LC Circuits

• Consider the LC and RC series circuits shown:
• Suppose that at \( t = 0 \) the capacitor is charged to a value of \( Q_0 \).

Is there a qualitative difference in the time development of the currents produced in these two cases? Why?

LC Oscillations

Kirchhoff’s loop rule

\[
V_L + V_C = L \frac{dI}{dt} + \frac{Q}{C} = 0
\]

Define:

- \( Q = Q_0 \cos(\omega_0 t + \phi) \)
- \( I = -\omega_0 Q_0 \sin(\omega_0 t + \phi) \)

\[
\frac{dQ}{dt} = -\omega_0^2 Q_0 \cos(\omega_0 t + \phi)
\]

\[
\frac{dI}{dt} = -\omega_0^2 Q_0 \cos(\omega_0 t + \phi)
\]

Substitute into Loop Rule:

\[-L \omega_0^2 Q_0 \cos(\omega_0 t + \phi) + \frac{Q_0}{C} \cos(\omega_0 t + \phi) = 0\]

\[\omega_0^2 = \frac{1}{LC}\]

\[\Rightarrow \quad \omega_0 = \frac{1}{\sqrt{LC}}\]

Oscillation frequency \( \omega_0 \) has been found from the loop equation.

The other unknowns ( \( Q_0, \phi \) ) are found from the initial conditions. e.g. in our original example we took as given, initial values for the charge (\( Q_i \)) and current (\( 0 \)). For these values: \( Q_0 = Q_i, \phi = 0 \).

Question: Does this solution conserve energy?
Energy Check

Energy in Capacitor

\[ U_E(t) = \frac{1}{2C} Q_0^2 \cos^2(\omega_0 t + \phi) \]

Energy in Inductor

\[ U_B(t) = \frac{1}{2} \left( \frac{1}{\sqrt{LC}} \right)^2 Q_0^2 \sin^2(\omega_0 t + \phi) \]

\[ \omega_0 = \frac{1}{\sqrt{LC}} \]

\[ U_E(t) + U_B(t) = \frac{Q_0^2}{2C} \]

Therefore,

LC Oscillations with Finite R

If \( L \) has finite \( R \), then energy will be dissipated in \( R \) and the oscillations will become damped.

\[ Q \]

\[ t \]

\[ R = 0 \]

\[ \omega_0 = \frac{1}{\sqrt{LC}} \]

\[ Q \]

\[ t \]

\[ R \neq 0 \]

\[ \omega' = \sqrt{\omega_0^2 - \left( \frac{R}{2L} \right)^2} \]

Driven Oscillations

- An LC circuit is a natural oscillator.

\[ \omega_{\text{resonance}} = \frac{1}{\sqrt{LC}} \text{ in absence of resistive loss} \]

- In a real LC circuit, we must account for the resistance of the inductor. This resistance will damp out the oscillations.
AC Circuits: Series LCR

- Statement of problem:
  Given $\varepsilon = \varepsilon_m \sin\omega t$, find $I(t)$.
  Everything else will follow.

Kirchhoff’s loop rule

$$V_L + V_R + V_C = \varepsilon$$

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = \varepsilon_m \sin(\omega t)$$

Phasors: LCR

$$\varepsilon = \varepsilon_m \sin\omega t$$

$$I = I_m \sin(\omega t - \phi)$$

$$\frac{dI}{dt} = I_m \omega \cos(\omega t - \phi)$$

$$V_R = RI = RI_m \sin(\omega t - \phi)$$

$$V_C = \frac{Q}{C} = -\frac{1}{\omega C} I_m \cos(\omega t - \phi)$$

$$V_L = L \frac{dI}{dt} = \omega I_m \cos(\omega t - \phi)$$

- From these equations, we can draw the phasor diagram at the right.

**Phasors: LCR**

$$\tan \phi = \frac{X_L - X_C}{R}$$

$$\varepsilon_m^2 = I_m^2 \left( R^2 + (X_L - X_C)^2 \right)$$

$$I_m = \frac{\varepsilon_m}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\varepsilon_m}{Z}$$
Resonance

- For fixed \( R, C, L \) the current \( I_m \) will be a maximum at the resonant frequency \( \omega_0 \) which makes the impedance \( Z \) purely resistive.
  
  \[
  I_m = \frac{\varepsilon_m}{Z} = \frac{\varepsilon_m}{\sqrt{R^2 + (X_L - X_C)^2}}
  \]
  reaches a maximum when:
  
  \[
  X_L = X_C
  \]
  
  the frequency at which this condition is obtained is given from:
  
  \[
  \omega_0 L = \frac{1}{\omega_0 C}
  \]
  \[
  \omega_0 = \frac{1}{\sqrt{LC}}
  \]
  
  - Note that the resonant frequency is identical to the natural frequency of the LC circuit by itself!
  - At this frequency, the current and the driving voltage are in phase!
  
  \[
  \tan \phi = \frac{X_L - X_C}{R} = 0
  \]

Power in LCR Circuit

- The power supplied by the \emf\ in a series LCR circuit depends on the frequency \( \omega \). The maximum power is supplied at the resonant frequency \( \omega_0 \).
  
  - The instantaneous power (for some frequency, \( \omega \)) delivered at time \( t \) is given by:
    
    \[
    P(t) = \varepsilon(t) I(t) = (\varepsilon_m \sin \omega t)(I_m \sin(\omega t - \phi)) = I^2(t)R
    \]
  
  - The most useful quantity to consider here is not the instantaneous power but rather the average power delivered in a cycle.
    
    \[
    \langle P(t) \rangle = \varepsilon_m I_m \langle \sin \omega t \sin(\omega t - \phi) \rangle
    \]
    
    \[
    \langle P(t) \rangle = \frac{1}{2} V_m I_m \cos \phi
    \]

- This result is often rewritten in terms of rms values:
  
  \[
  \varepsilon_{rms} = \frac{1}{\sqrt{2}} \varepsilon_m \quad I_{rms} = \frac{1}{\sqrt{2}} I_m \quad \langle P(t) \rangle = \varepsilon_{rms} I_{rms} \cos \phi
  \]
  
  - Power delivered depends on the phase, \( \phi \), the “power factor”
  - Phase depends on the values of \( L, C, R \), and \( \omega \) and therefore...
    
    \[
    \tan \phi = \frac{X_L - X_C}{R} \quad \cos \phi = \frac{R}{Z}
    \]