Our Study of Magnetism

- Lorentz Force Equation
  \[ \vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \]

- Motion in a uniform B-field
  \[ r_c = \frac{mv}{qB} \]

- Forces on charges moving in wires
  \[ d\vec{F} = Idl \times \vec{B} \]

- Magnetic dipole
  \[ \vec{\mu} = \vec{A}i \]
  \[ \vec{\tau} = \vec{\mu} \times \vec{B} \]
  \[ U = -\vec{\mu} \cdot \vec{B} \]

- Today: fundamentals of how currents generate magnetic fields
LECTURE 14

- **Fundamental Laws for Calculating B-field**
  - Biot-Savart Law ("brute force")
  - Ampere’s Law ("high symmetry")

- **Example: B-field of an Infinite Straight Wire**
  - from Biot-Savart Law
  - from Ampere’s Law

- **Other examples**
Biot-Savart Law: B-field due to a moving charge

\[ \vec{B} = \frac{\mu_0}{4\pi} \frac{q\vec{v} \times \hat{r}}{r^2} \]

Biot-Savart Law: B-field due to a current in a wire

\[ d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\,dl \times \hat{r}}{r^2} \]

Right Hand Rules

The magnetic field “curls” or “loops” around the wire. No dB field at P₂ since dℓ is parallel to r.
Observations about Biot-Savart Law

\[ d\vec{B} = \frac{\mu_o}{4\pi} \frac{Id\vec{l} \times \hat{r}}{r^2} \]

1. \( \hat{r} \) points from \( d\vec{l} \) to the field point.
2. \( d\vec{B} \perp d\vec{l} \) and \( d\vec{B} \perp \hat{r} \)
3. \( |d\vec{l} \times \hat{r}| = d\ell \sin(d\vec{l}, \hat{r}) \).
4. For a given current \( I \), \( d\vec{B} \) is maximum when \( d\vec{l} \perp \hat{r} \), and \( d\vec{B} = 0 \) when \( d\vec{l} \parallel \hat{r} \).
Switch open: $I = 0$
Compass points north.

Switch closed: $I \neq 0$
\[ \vec{B} \perp \vec{I} \vec{d} \vec{l} \]
Magnetic Field of an Infinite Straight Wire

• Calculate field at point $P$ using Biot-Savart Law:

$$dB = \frac{\mu_o}{4\pi} \frac{Idl \sin \varphi}{r^2}$$

$$B = 2 \int_0^\infty dB = \frac{\mu_o I}{2\pi} \int_0^\infty \frac{\sin \varphi dl}{r^2}$$

$$r = \sqrt{s^2 + R^2} \quad \sin \varphi = \sin(\pi - \varphi) = \frac{R}{\sqrt{s^2 + R^2}}$$

$$B = \frac{\mu_o I}{2\pi} \int_0^\infty \frac{Rds}{(s^2 + R^2)^{3/2}}$$

$$\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2 \left(x^2 + a^2\right)^{1/2}}$$

$$B = \frac{\mu_o I R}{2\pi} \left(\frac{s}{R \left(s^2 + R^2\right)^{1/2}}\right)\bigg|_0^\infty = \frac{\mu_o I}{2\pi R} \left(\frac{s}{\left(s^2 + R^2\right)^{1/2}}\right)\bigg|_0^\infty = \frac{\mu_o I}{2\pi R}$$
Calculation of Electric Field

- Two ways to calculate

**Coulomb’s Law**

\[
\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}
\]

"Brute force"

**Gauss’ Law**

\[
\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = Q_{\text{inside}}
\]

"High symmetry"

What are the analogous equations for the Magnetic Field?
Calculation of Magnetic Field

• Two Ways to calculate

Biot-Savart Law

"Brute force"

Ampere’s Law

AMPERIAN LOOP INTEGRAL

"High symmetry“ also,
only the ENCLOSED Current

These are the analogous equations
Magnetic Field of an Infinite Straight Line

- Calculate field at distance $R$ from wire using **Ampere's Law**:

$$\oint \vec{B} \cdot d\vec{l} = \mu_o I_{enc}$$

Ampere's Law simplifies the calculation thanks to symmetry around the current! (axial/cylindrical)
Magnetic Field Lines of a Straight Wire

DEMO – 6B-03
A current $I$ flows in an infinite straight wire in the $+z$ direction (towards you). A concentric infinite cylinder of radius $R$ carries current $2I$ in the $-z$ direction.

What is the magnetic field $B_x(a)$ at point $a$, just outside the cylinder as shown?

(a) $B_x(a) < 0$       (b) $B_x(a) = 0$       (c) $B_x(a) > 0$
Example

A current $I$ flows in an infinite straight wire in the $+z$ direction (towards you). A concentric infinite cylinder of radius $R$ carries current $2I$ in the $-z$ direction.

What is the magnetic field $B_x(b)$ at point $b$, just inside the cylinder as shown?

- (a) $B_x(b) < 0$
- (b) $B_x(b) = 0$
- (c) $B_x(b) > 0$
Magnetic Field at the Center of a Current Loop

Practice both right hand rules here:

$$|d\vec{B}| = \frac{\mu_0 I}{4\pi} \frac{dl}{R^2}$$
Line Segment Combinations

Find \( \vec{B} \) at point P.

\[
d\vec{B} = \frac{\mu_0}{4\pi} \frac{I \, d\ell \times \hat{r}}{r^2} \quad \rightarrow \quad dB = \frac{\mu_0}{4\pi} \frac{I \, d\ell \, \sin \theta}{r^2}
\]

Superposition principle: \( \vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 \)
Double Arc

Find $B$ at point $P$.
Only the arcs contribute.
Magnetic Field on Axis of Current Loop

\[
\vec{d}\vec{B} = \frac{\mu_o}{4\pi} \frac{I d\vec{l} \times \hat{r}}{r^2}
\]
Magnetic Field Lines of a Current Loop
Solenoid (DEMO)

DEMO – 6B-04 & 05
Magnetic Field of a Solenoid

Step 1: Cut up the distribution into pieces

Step 2: Contribution of one piece

**Origin:** center of the solenoid

**One loop:**

$$B_z = \frac{\mu_0}{4\pi} \frac{2\pi R^2 I}{\left( R^2 + (d - z)^2 \right)^{3/2}}$$

Number of loops per meter: $N/L$

Number of loops in $\Delta z$: $(N/L) \Delta z$

Field due to $\Delta z$: 

$$\Delta B_z = \frac{\mu_0}{4\pi} \frac{2\pi R^2 I}{\left( R^2 + (d - z)^2 \right)^{3/2}} \frac{N}{L} \Delta z$$
Magnetic Field of a Solenoid

Step 3: Add up the contribution of all the pieces

\[
dB_z = \frac{\mu_0}{4\pi} \frac{2\pi R^2 I}{\left(R^2 + (d - z)^2\right)^{3/2}} \frac{N}{L} \, dz
\]

\[
B_z = \frac{\mu_0}{4\pi} \frac{2\pi R^2 NI}{L} \int_{-L/2}^{L/2} \frac{dz}{\left(R^2 + (d - z)^2\right)^{3/2}}
\]

Magnetic field of a solenoid:

\[
B_z = \frac{\mu_0}{4\pi} \frac{2\pi NI}{L} \left[ \frac{d + L/2}{\sqrt{(d + L/2)^2 + R^2}} - \frac{d - L/2}{\sqrt{(d - L/2)^2 + R^2}} \right]
\]
Magnetic Field of a Solenoid

\[ B_z = \frac{\mu_0}{4\pi} \frac{2\pi NI}{L} \left[ \frac{d + L/2}{\sqrt{(d + L/2)^2 + R^2}} - \frac{d - L/2}{\sqrt{(d - L/2)^2 + R^2}} \right] \]

Special case: \( R << L \), center of the solenoid (\( d \sim 0 \)):

\[ B_z \approx \frac{\mu_0}{4\pi} \frac{2\pi NI}{L} \left[ \frac{L/2}{\sqrt{(L/2)^2}} - \frac{-L/2}{\sqrt{(L/2)^2}} \right] = \frac{\mu_0}{4\pi} \frac{2\pi NI}{L} [2] \]

\[ B_z \approx \frac{\mu_0NI}{L} \quad \text{in the middle of a long solenoid} \]
Demos

6B-13
Earth’s Magnetic Field

Suppose that the Earth's magnetic field is produced by the current loop that is shown.

\[ R_{\text{Earth}} = 6.37 \times 10^6 \text{ m}, \ R_{\text{loop}} \approx 4 \times 10^6 \text{ m} \]
Earth’s Magnetic Field (continued)

How much current must the loop carry to produce a magnetic field of 0.7 gauss at one of the Earth’s poles?

\[ B_x \approx 0.7 \text{ G} = 0.7 \times 10^{-4} \quad T = \frac{\mu_0}{4\pi} \frac{2\pi R^2 I}{(x^2 + R^2)^{3/2}} \]

\[ I = \frac{2(x^2 + R^2)^{3/2}}{\mu_0 R^2} B_x \]

\[ x = 6.37 \times 10^6 \text{ m}, \quad R \approx 4 \times 10^6 \text{ m} \]

\[ I \approx \frac{2 \left[ (6.37 \times 10^6)^2 + (4 \times 10^6)^2 \right]^{3/2}}{4\pi \times 10^{-7} \times (4 \times 10^6)^2} \quad 0.7 \times 10^{-4} \]

\[ I \approx 3 \times 10^9 \text{ A} = 3 \text{ GA} \]

10kG = 1T
Earth’s Magnetic Field (continued)

1. It has decreased by ~ 6% in the last century.
2. Last polarity reversal was ~ 750,000 years ago.
3. Time between reversals has varied between 20K and 37M years. Average is 500K years.
4. Inner temperature is between 3,900 and 7,200°C degrees. It has cooled about 110°C in the last 4 billion years.
5. There is some agreement that convective motion of the molten part of the earth is generating the magnetic field.