I.E.) Cartesian Coordinates and Rotations

Vectors have magnitude and direction and obey the vector algebra of above. In components, that is wrt a particular Cartesian coordinate system, vectors are specified by their projection onto the 3 axes. The manipulations of vector addition and multiplication by a scalar have a simple form; add respective components or multiply each by the scalar

$$\vec{A}+\vec{B} = (A_x+B_x)\hat{i} + (A_y+B_y)\hat{j} + (A_z+B_z)\hat{k}$$
$$\alpha\vec{A} = \alpha A_x\hat{i} + \alpha A_y\hat{j} + \alpha A_z\hat{k}.$$

Likewise the scalar and vector products are simply given by

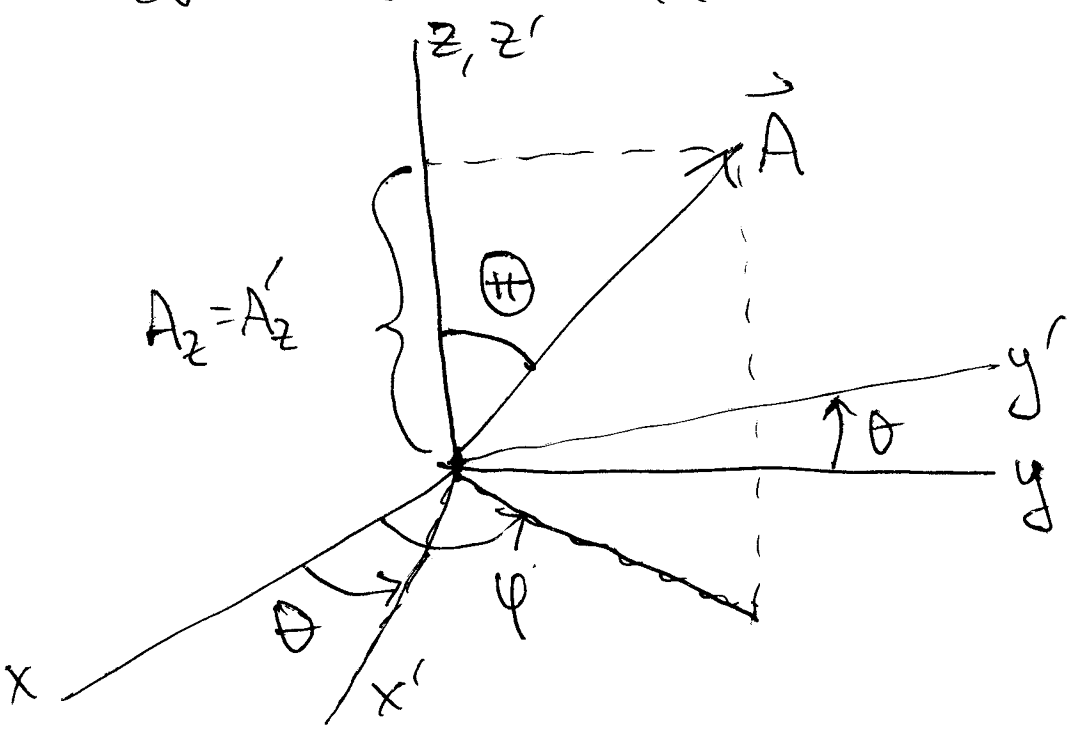
$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$
$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad \text{from}$$

which their properties are easily demonstrated.

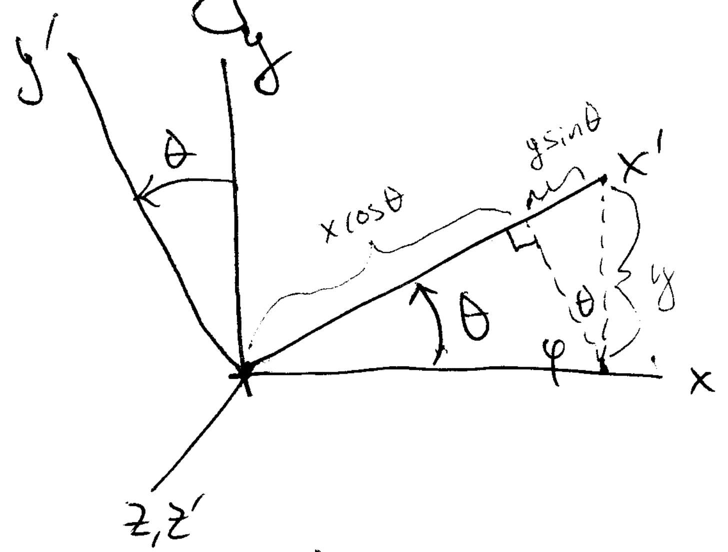
Each choice of Cartesian coordinate system is as good as the other. Since vectors have an independent

I.E.) identity, independent of how we label points in space, as we have seen from our geometric point of view, their properties and products etc.
have the same rules and consequences in each frame. Of course the components of the vector will be different in each frame, since the direction of the vector is described differently, but just in the right way for all the vector properties to be maintained:

For example consider two frames related by a rotation through angle θ about their common z-axis



I.E.) Looking down the z-axis we see that



$$\boxed{\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \\ z' &= z \end{aligned}}$$

The position coords. of point P are thus related

Likewise $\vec{A}$ in the S-frame has the component form

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

In the S'-frame $\vec{A} = A_x' \hat{i}' + A_y' \hat{j}' + A_z' \hat{k}'$

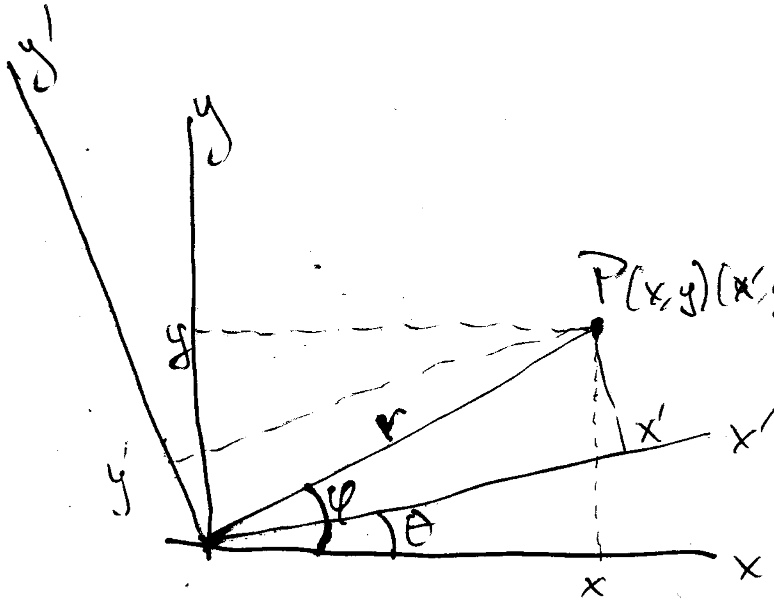
where $A_{x,y,z}$ are the projections on x, y, z axes and A'_{xyz} are the projections on the x', y', z' axes. Since z, z' are the same

$$A_z = A_z' .$$

Now
$$A_x = A \sin\Theta \cos\varphi$$
$$A_y = A \sin\Theta \sin\varphi$$

with $A = |\vec{A}|$. But in the S' frame

Simpler derivation:

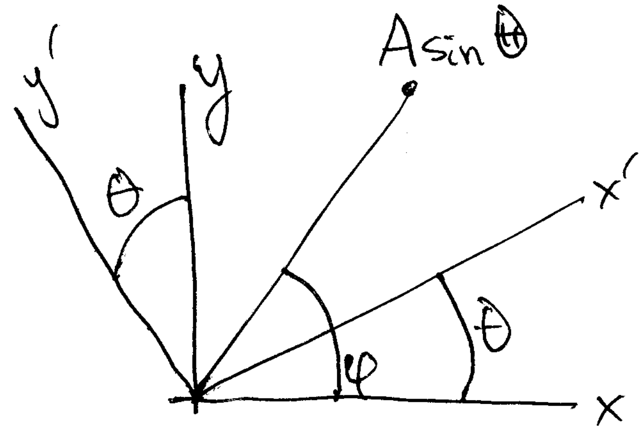


$x = r\cos\varphi$
$y = r\sin\varphi$

$x' = r\cos(\varphi-\theta) = r c\varphi c\theta + r s\varphi s\theta$
$y' = r\sin(\varphi-\theta) = r\, s\varphi c\theta - r c\varphi s\theta$

$$\Rightarrow \boxed{\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}}$$

I.E.)



$$A'_x = A\sin\Theta\cos(\varphi-\theta)$$

$$A'_y = A\sin\Theta\sin(\varphi-\theta)$$

Expanding $A'_x = A\sin\Theta(\cos\varphi\cos\theta + \sin\varphi\sin\theta)$

$$= (A\sin\Theta\cos\varphi)\cos\theta + (A\sin\Theta\sin\varphi)\sin\theta$$

$$\boxed{A'_x = A_x\cos\theta + A_y\sin\theta}$$

Likewise $A'_y = (A\sin\Theta\sin\varphi)\cos\theta - (A\sin\Theta\cos\varphi)\sin\theta$

$$\boxed{A'_y = -A_x\sin\theta + A_y\cos\theta}$$

$$\boxed{A'_z = A_z}$$

I.E.) Thus vector components are related to each other in the 2-frames just as the position vector or coordinate axes are related. The vector's direction has geometric integrity — the same direction is described differently in the 2-frames.

We can make these Rotations of frame simpler by using a matrix notation for the coordinates and components

group the components or coordinates as a column vector

$$\underline{X} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \equiv \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \equiv A$$

Then $\underline{X}' = R(\theta)\underline{X}$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

I.E.) or

$$A' = R(\theta) A$$

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix} = \begin{pmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Note: as required by a rotation the length of the ~~sec~~ vector is unchanged, it is only the [illegible] description of its direction in that frame that changes, so

$$\vec{A}' \cdot \vec{A}' = A_x'^2 + A_y'^2 + A_z'^2$$

$$= (A_x \cos\theta + A_y \sin\theta)^2$$

$$+ (-A_x \sin\theta + A_y \cos\theta)^2$$

$$+ A_z^2$$

$$= A_x^2 \cos^2\theta + A_y^2 \sin^2\theta + 2 \cancel{A_x A_y} s\theta c\theta$$

$$+ A_x^2 \sin^2\theta + A_y^2 \cos^2\theta - 2 \cancel{A_x A_y} s\theta c\theta$$

$$+ A_z^2$$

$$= A_x^2 + A_y^2 + A_z^2 = \vec{A} \cdot \vec{A}$$

I.E.) So the rotation leaves the magnitude of $\vec{A}$ unchanged $|\vec{A}| = |\vec{A}'|$.

From the matrix point of view we have

$$\vec{A}' \cdot \vec{A}' = A'^T A' = A^T R^T(\theta) R(\theta) A$$

but

$$R^T(\theta) = \begin{pmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = R^{-1}(\theta) \; !$$

(check $R^T R = \begin{pmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} c^2 + s^2 & +cs - sc & 0 \\ sc - cs & s^2 + c^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Big) .$$

$$\Rightarrow \boxed{\vec{A}' \cdot \vec{A}' = A'^T A' = A^T A = \vec{A} \cdot \vec{A}}$$

I.E.) So we can state this property of rotations, actually its their definition, as
Rotation of coord. =
The most general linear transformation that leaves the distance between 2 points (magnitude of a vector) in space unchanged i.e.

$$\vec{A}' \cdot \vec{A}' = A^T R^T R A \equiv A^T A = \vec{A} \cdot \vec{A}$$

$$\Rightarrow \quad R^T R = \underline{1}$$

$$\Rightarrow \quad R^T = R^{-1} \Rightarrow R \text{ is an}$$
(3×3) orthogonal matrix. Note also

$$R = R_1 R_2$$

$$R^T R = R_2^T R_1^T R_1 R_2$$
$$= 1 \Rightarrow R \text{ is orthognal}$$
if R_1, R_2 are. Hence set of all 3×3 orthog. matrices form a group under matrix mult: SO(3).

S = special = det R = 1 So

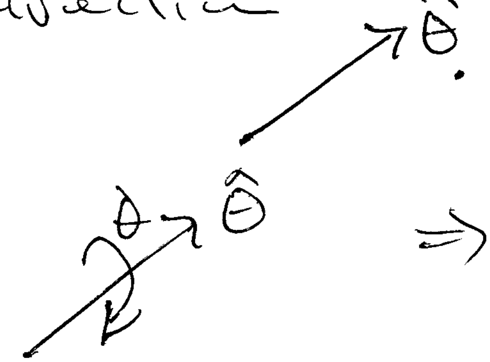
$$\det R^T R = \det 1 = 1 \Rightarrow (\det R)^2 = 1$$

I.E.) $\det R = \pm 1$. Right hand coord. systems; choose $\det R = +1$.

Now R is $3\times 3 \Rightarrow 9$ elements; but

$R^T = R^{-1}$, $\det R = 1$ $\Rightarrow$ 6 relations $\Rightarrow$ only 3 independent matrix elements.

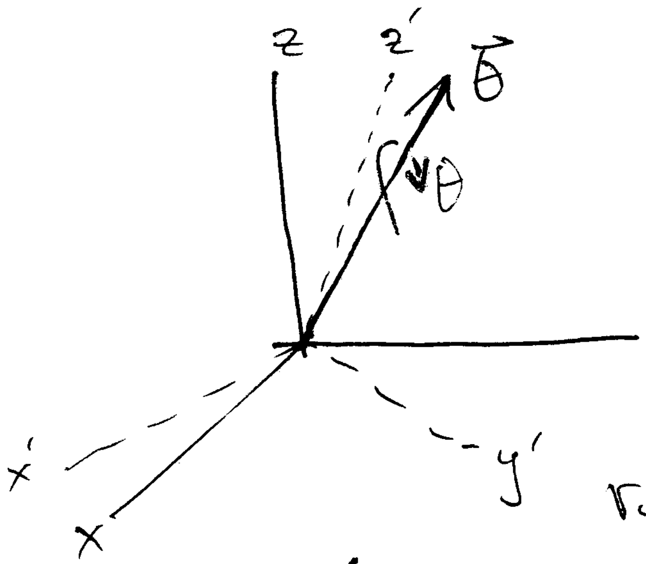
Thus Euler theorem: specify rotation by direction $\hat{\theta}$ and $\measuredangle$ of rotation



$\Rightarrow \vec{\Theta} = \theta\hat{\theta}$.

So $R = R(\vec{\Theta})$ 3 components.
(could choose Euler angles)

I.E)



$$\underline{X}' = R(\vec{\theta})\,\underline{X}$$

any vector ~~has~~ components in the 2 frames related by

$$A' = R(\vec{\theta})\,A$$

In fact this is the <u>definition</u> of a scalar, vector, tensor

1) Scalar: $S' = S$

2) Vector: $A' = R(\vec{\theta})\,A$

i.e. $$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix}_i = \Big(R(\vec{\theta})\Big)_{ij} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}_j$$

$$A_i' = \sum_{j=1}^{3} R(\vec{\theta})_{ij}\, A_j$$

$$= R_{ij}(\vec{\theta})\, A_j$$

Repeated indices are summed : Einstein summation convention.

I.E.) 3) Rank 2 Tensor:

$$T'_{ij} \equiv R_{ik} R_{jl} T_{kl}$$

(Inertial Tensor I_{ij}
Quadrupole Moment tensor Q_{ij})

4) General Rank n tensor

$$T'_{i_1 \cdots i_n} = R_{i_1 j_1} \cdots R_{i_n j_n} T_{j_1 \cdots j_n}$$

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2 special tensors which are constant, the same in all frames

a) 2nd rank Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\begin{aligned} \delta'_{ij} &= R_{ik} R_{jl} \delta_{kl} \\ &= R_{ik} R_{jk} = R_{ik} R^T_{kj} \\ &= (RR^T)_{ij} = \delta_{ij} \quad \checkmark \end{aligned}$$

δ_{ij} is invariant under Rotations

I. E.) b.) Levi-Civita tensor, permutation tensor, anti-symmetric tensor of rank 3

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise (if any 2 or 3 indices are equal } i=j,\ i=k,\ j=k \text{ or } i=j=k.) \end{cases}$$

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elements

$\epsilon_{123} = +1 = \epsilon_{231} = \epsilon_{312}$ 3

$\epsilon_{213} = -1 = \epsilon_{132} = \epsilon_{321}$ 3

$\epsilon_{iij} = 0$ $(j \neq i)$ 6

$\epsilon_{iji} = 0$ $i \neq j$ 6

$\epsilon_{jii} = 0$ 6

$\epsilon_{iii} = 0$ 3

$27 \checkmark$

$= 3^3.$

I. E. b.)

$$\epsilon'_{ijk} = R_{il} R_{jm} R_{kn}\, \epsilon_{lmn}$$
$$= \epsilon_{ijk} \quad \text{(verify)}$$

Further properties

1) $\epsilon_{abc}\, \epsilon_{ijk} = \delta_{ai}\delta_{bj}\delta_{ck} - \delta_{ai}\delta_{bk}\delta_{cj} + \delta_{aj}\delta_{bk}\delta_{ci} - \delta_{aj}\delta_{bi}\delta_{ck} + \delta_{ak}\delta_{bi}\delta_{cj} - \delta_{ak}\delta_{bj}\delta_{ci}$

$$= \begin{vmatrix} \delta_{ai} & \delta_{aj} & \delta_{ak} \\ \delta_{bi} & \delta_{bj} & \delta_{bk} \\ \delta_{ci} & \delta_{cj} & \delta_{ck} \end{vmatrix}$$

$\Rightarrow$

a) $\epsilon_{abc}\,\epsilon_{ijc} = \delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}$
$$= \begin{vmatrix} \delta_{ai} & \delta_{aj} \\ \delta_{bi} & \delta_{bj} \end{vmatrix}$$

b) $\epsilon_{abc}\,\epsilon_{ibc} = 2\delta_{ai}$

c) $\epsilon_{abc}\,\epsilon_{abc} = 3! = 6$

I. E. b. 2) If M_{ij} is a 3×3 matrix, then

$$\det M = \frac{1}{6}\,\epsilon_{ijk}\,\epsilon_{lmn}\,M_{il}\,M_{jm}\,M_{kn}$$
$$= \epsilon_{ijk}\,M_{i1}\,M_{j2}\,M_{k3}$$

3) $(\vec{A}\times\vec{B})_i = \epsilon_{ijk}\,A_j B_k$

4) $\vec{A}\cdot(\vec{B}\times\vec{C}) = \epsilon_{ijk}\,A_i B_j C_k$

$= \epsilon_{jki}\,B_j C_k A_i$ since $\epsilon_{ijk} = \epsilon_{jki}$

$= \vec{B}\cdot(\vec{C}\times\vec{A})$ etc.

5) $(\vec{\nabla}\times\vec{A})_i = \epsilon_{ijk}\,\partial_j A_k$

Recall the vector triple product identity we derived earlier

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Expanding in terms of components we have

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = \epsilon_{ijk} \epsilon_{klm} A_j B_l C_m$$

$$[\vec{B}(\vec{A} \cdot \vec{C})]_i = B_i A_j C_m \delta_{jm}$$

$$= \delta_{il} \delta_{jm} A_j B_l C_m$$

$$[\vec{C}(\vec{A} \cdot \vec{B})]_i = \delta_{im} \delta_{jl} A_j B_l C_m$$

Thus we find

$$0 = [\epsilon_{ijk} \epsilon_{lmk} - (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl})] A_j B_l C_m$$

Since $\vec{A}, \vec{B}, \vec{C}$ are independent, this implies the 81 different terms

$$\epsilon_{ijk} \epsilon_{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} .$$