Maxwell Equations: Electromagnetic Waves

Maxwell’s Equations contain the wave equation
The velocity of electromagnetic waves:
\[ c = 2.99792458 \times 10^8 \, \text{m/s} \]
The relationship between \( E \) and \( B \) in an EM wave
Energy in EM waves: the Poynting vector

The equations so far.....

Gauss’ Law for E Fields
\[ \oint_S E_n \, dA = \frac{1}{\varepsilon_0} Q_{\text{inside}} \]

Gauss’ Law for B Fields
\[ \oint_S B_n \, dA = 0 \]

Faraday’s Law
\[ \oint_c \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S B_n \, dA \]

Ampere’s Law
\[ \oint_c \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \]

A problem with Ampere’s Law
Consider a wire and a capacitor. \( C \) is a loop.
• Time dependent situation: current flows in the wire as the capacitor charges up or down.

Ampere’s Law for \( S_1 \):
\[ \oint_{S_1} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \]

Ampere’s Law for \( S_2 \):
\[ \oint_{S_2} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I = 0 \]
Because no current is enclosed by \( S_2 \).

Maxwell’s Displacement Current, \( I_d \)
\[ I_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \frac{d}{dt} \oint_c \mathbf{E} \cdot d\mathbf{A} \]

Substituting into:
\[ \oint_c \mathbf{B} \cdot d\mathbf{l} = \mu_0 \left( I + I_d \right) \]

Means that a changing electric flux: \( \frac{d\Phi_E}{dt} \) results in a magnetic field, just as a changing magnetic flux: \( \frac{d\Phi_M}{dt} \) gives rise to an electric field.

Displacement Current:
\[ I_d = \frac{d}{dt} \left( \frac{Q}{\varepsilon_0} \right) \text{ where } Q = \varepsilon_0 E A \]
\[ q = \varepsilon_0 A E \]
\[ \frac{dq}{dt} = I = \varepsilon_0 A \frac{dE}{dt} = \varepsilon_0 \varepsilon_0 \frac{d\Phi_M}{dt} \]

The charge on the plates from current in circuit:
Calculating Displacement Current

Consider a parallel plate capacitor with circular plates of radius $R$. If charge is flowing onto one plate and off the other plate at a rate $I = \frac{dQ}{dt}$, what is $I_d$?

- The displacement current is not a current. It represents magnetic fields generated by time varying electric fields.

$$\Phi_E = EA = \sigma A = \frac{Q}{\varepsilon_o} = \varepsilon_o$$

$$I_d = \varepsilon_o \frac{d\Phi_E}{dt} = \frac{dQ}{dt} = I$$

Maxwell's Equations (1865)

\[ \oint_S E \cdot dA = \frac{1}{\varepsilon_0} Q_{\text{inside}} \] Gauss's law

\[ \oint_S B \cdot dA = 0 \] Sometimes called Gauss's law for magnetism

\[ \oint_C E \cdot d\ell = -\frac{d}{dt} \oint_S B \cdot dA = -\oint_S \frac{\partial B}{\partial t} \cdot dA \] Faraday's law

\[ \oint_C B \cdot d\ell = \mu_0 I + I_d \] Ampere - Maxwell law

in Systeme International (SI or mks) units

Calculating the B field

Example

\[ \int_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 (I + I_d) \]

\[ I_{\text{enc}} = I_d \frac{\pi r^2}{\pi R^2} \]

\[ B2\pi r = \mu_0 I_d \frac{\pi r^2}{\pi R^2} \]

\[ B = \frac{\mu_0 I_d r}{2\pi R^2} \text{ for } r \leq R \]

For $r > R$

\[ B = \frac{\mu_0 I_d r}{2\pi r} \]

Maxwell's Equations (Free Space)

- Note the symmetry of Maxwell's Equations in free space, when no charges or currents are present.

\[ \oint E \cdot dA = 0 \]

\[ \oint B \cdot dA = 0 \]

\[ \oint E \cdot d\ell = -\frac{d\Phi_E}{dt} \]

\[ \oint B \cdot d\ell = \mu_0 \varepsilon_0 \frac{d\Phi_B}{dt} \]

We can predict the existence of electromagnetic waves. Why? Because the wave equation is contained in these equations. Remember the wave equation.

\[ \frac{\partial^2 h}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 h}{\partial t^2} \]

$h$ is the variable that is changing in space ($x$) and time ($t$). $v$ is the velocity of the wave.
Review of Waves from Mechanics

The one-dimensional wave equation: \[ \frac{\partial^2 h}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 h}{\partial t^2} \]

has a general solution of the form:
\[ h(x,t) = h_1(x - vt) + h_2(x + vt) \]

A solution for waves traveling in the +x direction is:
\[ h(x,t) = A \cos(kx - \alpha t) \]

Wave Examples

Wave on a String:

\[ F \frac{d^2 y}{dx^2} = \mu \frac{d^2 y}{dt^2} \]

Electromagnetic Wave

What is waving?? The Electric & Magnetic Fields!!

Rewrite Maxwell's equations as equations of the form:

\[ \frac{\partial^2 E_x}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 E_x}{\partial t^2} \]

e.g., sqrt(tension/mass) is wave speed of a guitar string, proportional to frequency of fundamental

Four Step Plane Wave Derivation

**Step 1** Assume we have a plane wave propagating in z (i.e. \( E, B \) not functions of \( x \) or \( y \))

Example: \( E_x = E_0 \sin(kz - \omega t) \)

**Step 2** Apply Faraday's Law to infinitesimal loop in x-z plane

\[ \oint E \cdot d\ell = -\frac{d\phi_B}{dt} \]

\[ [E_x(z_2) - E_x(z_1)] \Delta x = \Delta x \Delta z \frac{\partial E_x}{\partial z} = -\Delta x \Delta z \frac{\partial B_y}{\partial t} \]

\[ \frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \]

**Step 3** Apply Ampere's Law to an infinitesimal loop in y-z plane

\[ \oint B \cdot d\ell = \mu_0 \varepsilon_0 \frac{d\phi_E}{dt} \]

\[ [B_y(z_2) - B_y(z_1)] \Delta y = \Delta y \Delta z \frac{\partial B_y}{\partial z} = \mu_0 \varepsilon_0 \Delta y \Delta z \frac{\partial E_x}{\partial t} \]

\[ \frac{\partial B_y}{\partial z} = \mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t} \]

**Step 4**: Use results from steps 2 and 3 to eliminate \( B_y \)

\[ \frac{\partial^2 B_y}{\partial z^2} = -\mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} \]

\[ \frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \quad \frac{\partial^2 B_y}{\partial z \partial t} = -\frac{\partial^2 E_x}{\partial z^2} \]

\[ \frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} \]
Velocity of Electromagnetic Waves

We derived the wave equation for $E_x$:

$$\frac{\partial^2 E_x}{\partial t^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

The velocity of electromagnetic waves in free space is:

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Putting in the measured values for $\mu_0$ & $\varepsilon_0$, we get:

$$v = 3.00 \times 10^8 \text{ m / s} \equiv c$$

This value is identical to the measured speed of light! We identify light as an electromagnetic wave.

Electromagnetic Spectrum

~1850: infrared, visible, and ultraviolet light were the only forms of electromagnetic waves known.

Visible light (human eye)

Sunscreen Absorbs UV

Viscible

Sunscreen

Without

Sunscreen

Visible

With Sunscreen

Without

Sunscreen

UV

http://upload.wikimedia.org/wikipedia/commons/thumb/0/0d/UV_and_Vis_Sunscreen.jpg/800px-UV_and_Vis_Sunscreen.jpg
What a Bee Sees:

These two photographs of a flower show how the evening primrose looks in two different wavelengths. The upper panel shows the flower as humans see it in visible light. The lower panel shows the primrose in UV, and reveals the "honey guides" (the dark areas) invisible to the human eye but seen by insects. The dark lines and patches guide the insect to the collection of nectar stored in the center and to the pollen on the anthers. Image courtesy Dr. Jeremy Burgess, Science Source/Photo Researchers.

How is B related to E?

We derived the wave equation for $E_x$:

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2}$$

We could have derived for $B_y$:

$$\frac{\partial^2 B_y}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 B_y}{\partial t^2}$$

How are $E_x$ and $B_y$ related in phase and magnitude?

Consider the harmonic solution:

$E_x = E_0 \sin(kz - \omega t)$

where $\omega = kc$

$$\frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z} = -kE_x \cos(kz - \omega t)$$

$B_y = \int dB_y = -kE_x \int \cos(kz - \omega t) dt$

$B_y = E \frac{k}{\omega} \sin(kz - \omega t)$

Wien's Displacement Law

Visible spectrum

Infrared

Ultraviolet

$\lambda_{peak} \propto \frac{1}{T}$

Frequency (Hz)

Wavelength (nm)

E & B in Electromagnetic Waves

Plane Wave:

$E_x = E_0 \sin(kz - \omega t)$

$B_y = B_0 \sin(kz - \omega t)$

where $E_0 = cB_0$

The direction of propagation is given by the cross product

$\hat{s} = \hat{e} \times \hat{b}$

where $\{\hat{e}, \hat{b}\}$ are the unit vectors in the $(E, B)$ directions.

Nothing special about $(E_x, B_y)$; eg could have $(E_y, -B_x)$

Note cyclical relation:

$\hat{e} \times \hat{b} = \hat{s}$

$\hat{b} \times \hat{s} = \hat{e}$

$\hat{s} \times \hat{e} = \hat{b}$
Energy in Electromagnetic Waves

Electromagnetic waves contain energy. We know the energy density stored in $E$ and $B$ fields:

$$u_E = \frac{1}{2} \varepsilon_0 E^2 \quad u_B = \frac{1}{2} \mu_0 B^2$$

- In an EM wave, $B = E/c$ implies $u_B = \frac{1}{2} \mu_0 \varepsilon_0 E^2 = u_E$
- The total energy density in an EM wave = $u$, where $u = u_E + u_B = \varepsilon_0 E^2$

The intensity of a wave is defined as the average power ($P_{av} = \frac{u_{av}}{\Delta t}$) transmitted per unit area = average energy density times wave velocity:

$$I = c \varepsilon_0 \langle E^2 \rangle = \frac{\langle E^2 \rangle}{\mu_0 c}$$

- For ease in calculation define $Z_0$ as:
  $$Z_0 = \mu_0 c = 377 \Omega$$

The Poynting Vector

The direction of the propagation of the electromagnetic wave is given by:

$$\hat{s} = \hat{e} \times \hat{b}$$

This energy transport is defined by the Poynting vector $S$ as:

$$S = \frac{E \times B}{\mu_0}$$

$S$ has the direction of propagation of the wave

- The magnitude of $S$ is directly related to the energy being transported by the wave

$$\langle S \rangle = \frac{\langle E^2 \rangle}{Z_0} = \frac{E^2}{Z_0} = \frac{E^2}{\mu_0 c}$$

- The intensity for harmonic waves is then given by:

 Characteristics

1. $\hat{E} \perp \hat{B}$
2. If $E = E_0 \sin(kx - \omega t)$, $B = B_0 \sin(kx - \omega t)$.
3. The Poynting vector $\hat{S} = \frac{\hat{E} \times \hat{B}}{\mu_0}$ is in the direction of propagation.
4. $\hat{E}$ and $\hat{B}$ are both $\perp \hat{s}$.
5. $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 2.99792458 \times 10^8$ m/s (exact)

Summary of Electromagnetic Radiation

- combined Faraday’s Law and Ampere’s Law
  - time varying $B$-field induces $E$-field
  - time varying $E$-field induces $B$-field

- Electric field and Magnetic field are perpendicular
- energy density
  - $u = \varepsilon_0 E^2 = \frac{B^2}{\mu_0}$
- Poynting Vector describes power flow
  $$\hat{S} = \hat{E} \times \hat{B} / \mu_0 = cu_k$$
- units: watts/m\(^2\)