ISOPERIMETRIC PROBLEMS
AND
LAGRANGE MULTIPLIERS
This is a class of problems where it is given to extremize one quantity subject to the constraint that another quantity remain fixed. For example: A farmer with a fixed amount of fence material wants to enclose the maximum possible area for his horse to graze.

Formulation: We are given to extremize the integral I

\[ I = \int_{x_1}^{x_2} dx \ f(x, y, y') \quad y(x_1) = y_1, \quad y(x_2) = y_2 \quad (1) \]

Subject to the constraint that some other integral J remains fixed:

\[ J = \int_{x_1}^{x_2} dx \ g(x, y, y') = \text{constant} \quad (2) \]

The solution to this problem requires Lagrange Multipliers which we review now.

**Review of Lagrange Multipliers:** [ARFKEN]

Consider the function \( f(x, y, z) \) and evaluate

\[ df = \left( \frac{\partial f}{\partial x} \right) dx + \left( \frac{\partial f}{\partial y} \right) dy + \left( \frac{\partial f}{\partial z} \right) dz \quad (3) \]

To find an extremum of \( f \) we set \( df = 0 \). Since the variations \( dx, dy, \) and \( dz \) are arbitrary, the only way that \( df = 0 \) can hold is if

\[ \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0 \quad (4) \]

Suppose now that we find that there is a constraint in the problem which can be expressed by some equation of the form

\[ g(x, y, z) = 0 \quad (5) \]
Because of this constraint, the variations $dx, dy, dz$ are no longer independent, which was the assumption needed to derive the condition in (4). Specifically

$$q(x, y, z) = 0 \Rightarrow 0 = (\frac{\partial q}{\partial x})dx + (\frac{\partial q}{\partial y})dy + (\frac{\partial q}{\partial z})dz \quad (6)$$

Since $\frac{\partial q}{\partial x}, \frac{\partial q}{\partial y},$ and $\frac{\partial q}{\partial z}$ are known, one can solve (6) explicitly for $dz$, for example, in terms of $dx$ and $dy$:

$$dz = -\left(\frac{\partial q}{\partial dz}\right)^{-1} \left[ (\frac{\partial q}{\partial x})dx + (\frac{\partial q}{\partial y})dy \right] \quad (7)$$

Because of this equation, $dz$ is dependent on $dx$ and $dy$ and the previous arguments to find the extremum are not valid.

One can of course eliminate $dz$ simply by using (7) to replace $dz$ everywhere. This can be done but is tedious.

There is another way to eliminate $dz$ using Lagrange multipliers: Using (3) & (6) form the function $f(x, y, z) + \lambda \cdot q(x, y, z)$. Then the extremum

$$df = 0 \quad (8)$$

can be rewritten as $df + \lambda dq = 0$, since $q(x, y, z) = 0 \Rightarrow dq = 0$.

This gives the following equation:

$$df(x, y, z) + \lambda dq(x, y, z) = 0 = \left( \frac{2f}{2x} + \lambda \frac{2g}{2x} \right)dx + \left( \frac{2f}{2y} + \lambda \frac{2g}{2y} \right)dy + \left( \frac{2f}{2z} + \lambda \frac{2g}{2z} \right)dz$$

(9)

Since $dz$ (for example) is not linearly independent it should not appear in (9), and one way of ensuring this is to choose $\lambda$ to make the coefficient of $dz$ vanish:

$$\frac{2f}{2z} + \lambda \frac{2g}{2z} = 0 \quad (10)$$
Having eliminated \( dz \), the expressions which give the extremum are now:

\[
\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0; \quad \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0
\]  

(11)

When these equations are solved, \( df = 0 \) and \( f(x, y, z) \) is an extremum subject to the constraint \( g(x, y, z) = 0 \).

**Summary:**

- We want to find the extremum of \( f(x, y, z) \) subject to the constraint \( g(x, y, z) = 0 \). Finding the extremum means finding \( x_0, y_0, z_0 \).

- Once we introduce the Lagrange multiplier \( \lambda \), we then have \( 4 \) unknowns to solve for: \( x_0, y_0, z_0, \lambda \).

- These 4 quantities are then determined by the following 4 equations:

\[
\begin{align*}
\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} &= 0 \quad \text{Eqs. (11) above} \quad (12a) \\
\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} &= 0 \quad (12b) \\
\frac{\partial f}{\partial z} + \lambda \frac{\partial g}{\partial z} &= 0 \quad \text{Eqs. (10)} \quad (12c) \\
g(x, y, z) &= 0 \quad \text{Eqs. (5)} \quad (12d)
\end{align*}
\]
Example: Application of Lagrange Multipliers in QM

Carlson: The ground state energy of a particle in rectangular QM box whose sides are \( a, b, c \) is given by

\[
E = E(a, b, c) = \frac{\hbar^2}{8m} \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right)
\]  
(12)

We wish to find the shape of the box (i.e., \( a, b, c \)) such that \( E \) is a minimum for a fixed volume.

\[
V = V(a, b, c) = abc = \text{constant} = k
\]  
(14)

Solution: In our previous notation let \( f(a, b, c) = E(a, b, c) \) and

\[
g(a, b, c) = V(a, b, c) - k = 0 = abc - k
\]  
(15)

We then solve:

\[
\frac{\partial E}{\partial a} + \lambda \frac{\partial V}{\partial a} = 0; \quad \frac{\partial E}{\partial b} + \lambda \frac{\partial V}{\partial b} = 0; \quad \frac{\partial E}{\partial c} + \lambda \frac{\partial V}{\partial c} = 0
\]  
(16)

This may be viewed as eliminating \( da \)

\[
\frac{\partial E}{\partial a} + \lambda \frac{\partial V}{\partial a} = -\frac{\hbar^2}{4ma^3} + \lambda abc = 0
\]  
(17a)

Similarly:

\[
\frac{\partial E}{\partial b} + \lambda \frac{\partial V}{\partial b} = 0 = -\frac{\hbar^2}{4mb^3} + \lambda abc = 0
\]  
(17b)

\[
\frac{\partial E}{\partial c} + \lambda \frac{\partial V}{\partial c} = 0 = -\frac{\hbar^2}{4mc^3} + \lambda abc = 0
\]  
(17c)

Multiply these equations in turn by \( a, b, c \) then gives:

\[
\lambda abc = \frac{\hbar^2}{4ma^2}; \quad \lambda abc = \frac{\hbar^2}{4mb^2}; \quad \lambda abc = \frac{\hbar^2}{4mc^2}
\]  
(18)

The solution to these equations is obviously \( a = b = c \)

\Rightarrow \text{Rectangular box} \rightarrow \text{Cube}
Note that we have solved the problem without having to actually determine $\alpha$. However, if we wish to solve for $\alpha$ to give it a physical interpretation we can write:

\[ \lambda_{abc} = \frac{\hbar^2}{4ma^2} \quad a=b=c \rightarrow \lambda a^3 = \frac{\hbar^2}{4ma^2} \rightarrow \lambda = \frac{\hbar^2}{4ma^2} \quad (20) \]

To interpret $\lambda$ we note from (13) & (16) that

\[ E = \frac{\hbar^2}{8m} \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \rightarrow \frac{3}{8} \frac{\hbar^2}{ma^2} \]

Hence the energy density is given by

\[ \frac{E}{V} = \frac{(3/8) \frac{\hbar^2}{ma^2}}{q^3} = \frac{3}{8} \frac{\hbar^2}{ma^2} \]

If we convert this to physical energy units, $E = 4\pi^2 E^*$ etc. then

\[ \lambda = \frac{3\pi^2}{2} \frac{E}{V} \quad \text{so} \quad \lambda \text{ is a measure of the energy density} \]
Example 2: text p. 57

Extremize \( f(x, y) = x^2 + 2xy \). Subject to the constraint \( x + y^2 = 4 \).

Solution: In this case the constraint is \( x + y^2 + 4 = 0 \equiv g(x, y) \).

From the preceding example and discussion, we want to extremize (i.e., minimize or maximize) the function \( J = f(x, y) - \lambda g(x, y) \).

[Note: We have previously used \( f + \lambda g \), whereas the text used \( f - \lambda g \). Either choice is purely conventional, since \( \lambda \) can itself be positive or negative.] Hence

\[
J(x, y) = f(x, y) - \lambda g(x, y) = x^2 + 2xy - \lambda(x + y^2 - 4) \tag{1}
\]

Once \( \lambda \) is included, we can now view the variations \( \partial J/\partial x \) and \( \partial J/\partial y \) are independent, so that

\[
\frac{\partial J}{\partial x} = 2x + 2y - 2\lambda x = 0 \quad \frac{\partial J}{\partial y} = 2x - 2\lambda y = 0 \tag{2}
\]

These two equations along with the original constraint equation

\( g = (x^2 + y^2 - 4) = 0 \) solve the problem as follows: From (2)

\[
2x - 2\lambda y = 0 \implies x = \lambda y \quad \text{(3)}
\]

Combining this with the first equation in (2) gives:

\[
0 = 2x + 2y - 2\lambda x = 2\lambda y + 2y - 2\lambda (\lambda y) = 0 \implies \lambda + 1 - \lambda^2 = 0 \implies \lambda (\lambda + 1 - \lambda^2) = 0 \tag{4}
\]

The solution of \( \lambda^2 - \lambda - 1 = 0 \) is \( \lambda = \frac{1 \pm \sqrt{5}}{2} \).
Once we find $\lambda \pm$ we can solve for the value(s) $(x_0, y_0)$ where the extrema are.

Using $x = 2y.$ ($\lambda$ is either $\lambda_+ \text{ or } \lambda_-,$) and $x^2 + y^2 = 4,$ we get:

$$x^2 = 2^2 y^2 \Rightarrow x^2 + y^2 = 2^2 y^2 + y^2 = 4 \Rightarrow y^2 = \frac{4}{1 + \lambda^2} \Rightarrow y = \pm \frac{2}{\sqrt{1 + \lambda^2}}$$

Lastly, $x = 2y \Rightarrow x = \frac{2\lambda}{\sqrt{1 + \lambda^2}}$ \hspace{1cm} (7)

**SIDE COMMENT:**

By design or not the solution plays an important role in art;

It is defined by:

![Golden Ratio Diagram](image.png)

The **Golden Ratio** is defined by the equation

$$r = \frac{a}{b} = \frac{a+b}{a} \Rightarrow \frac{a}{b} = 1 + \frac{b}{a} \Rightarrow r = 1 + \frac{1}{r} \Rightarrow r^2 - r - 1 = 0$$

(10)

Eq. (10) is the same as Eq. (5) above for $\lambda.$ The solution $\lambda_+$ gives the **Golden Ratio**: Numerically,

$$r = \frac{1 + \sqrt{5}}{2} = 1.618033989...$$

(11)