LECTURE 18

- RL Circuits
- Energy in Magnetic Fields

Self Inductance

The inductance of an inductor can be calculated from its geometry alone if the device is constructed from conductors and air.

- If extra material (eg iron core) is added, then we need to add some knowledge of materials as we did for capacitors (dielectrics) and resistors (resistivity)

- Archetypal inductor is a long solenoid, just as a pair of parallel plates is the archetypal capacitor

Calculation of Inductance (from last class)

*Long solenoid with N turns, radius r, length L:

\[
\Phi_M = NBA = N\mu_0 nIA
\]

\[
L = \frac{\Phi_M}{I} = N\mu_0 nA = \mu_0 n^2 lA
\]

\[
n = \frac{N}{l}
\]
**RL Circuits**

- At $t=0$, the switch is closed and the current $I$ starts to flow.

Kirchhoff’s Loop Rule

\[ \epsilon - IR - L \frac{dI}{dt} = 0 \]

Ordinary Differential Equation

\[ L \frac{dI}{dt} + IR = \epsilon \]

Initial Condition: $I(0) = 0$

**Solution for Charging**

\[ I(t) = \frac{\epsilon}{R} \left( 1 - e^{-\frac{t}{\tau_L}} \right) \]

where $\tau_L = \frac{L}{R}$

Initially, an inductor acts to oppose changes in current through it. A long time later, it acts like an ordinary connecting wire.

**RL Circuits: Charging**

- Find the current as a function of time.

\[ I = \frac{\epsilon}{R} \left( 1 - e^{-\frac{t}{\tau_L}} \right) = \frac{\epsilon}{R} \left( 1 - e^{-\frac{t}{\tau_{RL}}} \right) \]

\[ t = 0 \quad e^0 = 1 \quad I = 0 \]

\[ t \to \infty \quad e^{-\infty} = 0 \quad I = \frac{\epsilon}{R} \]

- What about potential differences?

For resistor:

\[ V_R = \epsilon (1 - e^{-\frac{t}{\tau_{RL}}}) \quad V_R(t = 0) = 0 \quad V_R(t \to \infty) = \epsilon \]

For inductor:

\[ V_L = -L \epsilon e^{-\frac{t}{\tau_L}} \quad V_L(t = 0) = \epsilon \quad V_L(t \to \infty) = 0 \]

**RL Circuits (\(\epsilon\) on): Charging**

- Why does $\tau_{RL}$ increase for larger $L$?

- Why does $\tau_{RL}$ decrease for larger $R$?

**Current**

\[ I = \frac{\epsilon}{R} \left( 1 - e^{-\frac{t}{\tau_L}} \right) \]

Max $= \frac{\epsilon}{R}$

63% Max at $t = L/R$

**Voltage on $L$**

\[ V_L = L \frac{dI}{dt} = \epsilon e^{-\frac{t}{\tau_L}} \]

Max $= \frac{\epsilon}{R}$

37% Max at $t = L/R$
RL Circuits: Discharging

After the switch has been in position for a long time, redefined to be $t=0$, it is moved to position b.

Kirchhoff’s Loop Rule: $-IR - L \frac{dI}{dt} = 0$

Ordinary Differential Equation

$$\frac{dI}{I} = -\frac{R}{L} dt$$

Solution for Discharging

$$I = \frac{E}{R} e^{-R/L}$$ where $\tau_L = \frac{L}{R}$

$$V_r = \frac{E}{R} e^{-R/L}, \quad V_L = -\frac{E}{R} e^{-R/L}$$

Inductor in Series

Consider two inductors in series. Both inductors will carry the same current $i$.

$$V = L \frac{di}{dt} = V_1 + V_2 = (L_1 + L_2) \frac{di}{dt}$$

$$L_{eq} = L_1 + L_2$$
Inductor in Parallel

Consider two inductors in series. Both inductors are at the same potential.

\[ I = I_1 + I_2 \]

\[ \frac{dI}{dt} = \frac{I_1}{L_1} + \frac{I_2}{L_2} \]

\[ \frac{V}{L_{EQ}} = \frac{dI}{dt} = \frac{V}{L_1} + \frac{V}{L_2} = V \left( \frac{1}{L_1} + \frac{1}{L_2} \right) \]

\[ \frac{1}{L_{EQ}} = \frac{1}{L_1} + \frac{1}{L_2} \]

Energy in an Inductor

- How much energy is stored in an inductor when a current is flowing through it?

- Start with loop rule:

\[ \epsilon = IR + L \frac{dI}{dt} \]

- Multiply this equation by \( I \):

\[ \epsilon I = I^2 R + LI \frac{dI}{dt} \]

- \( P_L \), the rate at which energy is being stored in the inductor:

\[ P_L = \frac{dU}{dt} = LI \frac{dI}{dt} \]

- Integrate this equation to find \( U \), the energy stored in the inductor when the current = \( I \):

\[ U = \int_0^I U \, dI = \int_0^I L \, dI \]

\[ U = \frac{1}{2} LI^2 \]

Where is the Energy Stored?

- Claim: (without proof) energy is stored in the magnetic field itself (just as in the capacitor / electric field case).

- To calculate this energy density, consider the uniform field generated by a long solenoid:

\[ B = \mu_0 \frac{N}{l} I \]

- The inductance \( L \) is:

\[ L = \mu_0 \frac{N^2}{l} \pi r^2 \]

- Energy \( U \):

\[ U = \frac{1}{2} LI^2 = \frac{1}{2} \left( \mu_0 \frac{N^2}{l} \pi r^2 \right) I^2 = \frac{1}{2} \pi r^2 \frac{B^2}{\mu_0} \]

- The energy density is found by dividing \( U \) by the volume containing the field:

\[ u = \frac{U}{\pi r^2 l} = \frac{1}{2} \frac{B^2}{\mu_0} \]

Mutual Inductance

A changing current in a coil induces an emf in an adjacent coil. The coupling between the coils is described by mutual inductance \( M \).

\[ M_{21} = \frac{N_2 \Phi_{21}}{I_1} \Rightarrow I_1 M_{21} = N_2 \Phi_{21} \]

\[ M_{21} \frac{dI_1}{dt} = N_2 \frac{d\Phi_{21}}{dt} \]

\[ \epsilon_2 = -M_{21} \frac{dI_1}{dt} \quad \text{and} \quad \epsilon_1 = -M_{21} \frac{dI_2}{dt} \]

\( M \) depends only on geometry of the coils (size, shape, number of turns, orientation, separation between the coils).
Mutual Inductance Applications

1. Cool it down
2. Move magnet

What will happen?

\[ \text{emf} = \frac{-d\Phi_{\text{mag}}}{dt} \]

The induced current in the loop produces an induced B field.

Superconductors

Resistivity \( \rho = 0 \) for temperature \( T < T_c \).

A type 1 superconductor is a perfect diamagnetic material with \( \chi_m = -1 \).

\[ \vec{B} = \vec{B}_{\text{app}} (1 + \chi_m) \]

Meissner effect (1939)

B field becomes 0 because superconducting currents on the surface of the superconductor create a magnetic field that cancel the applied one for \( T < T_c \).
Magnetic levitation of a type 1 superconductor

**DEMO**

Magnetic levitation:
Repulsion between the permanent magnetic producing the applied field and the magnetic field produced by the currents induced in the superconductor

Superconductor: $\text{YBa}_2\text{Cu}_3\text{O}_7$

Magnet: NdFeB