TAYLOR SERIES
Suppose that we know the value of a function $f(x)$ at some point $x$, but we would like to be able to compute the value at a nearby point $(x+a)$. This can be done via a Taylor series which can be written in a number of equivalent ways.

A formula (useful in Quantum Mechanics — see below!!) is

$$f(x+a) = e^{a \frac{d}{dx}} f(x) = \left[ 1 + a \frac{d}{dx} + \frac{a^2}{2!} \frac{d^2}{dx^2} + \ldots \right] f(x)$$  \hspace{1cm} (1)

Equivalently by interchanging $x \leftrightarrow a$,

$$f(a+x) = e^{x \frac{d}{da}} f(a) = \left[ 1 + \frac{x}{1!} \frac{d}{da} + \frac{x^2}{2!} \frac{d^2}{da^2} + \ldots \right] f(a)$$  \hspace{1cm} (2)

If we now shift $x$ by replacing $x \rightarrow x-a$ then (2) $\Rightarrow$

$$f(a+x-a) = f(x) = \left[ 1 + (x-a) \frac{d}{da} + \frac{(x-a)^2}{2!} \frac{d^2}{da^2} + \ldots \right] f(a)$$  \hspace{1cm} (3)

$$\therefore \ f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$$  \hspace{1cm} (4)

Both of these forms are useful in different contexts.

**Connection to Quantum Mechanics:** ($\hbar = $ Planck's constant)

In (1) write $a \frac{d}{dx} = \frac{i \hbar}{\hbar} a \frac{d}{da} = \frac{iaP}{\hbar}$

$$p = \frac{\hbar}{i} \frac{d}{dx}$$  \hspace{1cm} (5)

Hence (1) $\&$ (5) $\Rightarrow$

$$e^{iap/\hbar} f(x) = f(x+a)$$  \hspace{1cm} (6)

The momentum operator induces spatial translations.
The fact that the momentum operator "moved" (translates) the function \( f(x) \to f(x+a) \) is intimately associated with the **Heisenberg Uncertainty Principle**.

\[
\Delta p \Delta x \geq \hbar
\]

(7)

The close connection between momentum conservation and spatial translations is part of **Noether's Theorem**.

**Applications of Taylor Series:**

\[ f(x) = \frac{1}{1-x} = \frac{1}{2} \]

(8)

\[
f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots
\]

(9)

\[
f'(a) = \frac{df}{du} \left| \frac{du}{dx} \right| = \frac{-\frac{1}{u^2}}{x=0} = \frac{1}{(1-x)^2} \bigg|_{x=0} = +1
\]

(10)

\[
f''(a) = \frac{d^2f}{du^2} \left| \frac{du}{dx} \right| = \frac{-2}{(1-x)^3} \bigg|_{x=0} = +2
\]

(11)

\[
f(x) = \frac{1}{1-x} = f(0) + \frac{(x-0)}{1!} \cdot 1 + \frac{(x-0)^2}{2!} \cdot 2 + \ldots + x + x^2 + \ldots
\]

(12)

We will show later that for \(|x| < 1\) this series converges.

Hence for \(|x| < 1\) we can write:

\[
\frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n \quad |x| < 1
\]

(13)

Also

\[
\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^n \quad |x| < 1
\]
\[ f(x) = \ln x \]

\[ f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots \]  

(14)

Again the Taylor series starts by knowing a value \( a \) at which we know the function. Here we can choose \( x=1 \Rightarrow \ln 1 = 0 \).

Also from our previous results, for any value \( a \)

\[ D^n \ln a = \frac{(-1)^{n+1}}{a^n} (n-1)! \quad a=1 \Rightarrow (-1)^{n+1} (n-1)! \rightarrow 0! = 1 \]  

(15)

Hence the first few terms in the series expansion of \( \ln x \) are

\[ \ln x = \ln 1 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \ldots \]  

(16)

This series converges for \( |x-1| < 1 \Rightarrow 0 < x < 2 \). Another useful form for the expansion of \( \ln x \) follows from the replacement

\[ y = x-1 \Rightarrow x = 1 + y. \]  

Then (16) \( \Rightarrow \)

\[ \ln (1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots \]  

(17)

This formula is used in our derivation of the Rule of 72: \( \ln(1+y) \approx \frac{y}{2} \)

Typical applications are for small \( x \) near 1 so we expand about \( f(a)=1 \), which corresponds to \( x=a=1 \).

Then

\[ f(x) = f(a) + \frac{(x-a)}{1!} f'(x=a) + \frac{(x-a)^2}{2!} f''(x=a) + \ldots \]  

(18)

Note that (as a notation check) \( f(x=a) = f(a) + 0 + 0 + 0 \ldots \)

which is as expected.
\[
\frac{df}{du} = \frac{df}{dx} \quad \text{and} \quad u = 1 + x \quad \Rightarrow \quad \frac{du}{dx} = 1 \quad (19)
\]

\[
f'(x) = nu^{n-1} \cdot 1 = n(1+x)^{n-1} \quad \Rightarrow \quad f'(x=a=0) = n \quad (20)
\]

\[
f''(x) = n(n-1)u^{n-2} \cdot 4 \quad \Rightarrow \quad f''(x=a=0) = n(n-4) \ldots \quad (21)
\]

Hence \((1+x)^n = 1 + \frac{nx-0}{1!} + \frac{(x-0)^2}{2!} n(n-1) + \ldots \quad (22)
\]

To leading order, \((1+x)^n \approx 1 + nx \quad (23)\]

More generally, \((1\pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{1!} + \frac{n(n-1)(n-2)x^3}{2!} + \ldots \quad (24)\]

This series converges for \(x^2 < 1\).

**CHECK:**
For \(n = -1\), \((1\pm x)^{-1} = \frac{1}{1\pm x} = 1 \mp x + x^2 \mp x^3 \quad (25)\]

This agrees with the previous results in (13).

The preceding results also hold if \(n\) is replaced by a rational power \(p\), such as \(p = \frac{1}{2}\). This is useful, for example in the theory of relativity where we often encounter the expressions \(\sqrt{1-v^2/c^2}\) and \(\frac{1}{\sqrt{1-v^2/c^2}}\). Let \(x = v^2/c^2\), then

\[
\sqrt{1-v^2/c^2} \Rightarrow (1\mp x)^{1/2} = 1 \mp \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} x^3 \ldots \quad (26)
\]

\[
\therefore (1\mp x)^{1/2} = 1 \mp \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} x^3 \ldots \quad (27)
\]

Also:

\[
(1\mp x)^{-1/2} = \frac{1}{\sqrt{1\mp x}} = 1 \mp \frac{1}{2} x + \frac{3}{8} x^2 \pm \frac{5}{16} x^3 \ldots \quad (28)
\]

Hence \(\frac{\sqrt{\text{Mc}^2}}{\sqrt{1-v^2/c^2}} \approx \text{Mc}^2 (1 \pm \frac{v^2}{c^2}) \leq \text{Mc}^2 + \frac{1}{2} \text{mv}^2 \quad (29)\)
Comparison of the Two Forms of the Taylor Series

**Increment Form**

\[ f(x) = \text{known then} \]

\[ f(x + a) = e^{\frac{a}{dx}} f(x) = \left(1 + \frac{a}{1!} \frac{df}{dx} + \frac{a^2}{2!} \frac{d^2 f}{dx^2} + \ldots\right) f(x) \]  \hspace{1cm} (1)

\[ f(x + a) = f(x) + \frac{a}{1!} f'(x) + \frac{a^2}{2!} f''(x) + \ldots \]  \hspace{1cm} (2)

In this form we assume that we know the function \( f(x) \) at some point \( x \), and this allows us to evaluate the l.h.s. exactly. Then if we know \( f(x) \) we can compute \( f(x + a) \).

\[ \text{any} \ x \rightarrow \text{a specific value} \ (x = a) \]

**Taylor Series Form**

\[ f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots \]  \hspace{1cm} (3)

In this form we again assume that we know the function \( f(x) \), which now appears on the l.h.s. of (3). Here the main objective is to construct a series expansion of \( f(x) \) in terms of the (presumably small) quantities \( (x-a), \ (x-a)^2, \ldots \)

A very useful compendium of formulas for series is:

I. S. Gradshteyn & I. M. Ryzhik

"**Table of Integrals, Series and Products**"

(Academic Press, 1980)