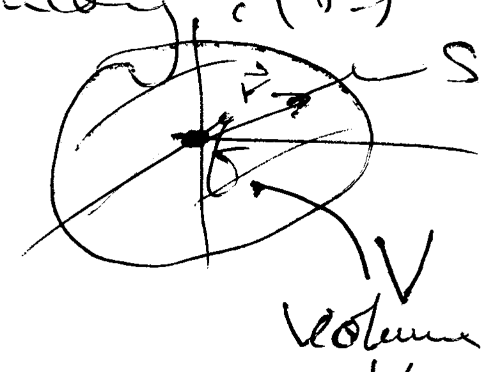
III.) Generalized Functions (Distributions)

A.) Our final mathematical subject will result from a further consideration of Gauss's divergence theorem

From our study of the solid angle we have that flux of the electric field due to a point charge q through any surface surrounding it is q/ϵ_0, i.e.



$$\int_S \vec{E}\cdot d\vec{S} = \frac{q}{4\pi\epsilon_0}\underbrace{\int \frac{\hat{r}}{r^2}\cdot d\vec{S}}_{=4\pi}$$
$$= q/\epsilon_0$$

In particular, we could have chosen a sphere of radius a surrounding the point charge by. So $d\vec{S} = a^2 \sin\theta\, d\theta\, d\varphi\, \hat{r}$

$$\Rightarrow \int_S \vec{E}\cdot d\vec{S} = \frac{q}{4\pi\epsilon_0}\underbrace{\underbrace{\int_{\varphi=0}^{2\pi} d\varphi}_{=2\pi}\underbrace{\int_{\theta=0}^{\pi} d\theta\, \frac{\cancel{a^2}}{\cancel{a^2}}\sin\theta}_{=2}}\;\underbrace{\hat{r}\cdot\hat{r}}_{=1}$$

$$= q/\epsilon_0.$$

The integral is independent of the radius of the sphere a!

III.) A.) On the other hand Gauss's theorem $\Rightarrow$

$$\oint_S \vec{E} \cdot d\vec{S} = \int_V \vec{\nabla} \cdot \vec{E} \, dV$$

$$= \frac{q}{4\pi\epsilon_0} \int_V \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) d^3 r$$

So we must calculate the divergence of $\frac{\hat{r}}{r^2}$. In spherical polar coordinates $\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r)$

$$+ \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta E_\theta) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} E_\varphi$$

But $\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r}$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{E} = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \frac{\partial}{\partial r} \left(\cancel{r^2} \frac{1}{\cancel{r^2}} \right) = 0}$$

$$\boxed{= 0} \quad \text{as long as } \boxed{r \neq 0}$$

So we find that if

$$\int_V \vec{\nabla} \cdot \vec{E} \, d^3 r \neq 0 \quad ; \text{ as Gauss's thm}$$

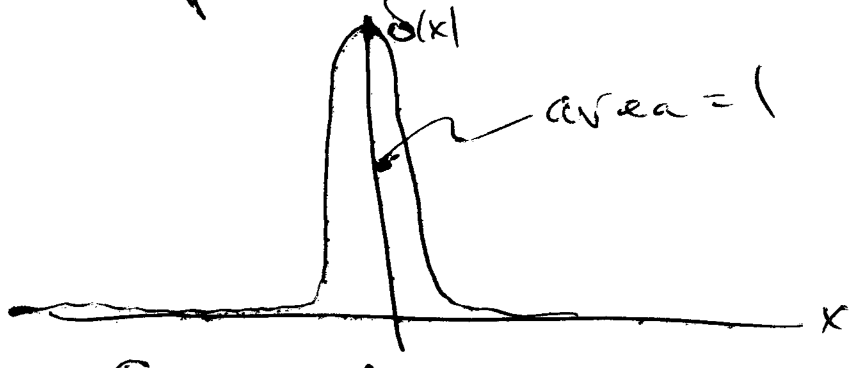
$$\Rightarrow \left(\text{i.e. } = q/\epsilon_0 \right), \text{ then}$$

III.A.) The entire contribution to the volume integral must come from $r=0$, where $\vec{E} \to \infty$ since $\vec{\nabla}\cdot\vec{E}=0$ for $r \neq 0$. Since Gauss's divergence theorem is correct, we see that the vol. integral is the surface integral. And that indeed was independent of the radius, we would get the same answer as we let $a \to 0$. Hence we are led to a new type of function — which is not a function at all. It is zero everywhere except at one point, $r=0$, yet its integral over volume containing the any singular point is a non-zero constant.
$[\vec{\nabla}\cdot\vec{E}=0,\ r\neq 0$ but $\int dV\, \vec{\nabla}\cdot\vec{E} = q/\epsilon_0.]$

Such a function is called a generalized function or a distribution. This example is that distribution called the Dirac delta function.

III. A) Note: we should not be surprised since a point charge $q \neq 0$ at the origin has zero charge density everywhere except at $\vec{r} = 0$, but yet $\int d^3r\, \rho(\vec{r}) = q$. Another example of a Dirac delta function.

1.3.B) So let's first introduce the Dirac delta function in one-dimension, $\delta(x)$. We can picture $\delta(x)$ is an infinitely high, infinitesimally narrow spike with area $= 1$



$$\delta(x) = \begin{cases} 0 & , \text{ if } x \neq 0 \\ \infty & , \text{ if } x = 0 \end{cases}$$ such

that $$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1.$$

III.B.) Since $\delta(x) = 0$ except at $x = 0$ we have that, for any ordinary function $f(x)$ say that's C^0,

$$f(x)\delta(x) = f(0)\delta(x).$$

This is understood to be the integral expression

$$\int_{-\infty}^{+\infty} dx\, f(x)\delta(x) = f(0)\int_{-\infty}^{+\infty} dx\, \delta(x)$$

$$= f(0).$$

Then the distribution $\delta(x)$ maps functions $f(x)$ to their value at $x = 0$. Generalized functions are always understood as integrals with ordinary functions $f(x)$, although for short hand we write $\delta(x)$.

<u>Note</u>: 1) $\int_{-\infty}^{+\infty} dx\, f(x)\delta(x) = \int_{-\epsilon}^{+\epsilon} dx\, f(x)\delta(x) = f(0)$

2) $\int_{-\infty}^{+\infty} dx\, f(x)\delta(x-a) \underset{y=x-a}{=} \int_{-\infty}^{+\infty} dy\, f(y+a)\delta(y)$

$= f(a)$ <u>i.e.</u> $\delta(x-a)f(x) = f(a)\delta(x-a)$

III.B.)

3) $\delta(f(x))$ suppose $f(x)$ has a zero at $x=a$

Taylor expand

$$f(x) = \cancel{f(a)}^{=0} + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \cdots$$

So

$$\delta(f(x)) = \delta\left((x-a)\left[f'(a) + \frac{(x-a)}{2}f''(a) + \cdots\right]\right)$$

but at $x=a$ all higher order terms vanish

$$= \delta((x-a)f'(a))$$

So

$$\int_{-\infty}^{+\infty} dx\, \delta(f(x))\, g(x) = \int_{-\infty}^{+\infty} dx\, \delta((x-a)f'(a))\, g(x)$$

$y = (x-a)f'(a)$

$$= \int_{-\infty}^{+\infty} dy \frac{1}{|f'(a)|} \delta(y)\, g\left(\frac{1}{f'(a)}y + a\right)$$

$$= \frac{1}{|f'(a)|} g(a) \Rightarrow$$

$$\boxed{\delta(f(x)) = \frac{1}{|f'(a)|}\delta(x-a)}$$

i.e. $y = f(x)$ $dy = f'(x)dx$; $y = 0$ at $x = a$

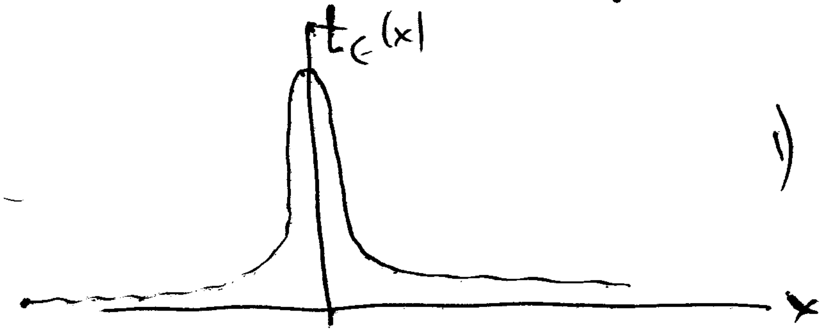
$$\int_{-\infty}^{+\infty} dx\, \delta(f(x))\, g(x) = \int \frac{dy}{|f'|}\delta(y)\, g(f^{-1}) = \frac{1}{|f'(a)|} g(x=a)$$

$$= \frac{1}{|f'(a)|}\int_{-\infty}^{+\infty} dx\, \delta(x-a)\, g(x) \Rightarrow \delta(f(x)) = \frac{1}{|f'(a)|}\delta(x-a)$$

III.B) Theorem: The Dirac delta function can be represented (non-uniquely) by limits of sequences of ordinary Riemann integrals of ordinary continuous functions.

ex. The Gaussian function

$$t_\epsilon(x) = \frac{1}{\sqrt{\pi}}\frac{1}{\epsilon}e^{-x^2/\epsilon^2}$$



Note:

1) $\lim\limits_{\epsilon\to 0^+} t_\epsilon(x) = \begin{cases} 0 & \text{if } x\neq 0 \\ \infty & \text{if } x=0 \end{cases}$

2) $\int_{-\infty}^{+\infty} dx\, t_\epsilon(x) = 1$ indep. of ϵ.

3) $\int_{-\infty}^{+\infty} dx\, t_\epsilon(x) f(x) = f(0) + f'(0)\int_{-\infty}^{+\infty} dx\, t_\epsilon(x)\, x + \dots$

Now

$$\int_{-\infty}^{+\infty} dx\, x^n t_\epsilon(x) = \begin{cases} \frac{1}{\sqrt{\pi}\epsilon}\left(\frac{-\partial}{\partial 1/\epsilon^2}\right)^{n/2}\int_{-\infty}^{+\infty} e^{-x^2/\epsilon^2}dx \\ \quad = \frac{1}{\sqrt{\pi}\epsilon}\left(\frac{-\partial}{\partial 1/\epsilon^2}\right)^{n/2}\sqrt{\pi}\,\epsilon \quad , \text{if } n=\text{even} \\ 0 \quad , \text{if } n=\text{odd} \end{cases}$$

III.B.3.) Now $\frac{\partial}{\partial 1/\epsilon^2} = -2\epsilon^3 \frac{\partial}{\partial \epsilon}$

So

$$\int_{-\infty}^{+\infty} x^n t_\epsilon(x)\,dx = \begin{cases} 0 & ;\text{ if } n \text{ odd} \\ \frac{1}{\sqrt{\pi}\,\epsilon}\left(2\epsilon^3 \cdot \frac{\partial}{\partial \epsilon}\right)^{n/2} \sqrt{\pi}\,\epsilon \sim O(\epsilon^2) & ;\text{ if } n = \text{even} \end{cases}$$

So $\lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} x^n t_\epsilon(x)\,dx \longrightarrow 0.$

$$\Rightarrow \lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} dx\, t_\epsilon(x)\, f(x) = f(0)$$

$$\Rightarrow \boxed{\delta(x) = \lim_{\epsilon \to 0^+} t_\epsilon(x)}\,.$$

4.) Fourier Transform:

$$f(x) = \int_{-\infty}^{+\infty} g(k)\, e^{ikx}\, \frac{dk}{2\pi}$$

with $g(k) = \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx$

III.B.4.) So

$$f(x) = \int_{-\infty}^{+\infty} dk\, e^{ikx} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx' f(x') e^{-ikx'}$$

interchange integral $= \int_{-\infty}^{+\infty} dx' f(x') \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ik(x-x')}$

$$= f(x)$$

$$\Rightarrow \boxed{\delta(x-x') = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ik(x-x')}}$$

III.)C.) The Three-Dimensional Dirac Delta Function:

For functions of 3-variables (x, y, z) we can generalize the Dirac Delta function by simply taking the product of the 3 one dimensional δ's

$$\delta^3(\vec{r}) \equiv \delta(x)\delta(y)\delta(z), \text{ or}$$

shifting the zero

$$\boxed{\delta^3(\vec{r}-\vec{r}_0) \equiv \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)}$$

III.C.) Thus we have the defining property

$$\int_{\substack{\text{all}\\\text{space}}} d^3r \, f(\vec{r})\, \delta^3(\vec{r}-\vec{r}_0) = f(\vec{r}_0) \, .$$

and in particular $\displaystyle\int_{\substack{\text{all}\\\text{space}}} d^3r \, \delta^3(\vec{r}-\vec{r}_0) = 1$.

Note: as long as the volume V contains the point $\vec{r}_0$ then

$$\int_V d^3r \, f(\vec{r})\, \delta^3(\vec{r}-\vec{r}_0) = f(\vec{r}_0) \, .$$

Ex. Recall we found that the flux of the $\vec{E}$ due to a point charge was given by

$$\frac{q}{\epsilon_0} = \oint_S \vec{E}\cdot d\vec{S} = \frac{q}{4\pi\epsilon_0}\oint_S \frac{\hat{r}}{r^2}\cdot d\vec{S}$$

$$\Rightarrow \quad 4\pi = \oint_S \frac{\hat{r}}{r^2}\cdot d\vec{S} = \int_V \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) d^3r$$

but $\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 0$ for $\vec{r} \neq 0$

and $\displaystyle\int_V d^3r \, \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi$

$$\Rightarrow \quad \boxed{\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(\vec{r})}$$

III.c) Or shifting the coordinates

$$\boxed{\vec{\nabla}_{\vec{r}} \cdot \left(\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \right) = 4\pi \delta^3(\vec{r}-\vec{r}_0)}$$

Ex The electric field due to a point charge q at $\vec{r}_0$ is

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}$$

Also for a charge density $\rho(\vec{r})$ the electric field is

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{\rho(\vec{r}')(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

Hence the charge density for a point charge q at $\vec{r}_0$ is simply

$$\boxed{\rho(\vec{r}) = q \, \delta^3(\vec{r}-\vec{r}_0)}$$

Since

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{q\,\delta^3(\vec{r}'-\vec{r}_0)(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3} \quad \text{as}$$

above.

III.c) Ex. finally recall

$$\vec{\nabla}\frac{1}{r} = \hat{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$$

$$\vec{\nabla}\cdot\vec{\nabla}\frac{1}{r} = \nabla^2\frac{1}{r} = \vec{\nabla}\cdot\left(-\frac{\vec{r}}{r^3}\right)$$

$$= -4\pi\delta^3(\vec{r})$$

So

$$\boxed{\nabla^2\frac{1}{r} = -4\pi\delta^3(\vec{r})}$$

$\underline{\nabla^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \vec{\nabla}\cdot\vec{\nabla}$ is called the <u>Laplacian operator</u>

III.D) <u>Helmholtz Theorem</u>:

If the divergence $D(\vec{r})$ and the curl $\vec{C}(\vec{r})$ of a vector function $\vec{F}(\vec{r})$ are specified, (i.e. $\vec{\nabla}\cdot\vec{F} \equiv D$ and $\vec{\nabla}\times\vec{F} \equiv \vec{C}$) and if they both go to zero faster than $1/r^2$ as $r\to\infty$, and if $\vec{F}(\vec{r})$ goes to zero as $r\to\infty$, then $\vec{F}$ is given uniquely by

III.D.) $\vec{F} = -\vec{\nabla} U + \vec{\nabla} \times \vec{W}$

where
$$U(\vec{r}) = \frac{1}{4\pi} \int_V \frac{D(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d^3r'$$
$$\vec{W}(\vec{r}) = \frac{1}{4\pi} \int_V \frac{\vec{C}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d^3r'$$
and V = all space.

Proof: 1) Existence is by checking the above construction explicitly:

a) $\vec{\nabla}\cdot\vec{F} = -\nabla^2 U + \underbrace{\cancel{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{W})}}_{0}$
$$= -\frac{1}{4\pi} \int_V D(\vec{r}\,') \underbrace{\nabla^2 \left(\frac{1}{|\vec{r}-\vec{r}\,'|} \right)}_{= -4\pi \delta^3(\vec{r}-\vec{r}\,')} d^3r'$$
$$\Rightarrow \boxed{D(\vec{r}) = \vec{\nabla}\cdot\vec{F} \quad \checkmark \text{ as required}}$$

b) $\vec{\nabla}\times\vec{F} = \underbrace{\cancel{-\vec{\nabla}\times\vec{\nabla} U}}_{0} + \vec{\nabla}\times(\vec{\nabla}\times\vec{W})$
$$= -\nabla^2 \vec{W} + \vec{\nabla}(\vec{\nabla}\cdot\vec{W})$$

III. D.) So first

$$-\nabla^2 \vec{W} = -\frac{1}{4\pi}\int_V \vec{C}(\vec{r}\,') \underbrace{\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)}_{= -4\pi\delta^3(\vec{r}-\vec{r}\,')} d^3r'$$

$$\boxed{-\nabla^2 \vec{W} = \vec{C}(\vec{r}).}$$

Now we must consider the second term

$$4\pi\, \vec{\nabla}\cdot\vec{W} = \int_V \vec{C}(\vec{r}\,')\cdot\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) d^3r'$$

change $\vec{r}\to\vec{r}\,'$ derivative

$$= -\int_V \vec{C}(\vec{r}\,')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) d^3r'$$

integrate by parts

$$= \int_V \frac{1}{|\vec{r}-\vec{r}\,'|}\vec{\nabla}'\cdot\vec{C}(\vec{r}\,')\, d^3r' - \int_V \vec{\nabla}'\cdot\left(\vec{C}(\vec{r}\,')\frac{1}{|\vec{r}-\vec{r}\,'|}\right) d^3r'$$

Gauss's theorem

$$= \int_V \frac{1}{|\vec{r}-\vec{r}\,'|}\vec{\nabla}'\cdot\vec{C}(\vec{r}\,')\, d^3r' - \oint_S \frac{1}{|\vec{r}-\vec{r}\,'|}\vec{C}(\vec{r}\,')\cdot d\vec{S}\,'$$

III b) Now we were given that $\vec{\nabla}\cdot\vec{F} = D$ while $\vec{\nabla}\times\vec{F} = \vec{C}$. $\Rightarrow$ $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) \equiv 0$
$= \vec{\nabla}\cdot\vec{C}$.

So

$$4\pi\vec{\nabla}\cdot\vec{W} = -\oint_S \frac{1}{|\vec{r}-\vec{r}'|}\vec{C}(\vec{r}')\cdot d\vec{S}'$$

$$\sim \int^{\infty} dr'\, r' \frac{1}{r'}\vec{C}(r')$$

So for this to be zero $\vec{C}(r')$ must go to zero as $r' \to \infty$ faster than $\frac{1}{r'}$

By assumption they do so

$$\boxed{\vec{\nabla}\cdot\vec{W} = 0.} \Rightarrow$$

$$\vec{\nabla}\times\vec{F} = -\nabla^2\vec{W} = \vec{C}(\vec{r}) \checkmark \text{ as required!}$$

Further we assumed the integrals for $U, \vec{W}$ in the first place existed but

$$U \sim \int^{\infty} r'^2 dr' \frac{D(r')}{r'}$$

$$\vec{W} \sim \int^{\infty} r'^2 dr' \frac{\vec{C}(r')}{r'}$$

Thus for these to converge at $r' \to \infty$

III. D.) D and $\vec{C}$ must go to zero as $r' \to \infty$ faster than $1/r'^2$, a stronger requirement than above and the one in our assumptions.

2) Uniqueness: If $\vec{C}, D \underset{r\to\infty}{\sim} \frac{1}{r^{2+\epsilon}}$

then $\vec{F}$ is not uniquely specified by its div. and curl. We could always add to $\vec{F}$ any vector function whose curl and div. vanish.

$$\vec{F}' = \vec{F} + \vec{\Lambda} \quad \text{where}$$

$$\vec{\nabla}\cdot\vec{\Lambda} = 0 \text{ and } \vec{\nabla}\times\vec{\Lambda} = 0 .$$

As we will <u>prove later</u>, there is no function $\vec{\Lambda}$ so that $\vec{\nabla}\cdot\vec{\Lambda} = 0$ and $\vec{\nabla}\times\vec{\Lambda} = 0$ <u>and</u> $\vec{\Lambda} \to 0$ as $r \to \infty$.

Thus we add to the assumptions of the theorem that $\vec{F} \to 0$ as $r \to \infty$ then it is <u>uniquely</u> given by its div. and curl.