Physics 24100
Electricity & Optics
Lecture 21
Maxwell’s Eqn’s; EM wave propagation
Spring 2019 Semester
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I-Clicker Quiz #20

An LC circuit has $C = 100 \ pF$ and $L = 100 \ \mu H$. The voltage oscillates with an amplitude of 500 mV. What is the amplitude of the current? (Hint: Consider the energy stored in C and in L.)

(a) 500 mA
(b) 50 mA
(c) 500 μA
(d) 50 μA
The equations so far.....

**Gauss' Law for E Fields**
\[ \oint_S \hat{n} \cdot \vec{E} \, dA = \frac{Q_{\text{inside}}}{\epsilon_0} \]

**Gauss' Law for B Fields**
\[ \oint_S \hat{n} \cdot \vec{B} \, dA = 0 \]

**Faraday's Law**
\[ \oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi_m}{dt} \]

**Ampere's Law**
\[ \oint_C \vec{B} \cdot d\vec{l} = \mu_0 I \]

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The Problem with Ampere's Law

Consider a wire and a capacitor. C is a loop around the wire.

\[ \oint_C \vec{B} \cdot d\vec{l} = \mu_0 I \]

- Time dependent situation: current flows in the wire as the capacitor charges up or down.

Current through \( S_1 \) is \( I \)
Current through \( S_2 \) is 0!

The current through \( S_2 \) is zero, but the electric flux is not zero. The electric flux changes as charge flows onto the capacitor.
Maxwell’s Displacement Current

• We can think of the changing electric flux through $S_2$ as if it were a current:

$$I_d = \varepsilon_0 \frac{d\phi_e}{dt} = \varepsilon_0 \frac{d}{dt} \int_{S_2} \vec{E} \cdot \hat{n} dA$$

$$\oint_{C} \vec{B} \cdot d\vec{l} = \mu_0 (I + I_d) = \mu_0 I + \mu_0 \varepsilon_0 \frac{d}{dt} \int_{S} \vec{E} \cdot \hat{n} dA$$

(When we apply Ampere’s Law, we evaluate $I$ & $I_d$ on just one surface, $S_1$ or $S_2$.)

This means that a changing electric flux $\frac{d\phi_e}{dt}$ results in a magnetic field, in just the same way that a changing magnetic flux $\frac{d\phi_m}{dt}$ results in an electric field (Faraday’s Law).

Calculating Displacement Current

*Parallel plate capacitor*

Gauss’s law: (Lecture 5)

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$$

Electric flux:

$$\phi_e = EA = \frac{Q}{\varepsilon_0}$$

Change in electric flux:

$$\frac{d\phi_e}{dt} = \frac{1}{\varepsilon_0} \frac{dQ}{dt} = \frac{I}{\varepsilon_0}$$

Displacement current:

$$I_d = \varepsilon_0 \frac{d\phi_e}{dt} = I$$
i-Clicker Question #1

Suppose you were able to charge a capacitor with constant current (does not change in time).

Does a $B$ field exist in between the plates of the capacitor?

A) NO
B) YES

Answer

Suppose you were able to charge a capacitor with constant current (does not change in time).

Does a $B$ field exist in between the plates of the capacitor?

A) YES  B) NO

Constant current $\rightarrow Q$ increases linearly with time. 

Therefore $E$ increases linearly with time ($E = Q/(A\varepsilon_0)$)

$dE/dt$ is not zero

$\rightarrow$ Displacement current is not zero

$\rightarrow B$ is not zero!
Maxwell Equations: Electromagnetic Waves

We can manipulate Maxwell’s Equations to derive a wave equation for EM waves.

The velocity of electromagnetic waves:
\[ c = 2.99792458 \times 10^8 \text{ m/s} \]

The relationship between E and B in an EM wave

Energy in EM waves: the Poynting vector

Maxwell’s Equations (1864)

\[ \oint_S \hat{n} \cdot \vec{E} \, dA = \frac{Q_{\text{inside}}}{\varepsilon_0} \quad \text{Gauss’ Law} \]

\[ \oint_S \hat{n} \cdot \vec{B} \, dA = 0 \quad \text{(sometimes called Gauss’ Law for Magnetism)} \]

\[ \oint_C \vec{E} \cdot d\vec{l} = -\frac{d\phi_m}{dt} \quad \text{Faraday’s Law} \]

\[ \oint_C \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\phi_e}{dt} \quad \text{Ampere’s Law} \]

in Systeme International (SI or mks) units
Maxwell’s Equations in Free Space

In “free space” where there are no electric charges or sources of current, Maxwell’s equations are quite symmetric:

\[ \oint_S \hat{n} \cdot \vec{E} \, dA = 0 \]
\[ \oint_S \hat{n} \cdot \vec{B} \, dA = 0 \]
\[ \oint_C \vec{E} \cdot d\vec{l} = -\frac{d\phi_m}{dt} \]
\[ \oint_C \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d\phi_e}{dt} \]

- Note the symmetry of Maxwell’s Equations in free space when no charges or currents are present.
- We can predict the existence of electromagnetic waves. Why? Because the wave equation is contained in these equations.

Maxwell’s Equations in Free Space

\[ \oint_C \vec{E} \cdot d\vec{l} = -\frac{d\phi_m}{dt} \]
**Faraday’s Law**
A changing magnetic flux induces an electric field.

\[ \oint_C \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d\phi_e}{dt} \]
**Ampere’s Law**
A changing electric flux induces a magnetic field.

Will this process continue indefinitely?
Light is an Electromagnetic Wave

Step 1 Assume we have a plane wave propagating in $z$ (i.e. $E$, $B$ not functions of $x$ or $y$)

Step 2 Apply Faraday’s Law to an infinitesimal loop in $x$-$z$ plane

- Faraday’s Law:
  \[
  \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_m}{dt} = -\frac{\partial B_y}{\partial t} \Delta x \Delta z
  \]
  \[
  \int_C \mathbf{E} \cdot d\mathbf{l} = E_x(z_2)\Delta x - E_x(z_1)\Delta x
  \approx \frac{\partial E_x}{\partial z} \Delta z \Delta x
  \]

Step 3 Apply Ampere’s Law to an infinitesimal loop in the $y$-$z$ plane:

- Ampere’s law:
  \[
  \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \epsilon_0 \frac{d\Phi_e}{dt} = \mu_0 \epsilon_0 \frac{\partial E_x}{\partial t} \Delta y \Delta z
  \]
  \[
  \int_C \mathbf{B} \cdot d\mathbf{l} = B_y(z_1)\Delta y - B_y(z_2)\Delta y
  \approx -\frac{\partial B_y}{\partial z} \Delta z \Delta y
  \]
Step 4: Use results from steps 2 and 3 to eliminate $B_y$

\[
\begin{align*}
\frac{\partial E_x}{\partial z} &= -\frac{\partial B_y}{\partial t} \\
- \frac{\partial B_y}{\partial z} &= \mu_0\varepsilon_0 \frac{\partial E_x}{\partial t}
\end{align*}
\]

Differentiate the first with respect to $z$:

\[
\frac{\partial^2 E_x}{\partial z^2} = -\frac{\partial^2 B_y}{\partial z \partial t}
\]

Differentiate the second with respect to $t$:

\[
- \frac{\partial^2 B_y}{\partial z \partial t} = \mu_0\varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}
\]

\[
\frac{\partial^2 E_x}{\partial z^2} = \mu_0\varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}
\]

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Review of Waves from Mechanics

The one-dimensional wave equation:

\[
\frac{\partial^2 h}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 h}{\partial t^2}
\]

has a general solution of the form:

\[
h(x, t) = h_1(x - vt) + h_2(x + vt)
\]

A solution for waves traveling in the $+x$ direction is:

\[
h(x, t) = A\cos(kx - \omega t)
\]

### Variable Definitions

- **Amplitude:** $A$
- **Wave Number:** $k = \frac{2\pi}{\lambda}$
- **Wavelength:** $\lambda$
- **Angular Frequency:** $\omega = \frac{2\pi}{T}$
- **Period:** $T$
- **Frequency:** $f = \frac{1}{T}$
- **Velocity:** $v = \frac{\omega}{k}$
Velocity of Electromagnetic Waves

\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}
\]

\[
\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}
\]

Speed of wave propagation is

\[
v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{(4\pi \times 10^{-7} N/A^2)(8.854 \times 10^{-12} C^2/N \cdot m)}}
\]

\[
= 2.998 \times 10^8 \text{ m/s}
\]

(speed of light)

How is B related to E?

\[
\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}
\]

- A solution is \(E_x(z, t) = E_0 \sin(kz - \omega t)\)
  where \(\omega = kc = 2\pi c / \lambda\)
- What is the magnetic field?
  \[
  \frac{\partial B_y}{\partial t} = - \frac{\partial E_x}{\partial z} = -kE_0 \cos(kz - \omega t)
  \]
  \[
  B_y(x, t) = \frac{k}{\omega} E_0 \sin(kx - \omega t)
  \]

How are \(E_x\) and \(B_y\) related in phase and magnitude?
Electromagnetic Waves

- \( \vec{E} \), \( \vec{B} \) and \( \vec{v} \) are mutually perpendicular.
- In general, the direction of wave propagation is \( \hat{s} = \vec{E} \times \vec{B} \)

i-Clicker Question #2

The magnetic field of a light wave oscillates parallel to a y axis and is given by \( B_y = B_m \sin(kz - \omega t) \). In what direction does this wave travel?

A. -y  
B. -z  
C. y  
D. z  
E. -x
The electromagnetic spectrum

- In 1850, the only known forms of electromagnetic waves were ultraviolet, visible and infrared.
- The human eye is only sensitive to a very narrow range of wavelengths:

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Discovery of Radio Waves

- **Electric Waves**
- **Research on the Propagation of Electric Action with Finite Velocity Through Space**
- **A. H. H. Hertz**
- **Macmillan and Co. and New York 1897**

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Energy in Electromagnetic Waves

Energy stored in electric and magnetic fields (lecture 17):

\[ u_e = \frac{1}{2} \varepsilon_0 E^2 \quad u_m = \frac{1}{2 \mu_0} B^2 \]

For an electromagnetic wave, \( B = E/c = E\sqrt{\mu_0\varepsilon_0} \)

\[ u_m = \frac{1}{2 \mu_0} B^2 = \frac{1}{2} \varepsilon_0 E^2 = u_e \]

The total energy density is

\[ u = u_m + u_e = \varepsilon_0 E^2 \]
Intensity of Electromagnetic Waves

- Intensity is defined as the average power transmitted per unit area.
  
  Intensity = Energy density × wave velocity

\[ I = \varepsilon_0 c \langle E^2 \rangle = \frac{\langle E^2 \rangle}{\mu_0 c} \]

\[ \mu_0 c = 377 \, \Omega \equiv Z_0 \]

(Impedance of free space)

Poynting Vector

- We can construct a vector from the intensity and the direction \( \hat{s} = \vec{E} \times \vec{B} \):
  
  \[ \hat{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} \]

  \[ \langle \hat{S} \rangle = \frac{\langle E^2 \rangle}{Z_0} = I \]

- This represents the flow of power in the direction \( \hat{s} \)
- Average electric field: \( E_{rms} = E_0/\sqrt{2} \)
  
  \[ \langle \hat{S} \rangle = \frac{(E_0)^2}{2Z_0} \]

- Units: Watts/m²
1. $\vec{E} \perp \vec{B}$

2. If $E = E_0 \sin(kx - \omega t)$, $B = B_0 \sin(kx - \omega t)$.

3. The Poynting vector $\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$ is in the direction of propagation.

4. $\vec{E}$ and $\vec{B}$ are both $\perp \vec{S}$.

5. $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \equiv 2.99792458 \times 10^8$ m/s (exact)

6. $E_0 = cB_0$

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Characteristics: a Review

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$\vec{S}$