Coexisting two-phase flow in correlated two-dimensional percolation

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The simultaneous percolation of two immiscible (noncrossing) phases in correlated two-dimensional percolation has a monotonically decreasing probability for increasing scale, leading to vanishing probability for infinite systems. However, an *increasing* probability for noncrossing coexistence with increasing observation size occurs for strongly correlated percolation models. When the correlations obey a tunable long-range power law that decays with distance as L^{-a}, an abrupt transition between coexistence and noncoexistence is observed when the criterion $a\nu - d < 0$ is satisfied. [S1063-651X(97)07710-6]

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The simultaneous flow of two immiscible fluids through the random topology of a single fracture is a key problem related to the recovery of energy-related hydrocarbons from partially saturated reservoirs [1], to the remediation of contaminant plumes in aquifers and ground water [2], and to the successful isolation of toxic or radioactive wastes [3]. The geometry of a single fracture is topologically two dimensional and is therefore governed by the percolation properties of two-dimensional systems. It has commonly been considered that coexisting percolation of two immiscible (noncrossing) phases in two dimensions is prohibited for large system sizes. Coexistence is defined as the simultaneous spanning of each phase across an infinite system. Noncrossing coexistence immediately implies that both phases must span in the same direction. Therefore statistical isotropy of the two-dimensional percolation system is also a critical issue for coexistence. For instance, the number of spanning clusters can be larger than unity in an anisotropic system, and increases for increasing anisotropy [4,5].

The requirements for noncrossing coexistence are much more severe than for crossing coexistence. For instance, in bond percolation, or in site percolation with next-nearest-neighbor connections, both phases reach the percolation threshold simultaneously and span even infinite realizations of the percolation system. Therefore crossing coexistence is firmly established. However, for many real percolation systems, such as those described in the first paragraph, the two phases cannot cross each other's path. The condition for noncrossing coexistence is satisfied trivially in three dimensions, while noncrossing coexistence is strictly disallowed in one dimension. Two dimensions is the critical dimensionality for noncrossing coexistence. Noncrossing coexistence is always possible in *finite* two-dimensional systems when it is supported by fluctuations or by correlations that span the finite system [6]. However, even in these cases the two-dimensional noncrossing coexistence probability is expected to *decrease* with increasing system size [7], and to vanish for infinite systems.

In this article, we show that the probability for noncrossing coexistence *increases* with increasing observation size when two-dimensional percolation contains long-range scaling correlations that decay sufficiently slowly. The condition that allows or disallows noncrossing coexistence for infinite two-dimensional systems is extracted from the two-point correlation function $g(L)$, evaluated *at threshold*. When the correlations decay with distance as a power law L^{-a}, long-range quenched disorder can modify some of the critical properties. The extended Harris criterion [8,9], which is an extension of work on the role of fluctuations in critical systems [10–13], states that correlations become relevant when $a\nu - 2 < 0$, for $a < d$, where d is the dimensionality and ν is the correlation length exponent. The extended Harris criterion has been applied to systems with long-range quenched disorder [9,14]. The numerical results shown here suggest that the extended Harris criterion may also determine whether coexisting percolation of noncrossing phases is, or is not, allowed in two dimensions for infinite size.

The case of two-dimensional continuum potentials is considered [15]. In this model a random potential field $V(x,y)$ spans the two-dimensional x-y plane. The occupancy condition is defined by a threshold. A position (x,y) is occupied by the A phase when $V(x,y) < V_c$, and is otherwise occupied by the B phase. Scaling correlations in the continuum potential $V(x,y)$ are produced using a hierarchical cascade [16]. The cascade originates within a square area of side L. Within this area, N positions $(x,y)_i$ are randomly chosen. Each position $(x,y)_i$ is the starting point for the next iteration in which N positions are again chosen randomly within an area of side L/b, where b is a scale factor greater than or equal to unity. The construction progresses iteratively. The iterations can be carried to infinity, but in practice are terminated at the Tth tier when the linear size $L/(b^T)$ falls below a selected cutoff L_c. A realization of this construction is therefore parametrized by T, N, and b, where T is the number of levels for the cascade (called tiers), N is the multiplicity of the cascade, and b is the scale factor between levels. The potential $V(x,y)$ is finally defined by the density of sites on the Tth tier. It has previously been shown that this iterative construction of the correlated potential $V(x,y)$ produces a multifractal topology in which the multifractal spectrum is tuned by changing T, N, and b [16]. The singular part of the two-point correlation function for this model, at threshold, decays with distance as a power law given by $g(L) \propto L^{-a}$. This hierarchical percolation construction produces a percolation model which has the same correlation length exponent ν as standard percolation [17].

To make an especially stringent test of coexistence in the two-dimensional continuum, percolation connections through saddle points are disallowed, which have been shown to strongly control the percolation properties in continuum percolation models [18]. In Monte Carlo simulations the saddle points are removed by projecting the continuum potential onto a discrete square lattice and allowing only nearest-neighbor connections; diagonal connections are not allowed, which also ensures that the A phase and the B phase cannot cross. This procedure forces an underestimation of the coexistence probability, placing a conservative bound on that probability. The persistence of coexistence under these stringent conditions should therefore be a strong indicator.

In the first study we compare a percolation system having long-range scaling correlations with a system that contains only short-range correlations. The long-range scaling system was a three-tier pattern with $N=46$ and $b=4.22$ projected onto a 300×300 discrete square lattice with a lower cutoff $L_c=4$. The short-range correlated system was a one-tier pattern with $N=10\,000$, $b=75$, and $L_c=4$. This one-tier pattern produces a potential with only short-range correlations and therefore serves as a control system.

Monte Carlo simulations of spanning probabilities were performed on 3000 realizations of each percolation system. We calculated the spanning probabilities $R(\phi,L)$ of both phases as a function of occupancy ϕ and size L for system sizes of 50×50, 100×100, 150×150, and 300×300 [16]. The percolation threshold $\phi^*(L)$ at finite size is defined as the fixed points of the spanning probability $R(\phi^*,L) = \phi^*(L)$ using the $M0$ condition [19] for spanning in at least one direction. The fixed points obey the scaling relationship $\phi^*(L) - \phi^*(\infty) \propto L^{-1/\nu}$, where $\nu = \frac{4}{3}$ is the correlation length exponent and $\phi^*(My) = \phi_c$ is the critical threshold for infinite size. When the fixed points are plotted against $L^{-1/\nu}$ the data describe a linear relationship that can be extrapolated to infinite size [16,19].

Finite-size scaling calculations were performed for both the long-range and short-range correlated patterns. The results are shown in Fig. 1 for both cases. For the smallest sample sizes, coexisting percolation of the A phase with the B phase occurs for both short-range and long-range correlations. Phase A begins to span before phase B ceases to span, allowing both phases to (statistically) span simultaneously (although not necessarily within the same pattern). For the larger sample sizes, there is a qualitative difference between the short-range and the long-range correlation cases. With only short-range correlations, the spanning probability evolves into a percolation gap with increasing sample size, in which neither phase spans the pattern. This gap is a result of our conservative rule that disallows saddle-point connections. In a completely random site percolation model, the percolation gap spans from 40% to 60%. In the one-tier pattern, the short-range correlations [20] bring the percolation thresholds close together, but the percolation gap remains. On the other hand, for the long-range correlated geometry, the coexistence region decreases in width, but never transforms into a percolation gap, even when extrapolated to infinite sample sizes, and even with our conservative rule that disallows percolation through saddle points.

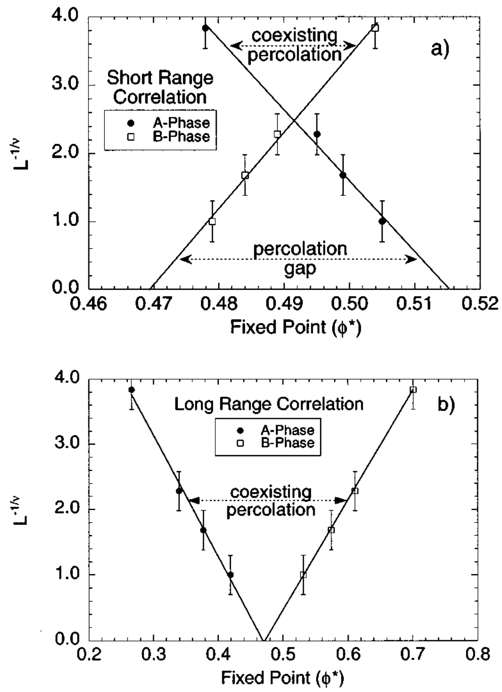

FIG. 1. Percolation fixed points ϕ^* for both phase A and phase B plotted vs $L^{-1/\nu}$ and extrapolated to infinite size for (a) $T=1$ (short-range correlations) and (b) $T=3$ (long-range correlations). For small scales, phase A turns on before phase B turns off, allowing coexisting percolation. For large scales, a percolation gap opens for $T=1$ in which phase B turns off before phase A turns on. No percolation gap is apparent for $T=3$, for which coexisting percolation persists up to infinite size.

While this lack of a percolation gap in a site-percolation model with only nearest-neighbor connections is certainly suggestive, we need to find more conclusive evidence that coexistence *in the same pattern* can persist to infinite size. Therefore during the Monte Carlo simulations we explicitly tabulated the cases of coexistence. The coexistence probabilities are shown in Fig. 2 for both the short-range and long-range correlated geometries as functions of the occupancy for increasing size.

The comparison shows strong qualitative as well as quantitative differences. The most important observation is that the coexistence probability in the short-range correlated case in Fig. 2(a) decreases with increasing size, as expected, vanishing for infinite size. In contrast, the coexistence probability for the long-range correlated case in Fig. 2(b) shows the opposite trend, with the probability *increasing* with increasing size. Therefore the probability that two phases simultaneously percolate in this two-dimensional system grows stronger with increasing size. In addition, the rate of increase of the coexistence probability is also increasing with increasing size, showing an acceleration of the effect as the observation size is increased. An extrapolation to infinite size shows a nonzero coexistence probability. Therefore the cascaded percolation model presented here is a percolation system in which coexistence persists from finite to infinite size.

As a final step, we were able to quantify the degree to

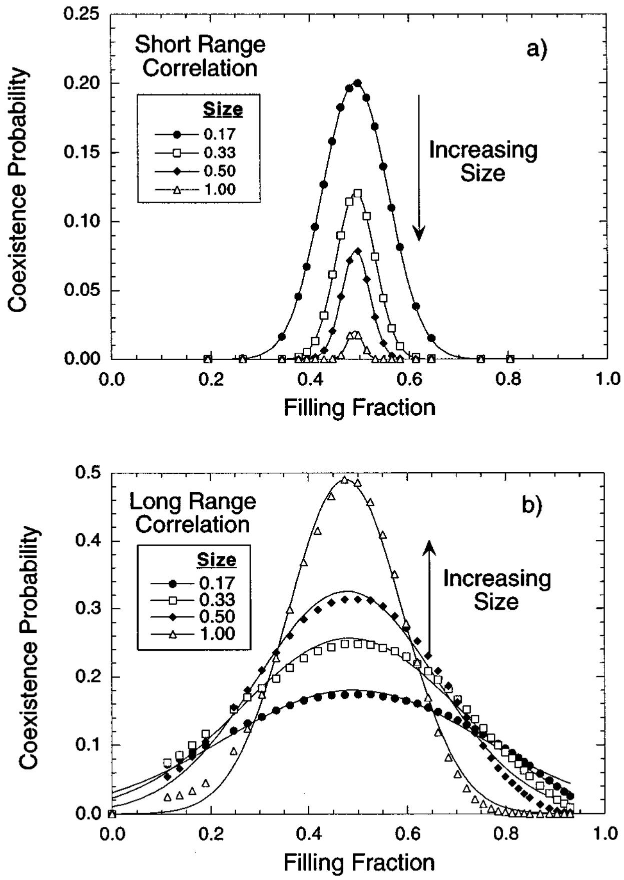

FIG. 2. The coexistence probabilities for (a) $T=1$ (short-range correlations), $N=10\,000$, and $b=75$; and (b) $T=5$ (long-range correlations), $N=10$, and $b=2.37$ as functions of observation size. The coexistence probability decreases with increasing size for short-range correlations, but increases with increasing size for long-range correlations.

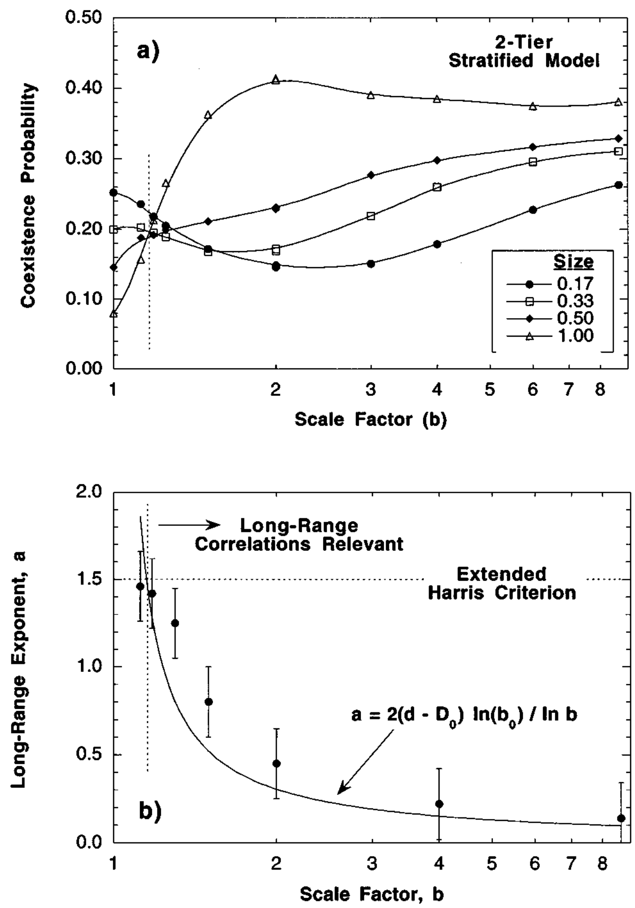

FIG. 3. (a) Coexistence probability at threshold as a function of scale factor b for $T=2$ cascade patterns showing abrupt crossover near $b=1.5$ from decreasing to increasing coexistence probability with increasing scale. (b) Two-point correlation exponent evaluated at threshold for the two-tier patterns (data), compared with the phenomenological dependence of the exponent a (solid line). The correlation exponent crosses the Harris criterion at $a_c \approx 1.5$, which coincides with the crossover behavior observed in (a).

which long-range correlation is necessary to allow coexistence by continuously varying the scale parameter b in the stratified percolation model from the short-range limit to the long-range correlated limit. Crossover behavior is expected to change the coexistence probability from the usual decreasing function to an increasing function of size. We performed Monte Carlo simulations on a two-tier cascade geometry in which the scale parameter is varied continuously from $b=1$ to 8.66. A scale factor of $b=1$ produces the geometry with short-range correlations only, while a scale factor of $b=8.66$ produces a long-range correlated geometry with correlations that fall off slowly with distance. The Monte Carlo coexistence probability is shown in Fig. 3(a) for the two-tier simulations as a function of scale factor for a family of sizes. There is a sudden crossover from the usual decreasing coexistence probability with increasing size to an increasing coexistence probability with increasing size. The crossover occurs when the scale factor is near $b_c=1.2$.

The strength of the long-range correlations at threshold is parametrized by the exponent a of the two-point correlation function $g(L) \propto L^{-a}$. The behavior of a is shown in Fig. 3(b) for comparison to Fig. 3(a). The long-range correlations fall off more slowly for increasing scale parameter b. The exponent a varies as

$$a=2(d-D_0)\,\frac{\ln(b_0)}{\ln b}, \tag{1}$$

where $d=2$ is the Euclidean dimensionality and $D_0=1.95$ is the fractal dimension of the hierarchical patterns produced by the scale factor $b_0=8.66$ [16]. The numerical values of the exponents were obtained by measuring the two-point correlation functions of the percolation patterns of the two-tier patterns, evaluated for the entire pattern at threshold. The crossover from increasing to decreasing coexistence probability clearly occurs when the exponent approaches the criterion $a_c \approx 2/\nu = 1.5$. The striking behavior we observed in Fig. 2(b) of increasing coexistence probability with increasing observation size occurs only when the extended Harris criterion $a\nu - d < 0$ is satisfied for long-range correlations (large scale factor b and slowly decaying correlations). Therefore we find that the special significance of the extended Harris criterion for our system pertains to the relevance of long-range order to support coexisting percolation in two dimensions. It should be noted that in our analysis we do not see a crossover in the correlation length exponent ν from the standard value to a long-range value predicted by Weinrib [9]. However, the importance of the extended Harris criterion remains strongly suggestive from Fig. 3.

To gain physical insight into the ability of long-range correlated systems to support coexisting percolation, we inspected many of the patterns that supported coexistence. In the cases of coexistence the percolation path of phase A (occupying low potential, or wetting phase) delineated the perimeter of phase B (occupying high potential, or non-wetting phase). Therefore, under sufficient long-range correlation, the two percolation paths become locked to each other and copercolate in the same direction, while for insufficient spatial correlations, the two paths become unrelated and cut each other off, preventing coexistence. One of the surprising results of this work is the abruptness at which this topographic locking between the two phases is initiated near $a_c = 1.5$, as seen in Fig. 3.

In conclusion, we have identified a two-dimensional percolation model that possesses long-range scaling geometry in which the probability for coexisting percolation of two immiscible phases increases with increasing size, demonstrating that there is coexistence in two-dimensional percolation, at least within the cascade model discussed here. An important extension of this work would be to define a broader class of scaling geometries which share this property. The agreement of our model with the extended Harris criterion may suggest that other systems satisfying the extended Harris criterion will support coexisting flow. Given the prevalence of correlations in many natural systems such as fractures [21], and in self-organized structures [22], the results described here should find broad applicability.

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