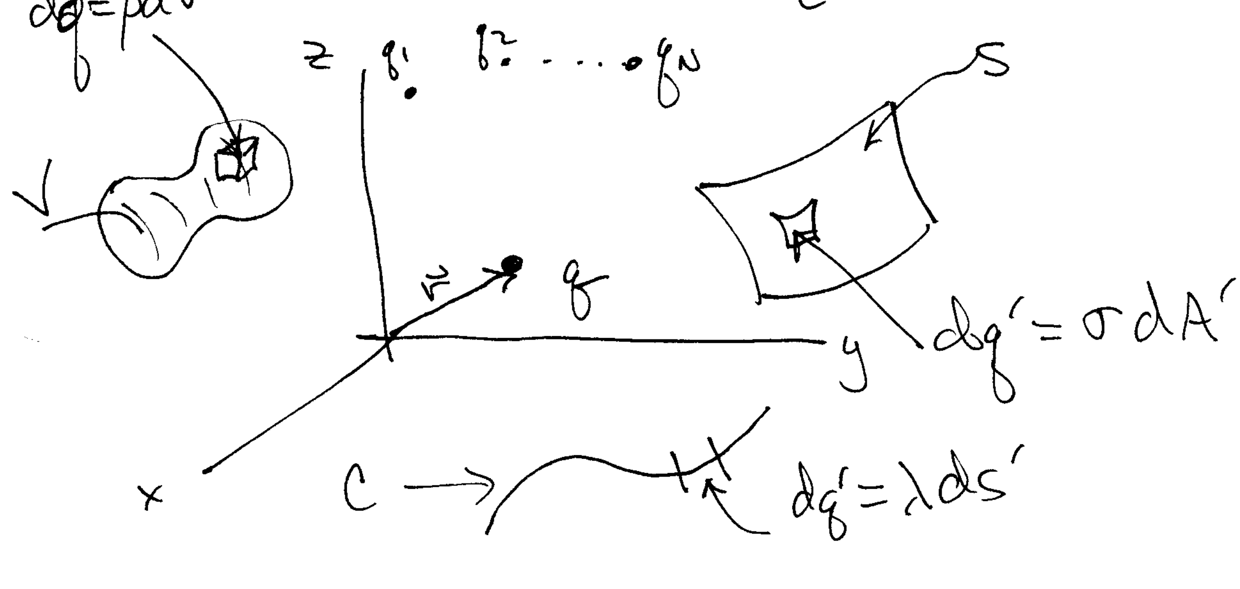
Physics 271 Electricity & Magnetism

Review Notes

I) Electrostatics

A) Coulomb's Law: Experimentally observed force between charges

$$\vec{F}_q = \frac{q}{4\pi\epsilon_0}\left\{\sum_{i=1}^{N} \frac{q_i}{|\vec{r}-\vec{r}_i|^2}\frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|} + \int_V \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\rho(\vec{r}\,')dV' + \int_S \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\sigma(\vec{r}\,')dA' + \int_C \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\lambda(\vec{r}\,')ds'\right\}$$



$$\left(\epsilon_0 = 8.85\times10^{-12}\frac{C^2}{N\cdot m^2} = 8.85\ pF/m\right)$$

I A 1) SI units $\frac{1}{4\pi\epsilon_0} = 8.99\times10^{9}\ \frac{N\cdot m^2}{C^2}$

2) Principle of superposition: Total force is vector sum of pairwise individual forces on q.

B) Electric Field: $\vec{E} = \lim_{q_0\to 0} \frac{\vec{F}_{q_0}}{q_0}$

with q_0 a positive test charge

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left\{\sum_{i=1}^{N} q_i \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3} + \int_V \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\rho(\vec{r}\,')\,dV' + \int_S \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\sigma(\vec{r}\,')\,dA' + \int_C \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\lambda(\vec{r}\,')\,ds'\right\}$$

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1) As did the force, $\vec{E}$ obeys the principle of superposition, the total Electric field is the vector sum of the individual charges' electric fields

IB2) Lines of Force (= field lines): a) end & start on charges, b) tangent to line is direction of $\vec{E}$ (as a positive test charge would move), c) # of lines / unit cross-sectional area is proportional to magnitude of $\vec{E}$ in that region, d) lines are continuous and do not cross in charge free regions of space

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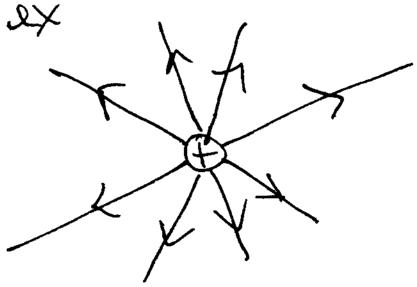

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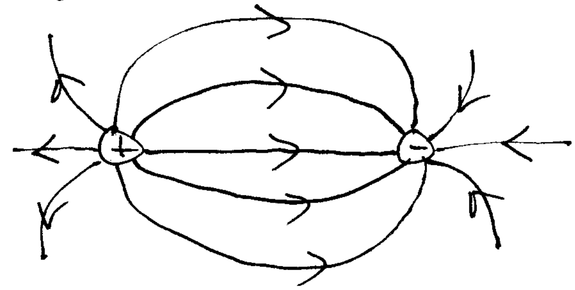
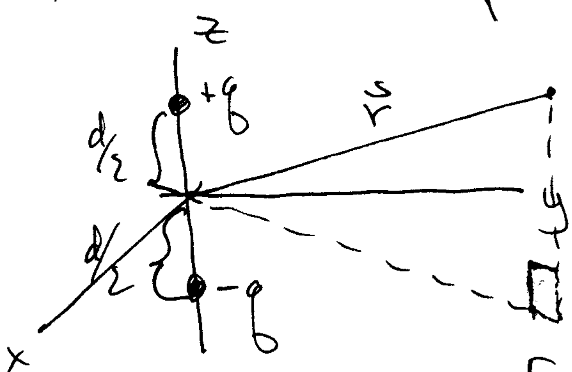


3) Electric Dipole:



$\vec{p} \equiv q d \hat{k}$, $\vec{E}$ given by $\vec{p}$

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0} \times \left[\frac{\vec{r} - \frac{d}{2}\hat{k}}{\left(x^2+y^2+(z-\frac{d}{2})^2\right)^{3/2}} - \frac{\vec{r} + \frac{d}{2}\hat{k}}{\left(x^2+y^2+(z+\frac{d}{2})^2\right)^{3/2}} \right]$$

IB3) $\Rightarrow$ a) $\vec{E}(x, y=0, z=0) = \dfrac{-\vec{p}}{4\pi\epsilon_0 \left(x^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}}$

b) $x \gg d$: Taylor Expand: $\epsilon \ll 1$

$$f(\epsilon) = f(0) + \epsilon f'(0) + \frac{\epsilon^2}{2} f''(0) + \ldots$$

$$\vec{E}(x, y=0, z=0) \overset{x \gg d}{\approx} \frac{-\vec{p}}{4\pi\epsilon_0 x^3}$$

c) General Result for $r \gg d$

$$\vec{E}(\vec{r}) \approx \frac{1}{4\pi\epsilon_0}\left[3\,\frac{\vec{p}\cdot\vec{r}}{r^5}\,\vec{r} - \frac{\vec{p}}{r^3}\right]$$

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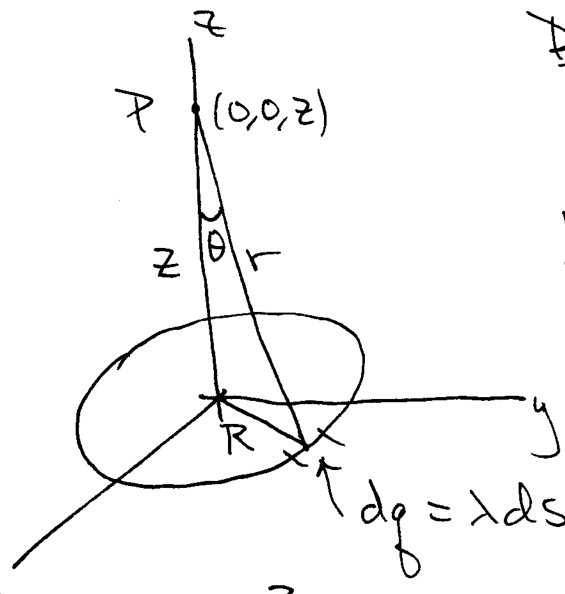
4) Ring of Charge:



By symmetry at P! $E_x = E_y = 0$

$$E_z = \int_{\text{ring}} dE_z$$

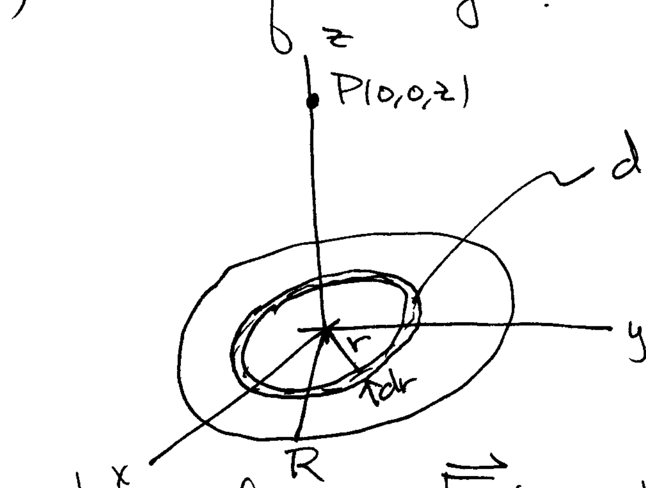
$$= \frac{\lambda}{4\pi\epsilon_0}\int_{\varphi=0}^{2\pi} \frac{R\,d\varphi}{(z^2+R^2)}\cos\theta$$

where $\cos\theta = \frac{z}{r}$

$$E_z \doteq \frac{q}{4\pi\epsilon_0}\,\frac{z}{(z^2+R^2)^{3/2}}$$

with $q = 2\pi R\lambda$

IB5) Disk of Charge:



$$dq = \sigma\, dA$$
$$= \sigma(2\pi r)\,dr$$

Ring of Charge

$$d\vec{E}(0,0,z) = \frac{dq\, z\, \hat{k}}{4\pi\epsilon_0 (z^2+r^2)^{3/2}}$$

integrate $\Rightarrow$
$$\vec{E}(0,0,z) = \frac{2\pi\sigma z\,\hat{k}}{4\pi\epsilon_0}\int_{r=0}^{R}\frac{r\,dr}{(z^2+r^2)^{3/2}}$$

$$\vec{E}(0,0,z) = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right]\hat{k} \quad (z>0)$$

a) Close to disk $R \gg z$: $\dfrac{z}{\sqrt{z^2+R^2}} = \dfrac{z/R}{\sqrt{1+(z/R)^2}} \simeq 0$

$$\Rightarrow \vec{E} = \frac{\sigma}{2\epsilon_0}\hat{k}$$

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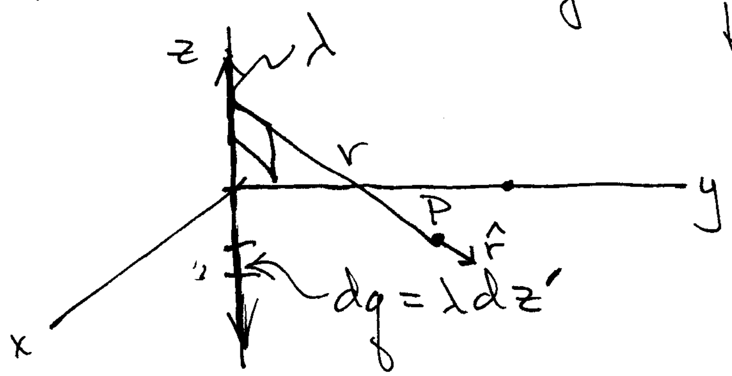
6) Line of Charge: By symmetry $\vec{E}$ is $\perp$ to z-axis



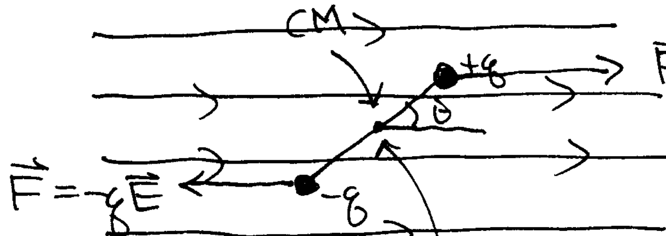
$$\vec{E} = E\,\hat{r}$$
$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$

I C) Given the electric field $\vec{E}(\vec{r})$ at a point in space $\vec{r}$, the electric force of a point charge q located at $\vec{r}$ is simply

$$\vec{F}_q = q\vec{E}.$$

If classical, non-relativistic particle, it will then move according to Newton's 2nd Law $\vec{F}_q = m\vec{a}$.

1) Dipole in a uniform $\vec{E}$-field



Torque tends to align $\vec{p}$ with $\vec{E}$

Torque about CM $\vec{\tau}$ points into page. $\otimes$

$$\vec{\tau} = \sum_i \vec{r}_i \times \vec{F}_i$$

$$|\vec{\tau}| = \tau = F\frac{d}{2}\sin\theta + F\frac{d}{2}\sin\theta$$
$$= Fd\sin\theta$$

$$F = qE \Rightarrow \tau = qdE\sin\theta$$
$$= pE\sin\theta$$

Which vectorially becomes

$$\boxed{\vec{\tau} = \vec{p}\times\vec{E}}$$

IC 1) Work done by electric field $= W = \int_{\theta_0}^{\theta} \vec{\tau} \cdot d\vec{\theta} = \int_{\theta_0}^{\theta} -\tau \, d\theta$

final ∡

initial ∡

$\vec{\tau}$ & $d\vec{\theta}$ opposite directions since ∡ θ decreases

$$W = -pE \int_{\theta_0}^{\theta} \sin\theta \, d\theta$$

$$W = pE\left[\cos\theta - \cos\theta_0\right]$$

Change in potential energy of system

$$\Delta U = U(\theta) - U(\theta_0) = -W$$

$$= -pE\left[\cos\theta - \cos\theta_0\right]$$

Choose arbitrary reference point $\theta = \frac{\pi}{2}$ for zero of PE : $U(\frac{\pi}{2}) \equiv 0$.

$$\Rightarrow \quad \boxed{U(\theta) = -pE\cos\theta = -\vec{p} \cdot \vec{E}}$$

PE is a minimum when $\vec{p}$ & $\vec{E}$ align.

I.D.) Gauss' Law:

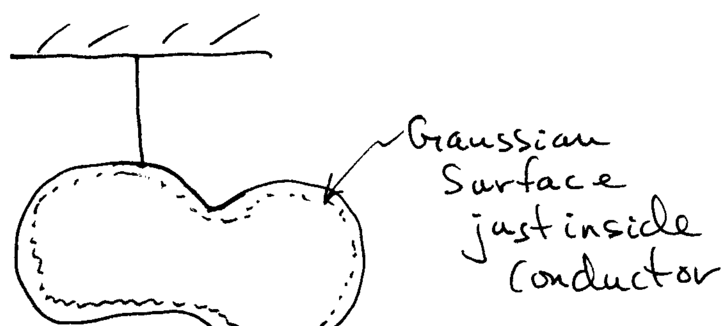
$$\oint_S \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0} Q_{enclosed}$$

S = closed Gaussian surface bounding volume V
$Q_{enclosed}$ = all charge inside Gaussian surface i.e. in volume V.

$$\Phi_E = \oint_S \vec{E}\cdot d\vec{A} = \text{flux of } \vec{E} \text{ through } S$$

1) Isolated conductor with excess charge q

a)

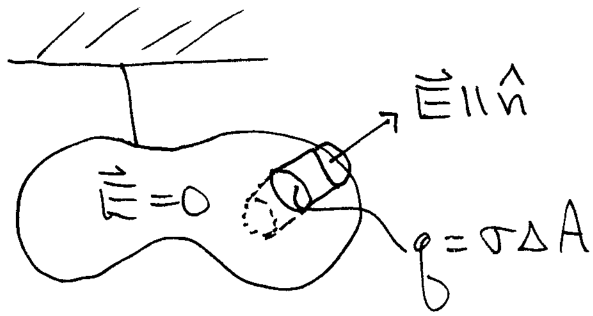


Experimental fact:
no currents in conductor $\Rightarrow \vec{E} = 0$ inside conductor

Hence $\Phi_E = 0 = \frac{1}{\epsilon_0} Q_{enclosed}$

$\Rightarrow$ All q on outer surface of conductor

I D 1 b)



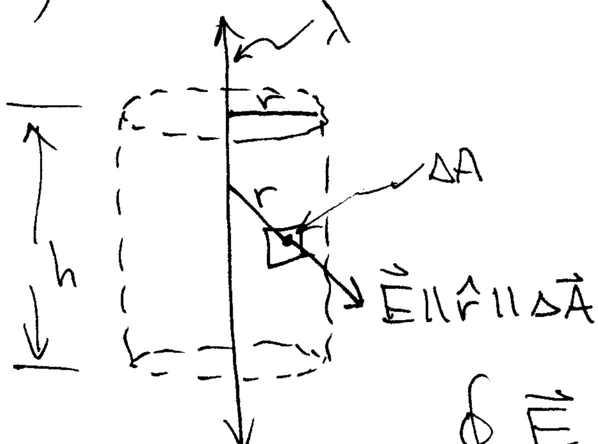
$\vec{E} \perp$ to surface (no currents)

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{enclosed}}{\epsilon_0}$$

$$E\Delta A = \frac{\sigma\Delta A}{\epsilon_0}$$

$$\Rightarrow \boxed{\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}}$$

2) Infinite line of charge:



Cylindrical symmetry

$$\Rightarrow \vec{E} = E(r)\hat{r}$$

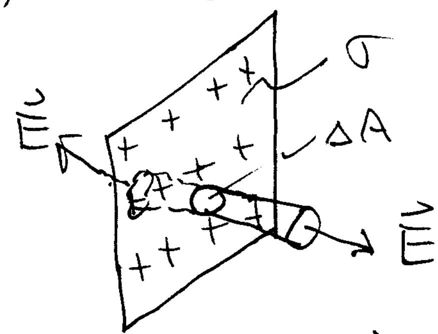
$$\oint_S \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0} Q_{enclosed}$$

$$0 + 0 + E(r)2\pi r h = \frac{1}{\epsilon_0}\lambda h$$

$\vec{E} \perp d\vec{A}$ on endcaps

$$\Rightarrow \boxed{E(r) = \frac{\lambda}{2\pi\epsilon_0 r}}$$

I D 3) Infinite Sheet of Charge



Planar symmetry $\vec{E} \perp$ plane

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} Q_{enclosed}$$

$$\underbrace{0}_{\text{cylindrical side}} + \underbrace{E\Delta A + E\Delta A}_{\text{2 endcaps}} = \frac{1}{\epsilon_0} \sigma \Delta A$$

$$\Rightarrow \quad E = \frac{\sigma}{2\epsilon_0} \quad \text{\& } \perp \text{ to sheet}$$

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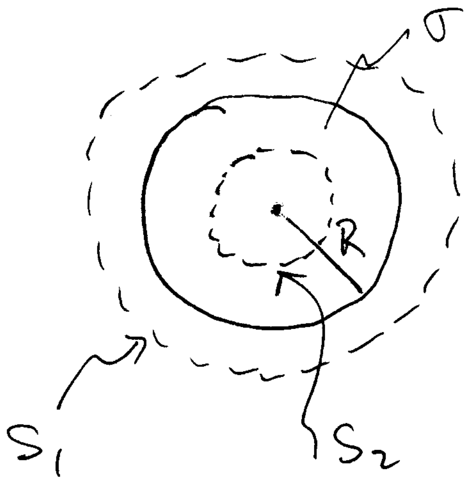
4) Spherical Shell of Charge



Spherical symmetry

$$\Rightarrow \quad \vec{E} = E(r)\,\hat{r}$$

1) $S_1 : r \geq R$

$$\oint_{S_1} \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} Q_{enclosed}$$

$$E(r)\, 4\pi r^2 = \frac{1}{\epsilon_0} \underbrace{4\pi R^2 \sigma}_{\equiv q}$$

$$\Rightarrow \quad \boxed{\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\,\hat{r}}$$

I D 4 2) S_2: $r<R$ No charge is enclosed in S_2

$$\oint_{S_2} \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0} Q_{enclosed} = 0$$

$$\parallel$$

$$E(r)\oint_{S_2} dA = E(r)\,4\pi r^2 = 0$$

$$\Rightarrow \boxed{\vec{E} = 0 \text{ for } r<R \text{ inside shell.}}$$

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5) Spherically Symmetric Charge Distribution:

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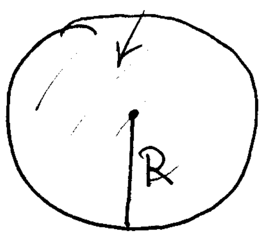


Spherical symmetry $\Rightarrow$

$$\vec{E} = E(r)\,\hat{r}$$

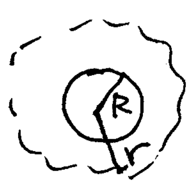
1) $r \geq R$ a) Add up shells $dq = 4\pi r'^2 dr' \rho$

$$E = \int_0^R \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2} = \frac{\rho}{4\pi\epsilon_0 r^2}\underbrace{\int_0^R 4\pi r'^2 dr'}_{=\frac{4}{3}\pi R^3}$$

$$\boxed{E = \frac{q}{4\pi\epsilon_0 r^2}\ ;\ r>R \quad q = \frac{4}{3}\pi R^3 \rho}$$

Acts like pt. q at center for $r \geq R$

b) Gauss' Law directly

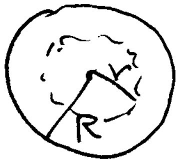


$$\oint_S \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0} Q_{enclosed}$$

$$= E(r)\,4\pi r^2 = \frac{1}{\epsilon_0} q$$

$$\Rightarrow \boxed{E = \frac{q}{4\pi\epsilon_0 r^2},\ r>R}$$

ID5 2) $r \leq R$ Gauss's law:



$$\oint_S \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} Q_{enclosed}$$

$$\| \qquad\qquad \|$$

$$E(r) \oint_S dA = \frac{1}{\epsilon_0}\left(\frac{4}{3}\pi r^3 \rho\right)$$

$$\|$$

$$E(r)\, 4\pi r^2 = \frac{1}{\epsilon_0}\left(\frac{4}{3}\pi R^3 \rho\right)\left(\frac{r^3}{R^3}\right)$$

$$\Rightarrow \boxed{E(r) = \frac{q}{4\pi\epsilon_0 R^2}\left(\frac{r}{R}\right), \quad r \leq R}$$

Note: $r = R$ both expressions must be $=$, and they are $E(R) = \frac{q}{4\pi\epsilon_0 R^2}$

E.) Potential Energy: Coulomb force is conservative

$$\Rightarrow$$

Work to go $a \to b$ $= W_{ab} = \int_a^b \vec{F} \cdot d\vec{s}$ is path independent

Potential Energy difference $\Delta U = U_b - U_a \equiv -W_{ab}$

$$\Delta U = U_b - U_a \equiv -\int_a^b \vec{F} \cdot d\vec{s}$$

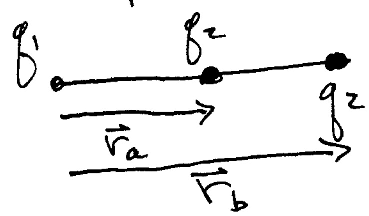
I.E) Potential Energy at a point $= U_b \equiv -W + U_a$

where U_a is an arbitrarily defined value at the reference point a. Usually choose $U_a \equiv 0$ at point a. For finite charges we choose $a = \infty$ i.e. all charges infinitely separated and $U_\infty \equiv 0$.

For $\vec{F} = q\vec{E}$ we have

$$\boxed{\Delta U = U_b - U_a = -q\int_a^b \vec{E}\cdot d\vec{s}}$$

1) Two point charges: q_1 fixed, q_2 moves from $\vec{r}_a$ to $\vec{r}_b$

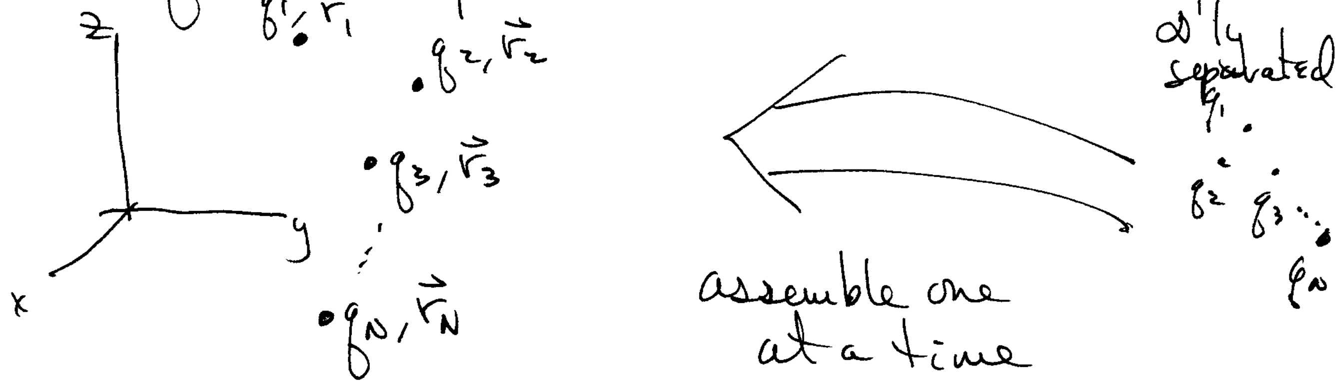


$$U_b - U_a = \frac{q_1 q_2}{4\pi\epsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$

let $r_a \to \infty$ and define $U_a = 0$ there

$$\Rightarrow \quad \boxed{U(r) = \frac{q_1 q_2}{4\pi\epsilon_0 r}} \qquad \left(\begin{array}{c} r = \text{separation} \\ \text{of } q_1 \,\&\, q_2 \end{array}\right)$$

I.E.2) System of point charges: Potential energy equals the work done by an external force to assemble the charge distribution from ∞ separation to final locations



$$U = U_1 + U_2 + \cdots + U_N$$

$$= \sum_{i=1}^{N} U_i$$

U_i ← work done to bring q_i in when $q_1, \ldots, q_{i-1}$ are already in place.

$$U = \sum_{i=1}^{N} \sum_{j=1}^{i-1} \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}$$

skip $i=j$

$$= \frac{1}{2} \sum_{i=1}^{N} q_i \underbrace{\left(\sum_{j=1}^{N} \frac{q_j}{4\pi\epsilon_0 |\vec{r}_i - \vec{r}_j|} \right)}_{\equiv V(\vec{r}_i)}$$

$\equiv V(\vec{r}_i)$

= electric potential at $\vec{r}_i$ due to other charges $q_1 \cdots q_N$ excluding q_i

I.F.) Electric Potential: $V(\vec{r}) \equiv \lim_{q_0 \to 0} \frac{U(\vec{r})}{q_0}$

$V = PE/\text{unit charge}$

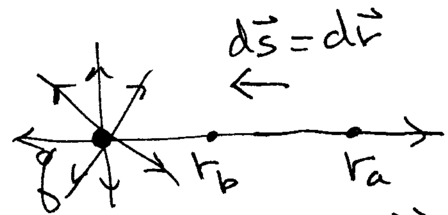
$$\boxed{\Delta V = V_b - V_a = -\int_a^b \vec{E} \cdot d\vec{s}}$$

1) SI units: $1\,\text{volt} \equiv \dfrac{1\,\text{Joule}}{1\,\text{Coulomb}} \Rightarrow E\,\text{units} = \dfrac{1\,\text{Volt}}{1\,\text{m}} = \dfrac{1\,\text{N}}{1\,\text{C}}$

2) Move charge q between 2 points at fixed potential difference ΔV then the PE difference is

$$\Delta U = q \Delta V.$$

3) Point Charge q:



$$V_b - V_a = -\int_a^b \vec{E} \cdot d\vec{s}$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3}$$

$$= -\int_{r_a}^{r_b} \frac{q}{4\pi\epsilon_0} \frac{dr}{r^2}$$

$$\boxed{V_b - V_a = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)}$$

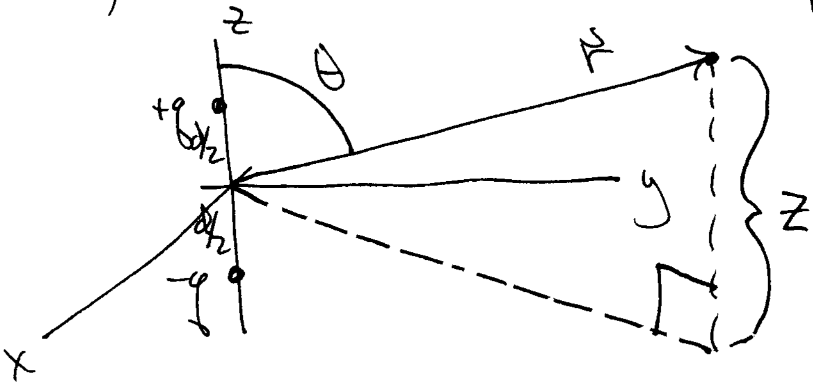
Convention: $r_a = \infty \quad V_a \equiv 0$

$$\boxed{V(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{r}}$$

I F4) Principle of superposition: $V = \sum_{i=1}^{N} V_i = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N} \frac{q_i}{r_i}$

Scalar addition of individual charge potentials.

5) Potential due to a dipole: use superposition of 2 pt. charges



$\hat{r}\cdot\hat{k} = \cos\theta = \frac{z}{r}$

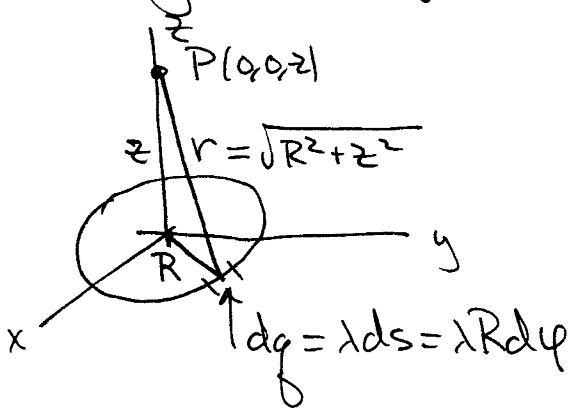
$$V = V_+ + V_- = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{r} - \frac{d}{2}\hat{k}|} - \frac{1}{|\vec{r} + \frac{d}{2}\hat{k}|}\right]$$

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0 r}\left[\frac{1}{\sqrt{1 - \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}}} - \frac{1}{\sqrt{1 + \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}}}\right]$$

a) $r >> d$: $\frac{1}{\sqrt{1+\epsilon}} = 1 - \frac{1}{2}\epsilon + \cdots$

$$\boxed{V = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3}}$$

I.F 6) Ring of charge



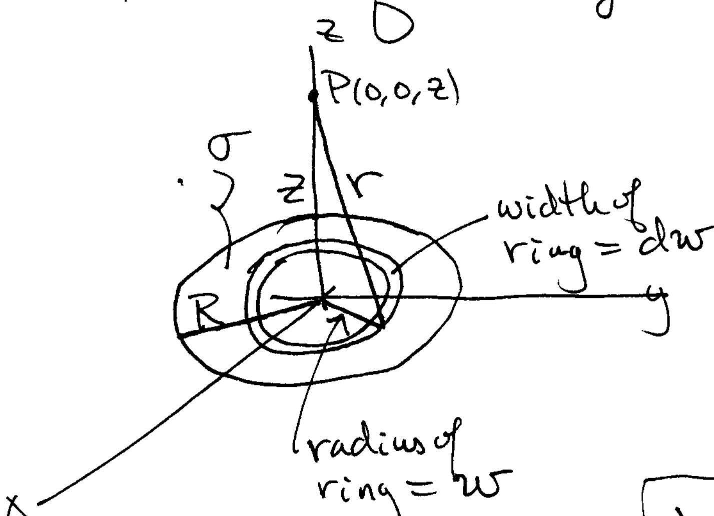
$$V(0,0,z)=\frac{1}{4\pi\epsilon_0}\int_{ring}\frac{dq}{r}$$

r is same around ring

$$=\frac{1}{4\pi\epsilon_0 r}\int_{ring} dq$$

$$\frac{\overbrace{q}^{=2\pi R\lambda}}{4\pi\epsilon_0\sqrt{R^2+z^2}} = \boxed{V(0,0,z)=\frac{2\pi R\lambda}{4\pi\epsilon_0\sqrt{R^2+z^2}}}$$

7) Disk of Charge



Ring of charge

$$dV(P)=\frac{dq}{4\pi\epsilon_0 r}$$

$r=\sqrt{w^2+z^2}$

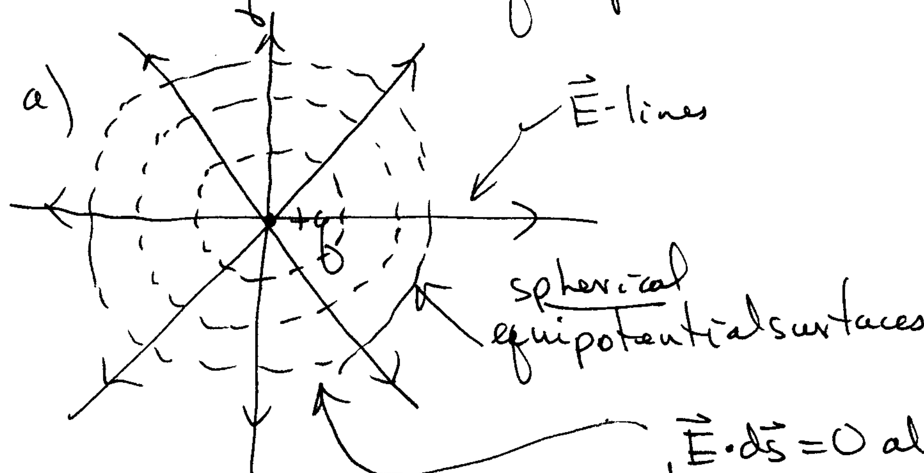
$$V(P)=\int_{w=0}^{R} dV(P)$$

$$\boxed{V(P)=\frac{\sigma}{2\epsilon_0}\left[\sqrt{R^2+z^2}-z\right]}$$

I.F. 8) Equipotential Surfaces: family of surfaces with each surface having the same electric potential.

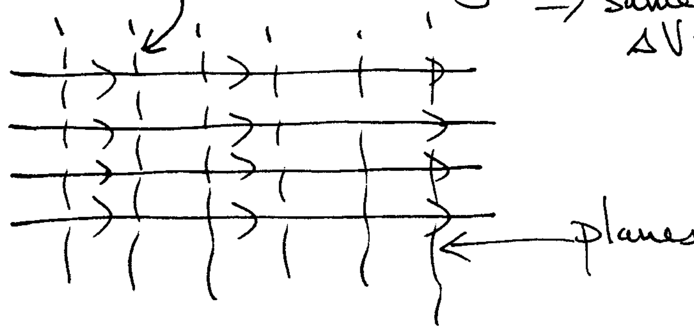
lines of $\vec{E} \perp$ equipotential surfaces

a)



$\vec{E} \cdot d\vec{s} = 0$ along surface $\Rightarrow$ same potential $\Delta V = 0$.

b) Uniform $\vec{E}$



9) Gradient of electric potential.

$$dV = -\vec{E} \cdot d\vec{s} = \vec{\nabla} V \cdot d\vec{s} \Rightarrow \boxed{\vec{E} = -\vec{\nabla} V}$$

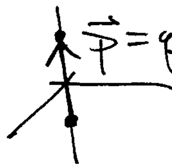
$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$E_x = -\frac{\partial V}{\partial x}$$

$$E_y = -\frac{\partial V}{\partial y}$$

$$E_z = -\frac{\partial V}{\partial z}$$

I.F.9a) Electric Dipole: $\boxed{V = \dfrac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3}}$



$\vec{p} = qd\hat{k} = p\hat{k}$

$$= \frac{pz}{4\pi\epsilon_0 (x^2+y^2+z^2)^{3/2}}$$

$$\vec{E} = -\vec{\nabla} V$$

$$E_x = -\frac{\partial V}{\partial x} = \frac{3pzx}{4\pi\epsilon_0 r^5}$$

$$E_y = -\frac{\partial V}{\partial y} = \frac{3pzy}{4\pi\epsilon_0 r^5}$$

$$E_z = -\frac{\partial V}{\partial z} = -\frac{p}{4\pi\epsilon_0 r^3} + \frac{3pz^2}{4\pi\epsilon_0 r^5}$$

$$\Rightarrow \quad \boxed{\vec{E} = \frac{3(\vec{p}\cdot\vec{r})\,\vec{r}}{4\pi\epsilon_0 r^5} - \frac{\vec{p}}{4\pi\epsilon_0 r^3}}$$

---

10) All isolated conductor is an equipotential.

I G) Capacitors: two conducting plates carrying equal but opposite charge; plates are equipotentials.

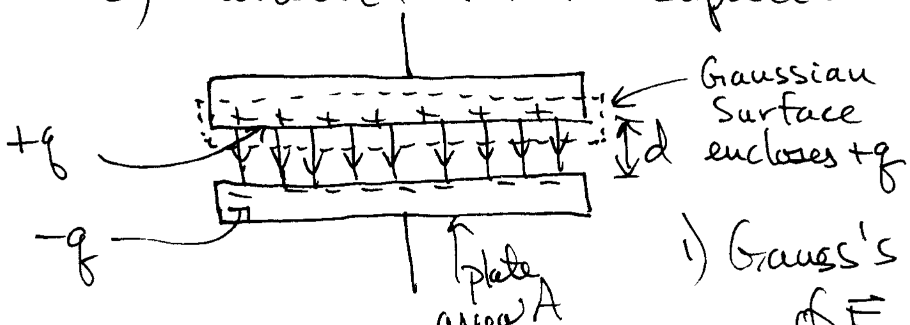
$$\boxed{q = CV}$$

q = magnitude of charge
V = magnitude of Potential difference

C = capacitance of capacitor

1) SI units of capacitance $1\,\text{Farad} = \dfrac{1\,\text{Coulomb}}{1\,\text{Volt}}$.

2) Parallel Plate Capacitor: Neglect fringing



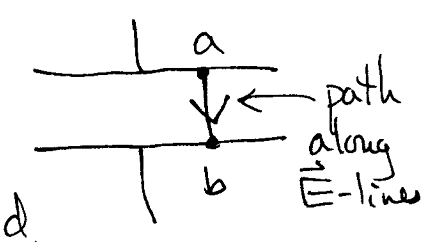
1) Gauss's Law

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0}q$$

$$EA = \frac{q}{\epsilon_0}$$

$$E = \frac{q/A}{\epsilon_0} \left(= \frac{\sigma}{\epsilon_0}\right)$$

2) Electric Potential Difference



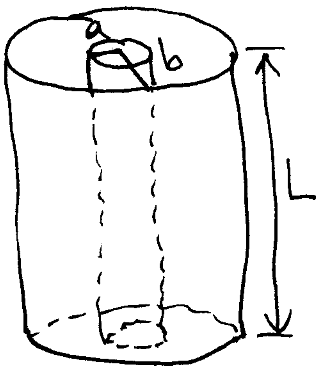
$$-V = V_b - V_a = -\int_a^b \vec{E}\cdot d\vec{s}$$

$$\Rightarrow +V = +\int_a^b \vec{E}\cdot d\vec{s} = E\int_0^d ds = Ed$$

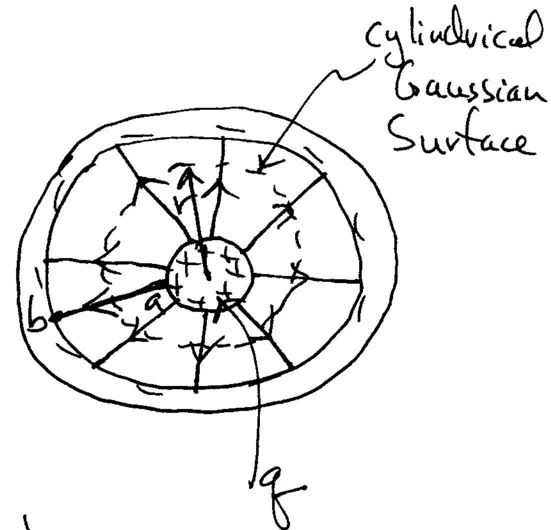
IG 2.2) $V = Ed = \left(\frac{q}{\epsilon_0 A}\right) d = \left(\frac{d}{\epsilon_0 A}\right) q = \frac{1}{C} q$

$$\Rightarrow \boxed{C = \frac{\epsilon_0 A}{d}} \quad \boxed{\text{$\parallel$-plate capacitor}}$$

3) Cylindrical Capacitor:



$L \gg b$ to ignore fringing



1) Gauss' Law

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} q$$

$$\parallel$$

$$0 + 0 + E(r) \underbrace{2\pi r L}_{\text{cylindrical area at radius } r}$$

endcaps $\vec{E} \perp d\vec{A}$

$$\Rightarrow \quad E(r) = \frac{q}{2\pi\epsilon_0 r L}$$

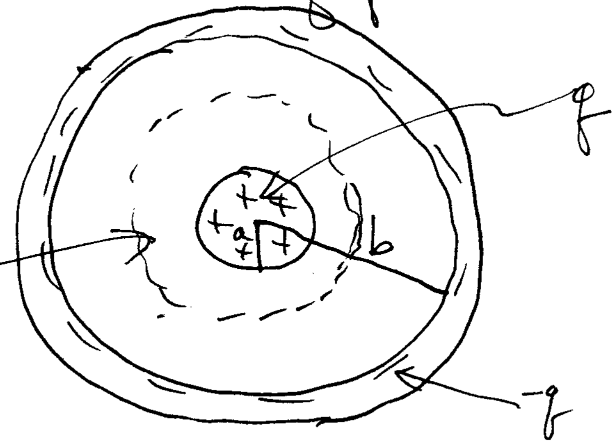
2) Potential Difference

$$V = \int_a^b \vec{E} \cdot d\vec{s} = \frac{q}{2\pi\epsilon_0 L} \int_a^b \frac{dr}{r}$$

$$= \frac{q}{2\pi\epsilon_0 L} \ln\left(\frac{b}{a}\right)$$

IG3) $\Rightarrow$ $\boxed{C = 2\pi\epsilon_0 \dfrac{L}{\ln(b/a)}}$ cylindrical capacitor

4) Spherical Capacitor: Cross-Section of sphere



Spherical Gaussian Surface

1) Gauss' Law

$$\oint_S \vec{E}\cdot d\vec{A} = q/\epsilon_0$$

$$\| $$

$$E(r)4\pi r^2$$

$$\Rightarrow \quad \vec{E}(r) = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$$

2) Potential difference:

$$V = \int_a^b \vec{E}\cdot d\vec{s} = \int_a^b E(r)dr = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

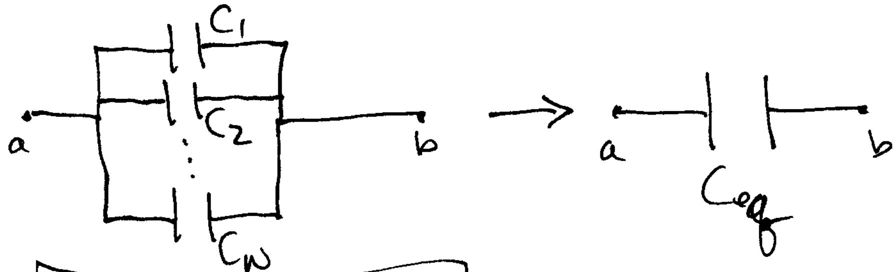
3) $\boxed{C = q/V = 4\pi\epsilon_0\left(\dfrac{ab}{b-a}\right)}$ Spherical Capacitor

5) Isolated Sphere: let $b \to \infty$ above and call $a = R$

$\boxed{C = 4\pi\epsilon_0 R}$ isolated sphere

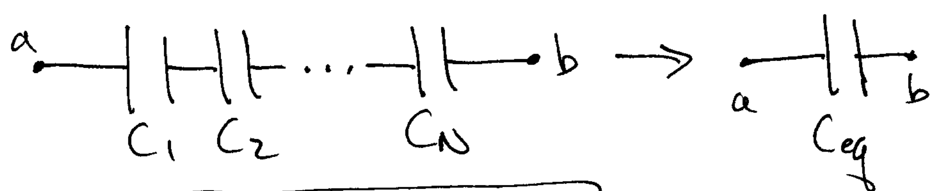
I.G.6) Equivalent Capacitance:

a) Capacitors in Parallel



$$\boxed{C_{eq} = \sum_{i=1}^{N} C_i}$$

b) Capacitors in Series



$$\boxed{\frac{1}{C_{eq}} = \sum_{i=1}^{N} \frac{1}{C_i}}$$

H.) Energy Storage in an Electric Field

Charge capacitor, at time t', q' charge is on plates with potential difference V' across them, so add dq' more charge the increase in PE is

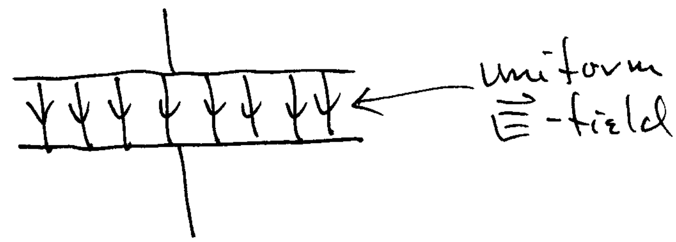
$$dU = dq' V' \qquad \text{but } V' = q'/C$$

I.H) $dU = \frac{q'\,dq'}{C}$; Continue until totally charged $\Rightarrow$

$$U = \int_0^U dU = \int_0^q \frac{1}{C} q'\,dq' = \frac{1}{2} q^2/C$$

$$\boxed{U = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2} C V^2}$$ since $q = CV$

1) $\|$-plate:



Energy density $\equiv u$ = const. between plates since $\vec{E}$ is uniform

$$u = \frac{U}{Ad} = \frac{\frac{1}{2} C V^2}{Ad} = \boxed{\frac{1}{2}\epsilon_0 E^2 = u}$$

since $C = \frac{\epsilon_0 A}{d}$ and $E = \frac{V}{d}$

If $\vec{E}$ exists at a point in space, then the <u>energy density</u> stored at that point is

$$\boxed{u = \frac{1}{2}\epsilon_0 |\vec{E}|^2}$$

general result

IH2) Isolated Conducting sphere with charge q

1) $U = \dfrac{q^2}{2C} = \dfrac{1}{2}\dfrac{q^2}{4\pi\epsilon_0 R}$

Total energy stored in $\vec{E}$-field

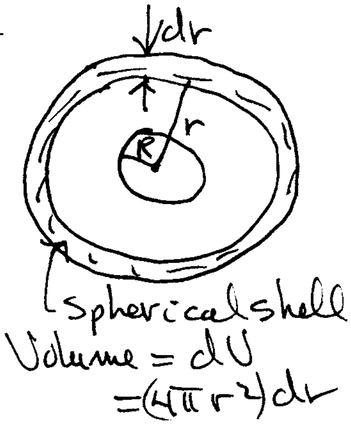
2) Energy density at point r from center

$$u = \frac{1}{2}\epsilon_0 E^2_{(r)} = \frac{q^2}{32\pi^2\epsilon_0 r^4}$$

3) One Half energy stored within radius R_0

$$dU = u\,dV = u(r)\,4\pi r^2 dr$$

$$= \frac{q}{8\pi\epsilon_0}\,\frac{dr}{r^2}$$



Require

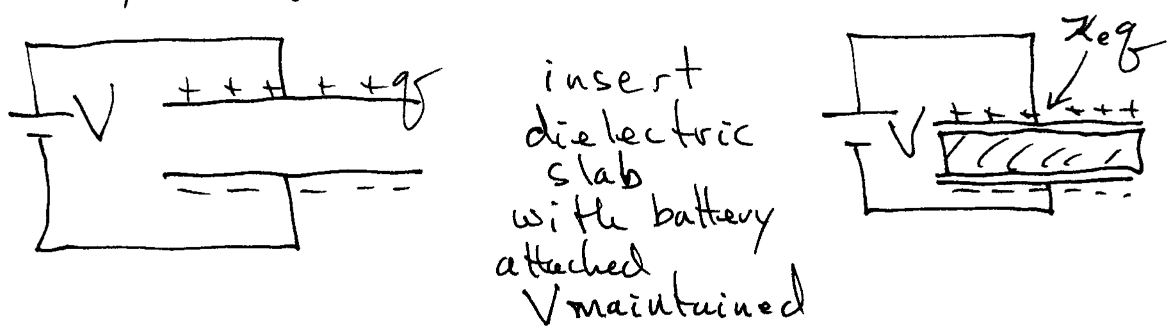
$$\int_R^{R_0} dU = \frac{1}{2}\int_R^{\infty} dU$$

$$\Rightarrow \boxed{R_0 = 2R}$$

I.I.) Dielectrics: Intermediate materials between conductors and insulators: the charged particles in dielectrics displace their positions slightly in response to an $\vec{E}$-field.

1) Faraday: Experimental results

a) V fixed



For same V, stored charge q increases

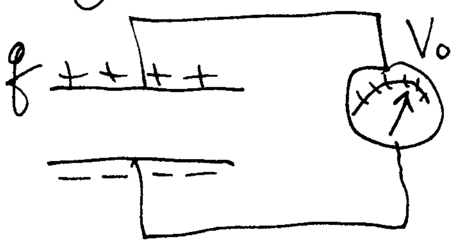
$\Rightarrow$ Capacitance $C = q/V$ increases

Define dielectric constant $\kappa_e \equiv \dfrac{C}{C_0}$ ← C: with dielectric; C_0: Vacuum between plates

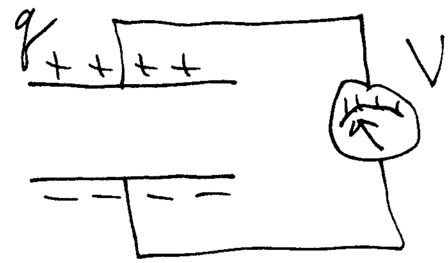
$(\epsilon \equiv \kappa_e \epsilon_0$ = permittivity of material)

So $q = CV = \kappa_e C_0 V = \kappa_e q_0$.

II 1b) <u>q - fixed</u>



insert Dielectric slab



$$q = C_o V_o$$

$$q = CV$$
but $C = \kappa_e C_o$
$$q = \kappa_e C_o V$$

$\Rightarrow$

$$q = C_o V_o = \kappa_e C_o V \quad \Rightarrow \quad \boxed{V = \frac{1}{\kappa_e} V_o}$$

for fixed charge q, the voltage decreases as the slab is inserted.

---

2) Fill Capacitors with dielectric $\kappa_e \Rightarrow$
$C = \kappa_e C_o$

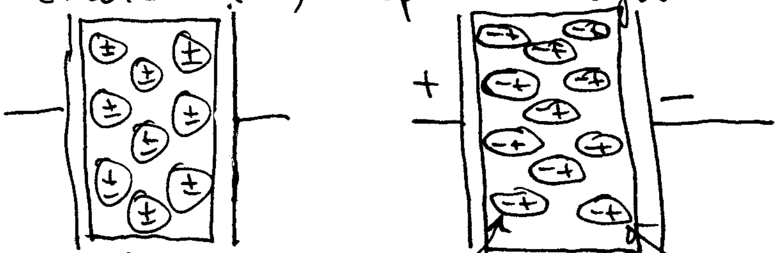
a) $\parallel$-plate : $C = \frac{\kappa_e \epsilon_o A}{d}$

b) Cylindrical : $C = 2\pi \kappa_e \epsilon_o \frac{L}{\ln(b/a)}$

c) Spherical : $C = 4\pi \kappa_e \epsilon_o \frac{ab}{b-a}$

d) <u>Isolated Sphere</u> : $C = 4\pi \kappa_e \epsilon_o R$.

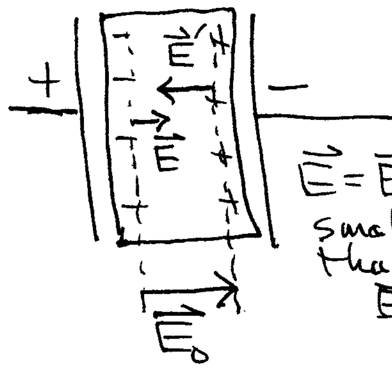
I.I.3) Microscopic View of Dielectrics: Atomic (molecular) dipoles align in $\vec{E}$-field.



$\vec{E}_0 = 0$ $\qquad$ $\vec{E}_0 \neq 0$ $\longrightarrow$

induced surface charges on dielectric

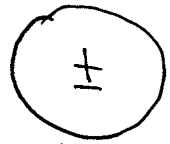
Alignment tends to lower $\vec{E}$-field in dielectric



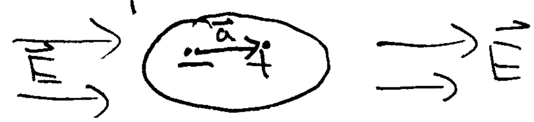
$\vec{E} = \vec{E}_0 + \vec{E}'$ smaller than $\vec{E}_0$

a) Molecular dipole moment $\vec{p}$ & Polarization Charge

ex. Atomic # = Z



$\vec{E} = 0$

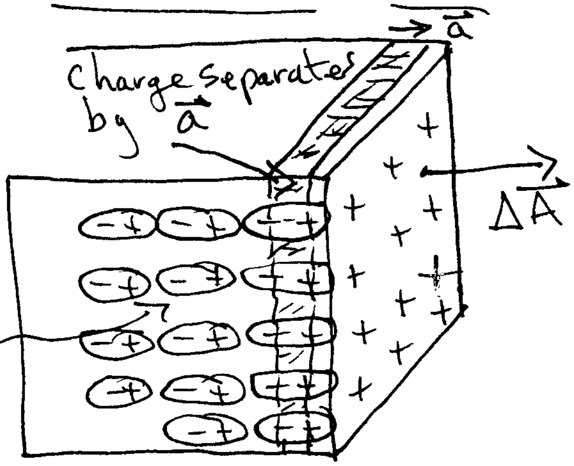


$\vec{p} = Ze\vec{a}$

Polarization Vector $= \vec{P} \equiv N\vec{p} = \dfrac{\text{dipole moment}}{\text{Volume}}$

N: $\dfrac{\#\text{ of atoms}}{\text{Volume}}$

$\left(= \dfrac{1}{\Delta V} \sum_{\substack{\text{molecules} \\ \text{in } \Delta V}} \vec{p} \right.$ sum molecular dipole moments in macro small but micro large volume ΔV. This depends on size ΔV, so divide by ΔV $\left. \right)$



averages to zero charge (neutral) inside

$\vec{E}_0$ uniform

- II.3a) Negative charges shift away from surface, no new negative charge enters surface since it is the external surface of the dielectric. Hence the box at the surface gains positive charge equal to the amount of negative charge that left the surface box

$$= \underbrace{N}_{\frac{\text{atoms}}{\text{Vol.}}} \underbrace{Ze}_{\frac{\text{Charge}}{\text{atom}}} \underbrace{(a \Delta A)}_{\text{Vol. of box}}$$

Hence the surface acquires a Polarization surface charge density

$$\sigma_P = \frac{NZe\, a \Delta A}{\Delta A} = NZea = N|\vec{p}| = |\underline{P}|$$

or in terms of vectors and in general

$$\boxed{\sigma_P \Delta A = \vec{\underline{P}} \cdot \Delta \vec{A}}$$

I.I.4) Gauss' Law Re-Visited

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0}[q - q']$$

$$\parallel$$

$$EA$$

$$\Rightarrow \boxed{E = \frac{q}{\epsilon_0 A} - \frac{q'}{\epsilon_0 A}}$$

$$\left(\text{Vacuum: } E_0 = \frac{q}{\epsilon_0 A}\right)$$

a) Charges

$$E = \frac{1}{\kappa_e} E_0$$ (Faraday's Exp. linear dielectric matrials)

$$\frac{q}{\epsilon_0 A} - \frac{q'}{\epsilon_0 A} = \frac{1}{\kappa_e}\frac{q}{\epsilon_0 A}$$

$$\Rightarrow \boxed{q' = q\left(1 - \frac{1}{\kappa_e}\right) < q}$$

induced Polarization surface charge < free charge

likewise $\sigma_p = \sigma\left(1 - \frac{1}{\kappa_e}\right)$

$\left(\begin{array}{l}\sigma_p = q'/A \\ \sigma = q/A\end{array}\right)$

b) Gauss' Law

$$\epsilon_0 \oint_S \vec{E}\cdot d\vec{A} = q - q' = q/\kappa_e$$

$$\Rightarrow \boxed{\epsilon_0 \oint_S \kappa_e \vec{E}\cdot d\vec{A} = q}$$

free charge only.

II.4. c) Electric Displacement Vector $\vec{D}$

1) $E = \frac{q}{\epsilon_0 A} - \frac{q'}{\epsilon_0 A} = \frac{\sigma}{\epsilon_0} - \frac{\sigma_p}{\epsilon_0}$

$= \frac{\sigma}{\epsilon_0} - \frac{P}{\epsilon_0} \Rightarrow \sigma = \epsilon_0 E + P$

$\sigma = \frac{q}{A}$

free charge density

Define: $\boxed{\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}}$

So $q/A = \sigma = D$; D is given by free charge only.

Now $\sigma = \kappa_e \epsilon_0 E$ hence

$$\boxed{\vec{D} = \kappa_e \epsilon_0 \vec{E}}$$

2) Gauss' Law for $\vec{D}$

$$\boxed{\oint_S \vec{D} \cdot d\vec{A} = q}$$

d.) Constitutive Equation: Linear, isotropic dielectrics (Faraday's exp.)

$$\vec{P} = \epsilon_0 \chi_E \vec{E}$$

↑ electric susceptibility

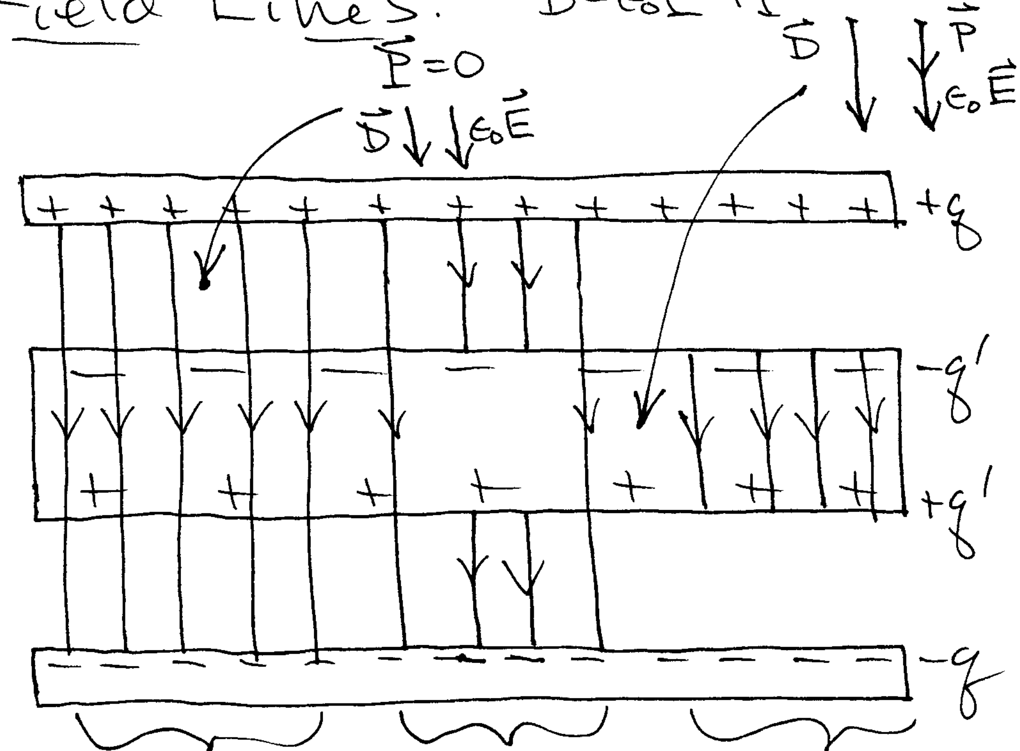
I.I.4.d) Now $\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}$

$$= \underbrace{\epsilon_0 (1 + \chi_E)}_{\equiv \epsilon \,=\, \epsilon_0 \kappa_e \,=\, \text{permittivity}} \vec{E}$$

Thus $\boxed{\kappa_e = \epsilon / \epsilon_0 = 1 + \chi_E}$ = dielectric constant

hence $\vec{P} = \epsilon_0 (\kappa_e - 1) \vec{E}$.

e) Field Lines: $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$



$\vec{D}$ involves "free" charges q

$\epsilon_0 \vec{E}$ involves ALL Charges q & q'

$\vec{P}$ involves polarization charges q'

I.I.5) Energy stored in field - Re-Visited

Dielectric filled capacitor $q = CV$, $C = \kappa_e C_0$

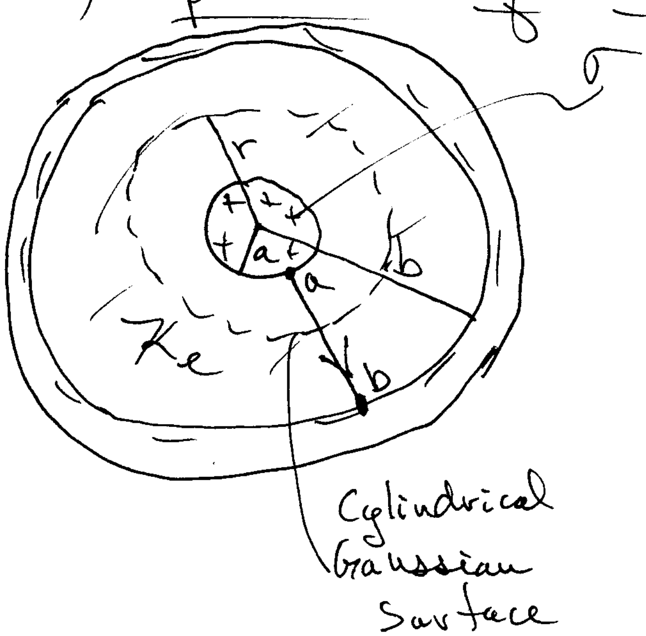
$$\Rightarrow \quad U = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2} C V^2$$ as previously

energy density stored in fields $\Rightarrow$ $$\boxed{u = \frac{1}{2}\vec{D}\cdot\vec{E}}$$

Hence energy stored in fields

$$\boxed{U = \frac{1}{2}\int_V \vec{D}\cdot\vec{E}\, dV}$$

6) Capacitance of Coaxial Cable



1) Gauss' Law

$$\oint_S \vec{D}\cdot d\vec{A} = Q_{free}$$

$$D(r) 2\pi r l = 2\pi a l \sigma$$

$$\Rightarrow D(r) = \frac{a\sigma}{r}$$

2) $$E(r) = \frac{1}{\epsilon_0 \chi_e} D(r)$$

$$= \frac{a\sigma}{\kappa_e \epsilon_0 r}$$

II.6) 3) Potential Difference:

$$V = \int_a^b \vec{E}\cdot d\vec{s} = \int_a^b E(r)\,dr$$

$$= \frac{a\sigma}{\chi_e \epsilon_0} \ln(b/a)$$

4) Capacitance $C = q/V$; $q = 2\pi a l \sigma$

$$\boxed{C = \frac{2\pi \chi_e \epsilon_0}{\ln(b/a)}\, l} = \kappa_e C_0$$

$$\boxed{C/l = \frac{2\pi \kappa_e \epsilon_0}{\ln(b/a)}} = \text{Capacitance / unit length.}$$

---

7) Forces on Charges

a) Isolated System: Q = fixed

$$dW = \vec{F}\cdot d\vec{s} = -dU = -\vec{\nabla} U \cdot d\vec{s}$$

Work performed by $\vec{E}$-field

loss in P.E.

$$\Rightarrow \boxed{\vec{F} = -\vec{\nabla} U \Big|_{Q=\text{fixed}}}$$

I.I.7 b) Energy source attached to system: fix potential difference: V-fixed

$$\vec{F} \cdot d\vec{s} = dW = dW_{battery} - dU$$

work performed by battery & $\vec{E}$-field

work performed by battery

PE gain

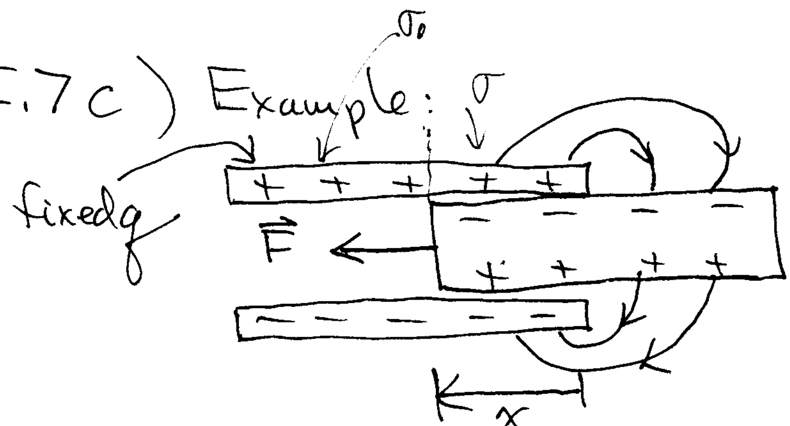
$$\left.\begin{aligned} dW_{battery} &= dq\,V \\ dU &= \tfrac{1}{2}\,dq\,V \end{aligned}\right\} \Rightarrow dW = \tfrac{1}{2}\,dq\,V = dU$$

So

$$\vec{F} \cdot d\vec{s} = +dU = +\vec{\nabla} U \cdot d\vec{s}$$

$$\Rightarrow \boxed{\vec{F} = +\vec{\nabla} U \Big|_{V=\text{fixed}}}$$

I.F.7c) Example:



q - fixed

dielectric slab pulled in by $\vec{E}$-field

$$U = \frac{1}{2}\int_{\substack{\text{capacitor}\\ \text{volume}}} \vec{D}\cdot\vec{E}\, dV = \frac{1}{2}\epsilon_0\chi_e E^2(xbd) + \frac{1}{2}\epsilon_0 E^2\left[(l-x)bd\right]$$

$$= \frac{1}{2}\epsilon_0 E^2 bd\left[\chi_e x + (l-x)\right]$$

q is fixed, but V changes with x

$$\begin{cases} q = \sigma x b + \sigma_0(l-x)b \\ E = \dfrac{\sigma}{\chi_e\epsilon_0} = \dfrac{\sigma_0}{\epsilon_0} = V/d \Rightarrow \sigma = \chi_e\sigma_0 \end{cases}$$

$$\Rightarrow \quad \sigma_0 = \frac{q}{\chi_e x b + (l-x)b}$$

Hence

$$U = \frac{1}{2}\frac{q^2}{C} \quad \text{with}$$

$$C = \frac{\epsilon_0 b}{d}\left[(l-x) + \chi_e x\right]$$

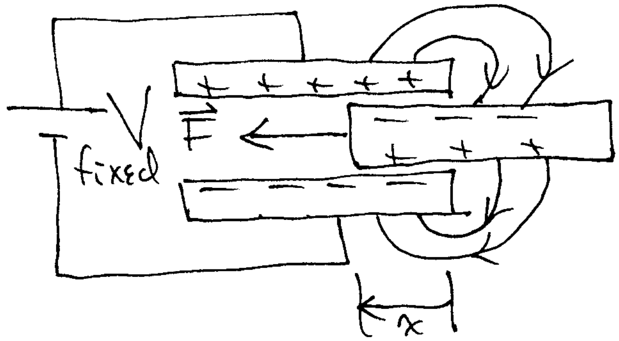
So

$$F = -\frac{dU}{dx} \quad \text{since } q \text{ is fixed}$$

$$\boxed{F = +\frac{q^2}{2C^2}\frac{dC}{dx} > 0}$$

(x-dependent force in q fixed case)

II 7d) Example: V-fixed



dielectric slab is pulled in.

$$E = V/d = \text{constant}$$

$$U = \frac{1}{2}\int_{\substack{\text{Capacitor}\\ \text{Volume}}} \vec{D}\cdot\vec{E}\, dV = \frac{1}{2}\epsilon_0 \chi_e E^2 (xbd) + \frac{1}{2}\epsilon_0 E^2 [(l-x)bd]$$

$$\boxed{U = \frac{1}{2}\epsilon_0 \left(\frac{V}{d}\right)^2 bd\,[\chi_e x + (l-x)] = \frac{1}{2}CV^2}$$

$$\boxed{F = +\frac{dU}{dx} = \frac{1}{2}V^2 \frac{dC}{dx} = \frac{1}{2}V^2 \frac{\epsilon_0(\chi_e - 1)b}{d} > 0}$$

F is a constant indep. of x in fixed V case

Note:

$$F\Big|_{q\,\text{fixed}} = \frac{q^2}{2C^2}\frac{dC}{dx} \quad \text{but } q = CV$$

$$= \frac{1}{2}V^2 \frac{dC}{dx} \quad \text{but } \underline{V = V(x)} \text{ in fixed } q \text{ case.}$$

$$F\Big|_{V\,\text{fixed}} = \frac{1}{2}V^2 \frac{dC}{dx} \quad \text{but } V = \text{constant in fixed } V \text{ case}$$

Similar form but different x-dependence